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Sharp large deviations for some hyperbolic systems

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Abstract. We prove a sharp large deviation principle concerning intervals shrinking with sub-exponential speed for certain models involving the Poincaré map related to a Markov family for an Axiom A flow restricted to a basic set Λ satisfying some additional regularity assumptions.

1. *Introduction*

1.1. *Main result.*

Let (X, f, μ) be an ergodic dynamical system, where $f : X \to X$ is a diffeomorphism and μ an ergodic probability measure. For an observable $\Psi : X \to \mathbb{R}$, Birkhoff's ergodic theorem says that

$$\frac{\Psi^n(x)}{n} = \frac{\Psi(x) + \Psi(f(x)) + \cdots + \Psi(f^{n-1}(x))}{n}$$

converges for almost all $x \in X$ with respect to μ to the mean value $M_\Psi = \int_X \Psi \, d\mu$ of Ψ over X. So, if a closed interval Δ does not contain the mean M_Ψ, then the measure of the set $\{x \in X : \Psi^n(x)/n \in \Delta\}$ for n sufficiently large should be small. The theory of large deviations provides exponential bounds for such measures.

For example, it follows from general large deviation principles (see [**Kif, OP, Y**]) that if X is a mixing basic set for an Axiom A diffeomorphism f, Φ and Ψ are Hölder continuous functions on X with equilibrium states m_Φ and m_Ψ, respectively, and m_Ψ is not the measure of maximal entropy of f on X, then there exists a real-analytic *rate function* $J : \mathrm{Int}(\mathcal{I}_\Psi) \longrightarrow [0, \infty)$, where

$$\mathcal{I}_\Psi = \left\{ \int \Psi \, dm : m \in \mathcal{M}_X \right\},$$

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$\mathcal{M}_X$ being the set of all f-invariant Borel probability measures on X, such that

$$\lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n}\log m_\Phi\left(\left\{x\in X:\frac{\Psi^n(x)}{n}\in(p-\delta,\,p+\delta)\right\}\right)=-J(p)$$
$$\text{for all } p\in \mathrm{Int}(\mathcal{I}_\Psi). \tag{1.1}$$

Since m_Ψ is not the measure of maximal entropy, Ψ is not cohomologous to a constant and the interval $\mathcal{I}_\Psi$ is non-trivial and closed. Moreover, $J(p)=0$ if and only if $p=\int \Psi\, dm_\Phi$.

Many results on large deviations for hyperbolic (discrete and continuous) dynamical systems have been established in both the uniformly hyperbolic case (see [**G, Kif, L, OP, Y**] and the references given there) and the non-uniformly hyperbolic case [**AP, MN, RY, PSY, PoS3**].

For shrinking intervals $(p-\delta_n,\,p+\delta_n)$ with $\delta_n\to 0$ as $n\to\infty$ it follows that we have an *upper bound*

$$\limsup_{n\to\infty}\frac{1}{n}\log m_\Phi\left(\left\{x\in X:\frac{\Psi^n(x)}{n}\in(p-\delta_n,\,p+\delta_n)\right\}\right)\le -J(p)$$
$$\text{for all } p\in \mathrm{Int}(\mathcal{I}_\Psi). \tag{1.2}$$

It is natural to study the question about the existence of a *lower bound* in (1.2). Recently, Pollicott and Sharp [**PoS2**] obtained a result of this kind in the case of a hyperbolic diffeomorphism $f:X\longrightarrow X$. Assuming that the Hölder continuous function Ψ satisfies a certain Diophantine condition related to three periodic orbits of f, m_Ψ is not the measure of maximal entropy of f, and the sequence $\{\delta_n\}$ of positive numbers is such that $1/\delta_n=O(n^{1+\kappa})$ as $n\to\infty$ for some appropriately chosen $\kappa>0$, they proved that

$$\lim_{n\to\infty}\frac{1}{n}\log m_\Phi\left(\left\{x\in X:\frac{\Psi^n(x)}{n}\in(p-\delta_n,\,p+\delta_n)\right\}\right)=-J(p) \tag{1.3}$$

for all $p\in\mathrm{Int}(\mathcal{I}_\Psi)$. As a consequence they derived a fluctuation theorem in a similar setup.

An apparently interesting question is whether one can go further and obtain the same lower bound as in (1.3) with a sequence $\{\delta_n\}$ converging much faster to 0. Our aim in this article is to obtain a class of examples where this holds in the case when $\delta_n\to 0$ with *sub-exponential speed*, i.e. when

$$\lim_{n\to\infty}\frac{\log\delta_n}{n}=0. \tag{1.4}$$

Moreover, we also show that for the class of functions we deal with, if

$$\lim_{n\to\infty}\frac{\log\delta_n}{n}=-\alpha_0$$

for some sufficiently small $\alpha_0>0$, the asymptotic (1.3) is not true and we have a lower bound $-J(p)-\alpha_0$. Thus our result in this situation is optimal and we indeed have sharp large deviations. To the best of our knowledge it seems that this is the first result with a precise limit different from $-J(p)$.

Unlike [**PoS2**], in our model the role of X is played by the union of all rectangles in a Markov family for an Axiom A flow restricted to a basic set Λ and f is just the corresponding Poincaré map.

We now proceed to state our assumptions and main result precisely. Let $\varphi_t:M\longrightarrow M$ ($t\in\mathbb{R}$) be a C^2 flow on a Riemannian manifold M and let Λ be a basic set for φ_t. It follows

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from the construction of Bowen [**B**] (cf. also Ratner [**Ra**]) that there exists a Markov family $\mathcal{R} = \{R_i\}_{i=1}^k$ of rectangles $R_i = [U_i, S_i]$ of arbitrarily small size $\chi > 0$ for the restriction of the flow φ_t to Λ (see §2 for terminology and definitions). Set $R = \bigcup_{i=1}^k R_i$. Let $\mathcal{P} : R \longrightarrow R$ and $\tau : R \longrightarrow [0, \infty)$ be the corresponding Poincaré map and *first return time*, respectively, so that $\varphi_{\tau(x)}(x) = \mathcal{P}(x)$. We can then model φ_t on Λ by using the so-called suspended flow on the suspension set $R_\tau = \{(x, t) : x \in R, 0 \le t \le \tau(x)\}$ (see, e.g., [**PP**, Ch. 6]).

Let $F, G : \Lambda \longrightarrow \mathbb{R}$ be continuous functions. We will assume that the representative of G on R_τ is *constant on stable leaves*, i.e. on each set of the form $\{([x, y], t) : y \in S_i\}$, where $i = 1, \ldots, k$, $x \in U_i$ and $t \in [0, \tau(x)]$.

Throughout this paper we assume the following.

Standing assumptions.

(A) φ_t is a mixing flow on a basic set Λ, φ_t and Λ satisfy the conditions (LNIC), (R_1) and (R_2) stated in §2 below and the local holonomy maps along stable laminations through Λ are uniformly Lipschitz.

(B) *$\mathcal{R} = \{R_i\}_{i=1}^k$ is a fixed Markov family of rectangles $R_i = [U_i, S_i]$ for the restriction of the flow φ_t to Λ, chosen so that the matrix $\mathcal{A}$ is irreducible.*

(C) *$F : \Lambda \longrightarrow \mathbb{R}$ is a Hölder continuous function, while $G : \Lambda \longrightarrow \mathbb{R}$ is Lipschitz and its representative in the suspension space R_τ is constant on stable leaves.*

For $T \ge 0$ and $x \in \Lambda$ set

$$G^T(x) = \int_0^T G(\varphi_t(x))\, dt,$$

and let

$$\mathcal{I}_0 = \left\{ \int_R G^{\tau(x)}(x)\, dm(x) : m \in \mathcal{M}_{\mathcal{P}} \right\},$$

where $\mathcal{M}_{\mathcal{P}}$ is the set of all $\mathcal{P}$-invariant Borel probability measures on R. For any function h on R, $x \in R$ and an integer $n \ge 1$ we set

$$h^n(x) = h(x) + h(\mathcal{P}(x)) + \cdots + h(\mathcal{P}^{n-1}(x)).$$

Under the standing assumptions above, in this paper we prove the following main result.

THEOREM 1. *There exists a constant $\mu_0 > 0$ with the following property: for any Lipschitz function $G > 0$ on Λ with $\mathrm{Lip}(G)/\min G \le \mu_0$ such that $G^{\tau(x)}(x)$ is a non-lattice function on R there exists a rate function $J : \mathcal{I}_0 \longrightarrow [0, \infty)$ with the following property: for every Hölder continuous function F on Λ there exists a constant $\rho = \rho(F, G) \in (0, 1)$ such that for any sequence $\{\delta_n\}$ of positive numbers decreasing to zero with*

$$\lim_{n\to\infty} \frac{\log \delta_n}{n} = -\alpha_0 \le 0 \tag{1.5}$$

for some $0 \le \alpha_0 \le -(\log \rho)/2$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log m\left(\left\{ x \in R : \frac{G^{\tau^n(x)}(x)}{n} \in (p - \delta_n, p + \delta_n) \right\} \right) = -J(p) - \alpha_0$$
$$\text{for all } p \in \mathrm{Int}(\mathcal{I}_0), \tag{1.6}$$

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where m is the equilibrium state of the function $F^{\tau(x)}(x)$ on R. In particular, for $\alpha_0 = 0$ we get

$$\lim_{n\to\infty} \frac{1}{n} \log m\left(\left\{x \in R : \frac{G^{\tau^n(x)}(x)}{n} \in (p-\delta_n,\ p+\delta_n)\right\}\right) = -J(p)$$
$$\text{for all } p \in \mathrm{Int}(\mathcal{I}_0). \tag{1.7}$$

The rate function J is explicitly defined in §3 below. The definition of 'non-lattice' is provided in §2.

It should be mentioned that in Theorem 1 it is enough to assume that G is essentially Lipschitz and F is essential Hölder continuous (see the definitions in §1.2.2 below)—the proof given below works without any changes in that case.

Notice that given any Lipschitz function $G > 0$ on Λ, the function $\widetilde{G} = G + c$ satisfies $\mathrm{Lip}(\widetilde{G})/\min \widetilde{G} \le \mu_0$ for any sufficiently large constant $c > 0$. Further remarks on the assumptions in Theorem 1 are given in the next sub-section.

1.2. *On the range of applicability of the main result.* We begin with some remarks concerning the standing assumptions.

1.2.1. *Remarks on the condition (A).* (a) It is well known that in general the maps $\Lambda \ni x \mapsto E^u(x)$ (or $E^s(x)$) are only Hölder continuous (see, e.g., [**Ha**] or [**PSW**]). The same applies to the so-called local stable and unstable holonomy maps (see §2 below for the definitions). The following *pinching condition* implies stronger regularity properties of these maps.

(P): *There exist constants $C > 0$ and $0 < \alpha \le \beta$ such that for every $x \in \Lambda$ we have*

$$\frac{1}{C} e^{\alpha_x t} \|u\| \le \| d\phi_t(x) \cdot u\| \le C\, e^{\beta_x t}\, \|u\|, \quad u \in E^u(x),\ t > 0,$$

for some constants α_x, $\beta_x > 0$ depending on x but independent of u and t with $\alpha \le \alpha_x \le \beta_x \le \beta$ and $2\alpha_x - \beta_x \ge \alpha$ for all $x \in \Lambda$.

For example in the case of contact flows φ_t, it follows from the results in [**Ha**] that assuming (P), the map $\Lambda \ni x \mapsto E^u(x)$ is $C^{1+\epsilon}$ with $\epsilon = 2\alpha/\beta - 1 > 0$ (in the sense that this map has a linearization at any $x \in \Lambda$ that depends Hölder continuously on x). The same applies to the map $\Lambda \ni x \mapsto E^s(x)$.

Notice that when $n = 2$ (then the local unstable manifolds are one-dimensional) the condition (P) is always satisfied. Geodesic flows on manifolds M of strictly negative curvature satisfy the pinching condition (P), provided the sectional curvature is between $-K_0$ and $-a\,K_0$ for some constants $K_0 > 0$ and $a \in (0,\ 1/4)$ [**HP**].

For open billiard flows in the exterior of a compact set K in $\mathbb{R}^n$ which is a finite union of disjoint strictly convex domains satisfying a certain no eclipse condition, the condition (P) is always satisfied when the minimal distance between distinct connected components of K is relatively large compared to the maximal sectional curvature of ∂K [**St3**]. In particular, for such billiards the standing assumption (A) is always satisfied [**St3**] (see also [**PeS1**]).

(b) As shown in [**St2**], the non-integrability condition (LNIC) always holds for contact flows φ_t when $\dim(M) = 3$, and also for transitive contact Anosov flows with Lipschitz

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stable/unstable holonomy maps. In fact, in the latter case (LNIC), (R_1) and (R_2) always hold [**St4**]. In particular, for geodesic flows on compact locally symmetric spaces the standing assumption (A) is always satisfied.

(c) It is proved in [**St5**] that the conditions (R_1) and (R_2) hold under some rather general assumptions for the flow ϕ_t and the basic set Λ. In particular, this is always the case under the pinching condition (P).

(d) We expect that further progress in the analysis of strong spectral estimates for iterations of the Ruelle operator will extend the setup of the dynamical systems for which we can apply our arguments. For example, it is natural to conjecture that for contact Anosov flows one should be able to obtain the result of Theorem 1 without assumption (A).

1.2.2. *Remarks on the condition* $\mathrm{Lip}(G)/\min G \le \mu_0$ *in Theorem 1.*

(a) Let $F, G : \Lambda \longrightarrow \mathbb{R}$ be Hölder continuous functions, and let

$$\Phi(x) = \int_0^{\tau(x)} F(\varphi_t(x))\,dt, \quad \Psi(x) = \int_0^{\tau(x)} G(\varphi_t(x))\,dt \tag{1.8}$$

for all $x \in R$. Then Φ and Ψ are *essentially Hölder* on R, i.e. for each $i \neq j$ they are Hölder continuous on $R_i \cap \mathcal{P}^{-1}(R_j)$ (whenever this is non-empty). In fact, for the latter it is enough to assume that F and G are *essentially Hölder continuous*, i.e. they are Hölder continuous on each *block*

$$B_{ij} = \{\varphi_t(x) : x \in R_i \cap \mathcal{P}^{-1}(R_j),\ 0 \le t \le \tau(x)\}$$

with $R_i \cap \mathcal{P}^{-1}(R_j) \neq \emptyset$. In a similar way one defines *essentially continuous* and *essentially Lipschitz functions* on Λ with respect to the Markov family $\mathcal{R}$.

It is easy to see that Theorem 1 can be stated in terms of the functions Φ and Ψ. Indeed, the measure m in Theorem 1 is m_Φ, the equilibrium state of Φ, and it is easy to show that

$$G^{\tau^n(x)}(x) = \Psi^n(x)$$

for any $x \in R$ (see §2.1 below). Moreover, $\hat{\tau} \le \tau(x) \le \hat{\tau}_0$ $(x \in R)$ for some constants $0 < \hat{\tau} < \hat{\tau}_0$, so $\tau^n(x)/n \in [\hat{\tau}, \hat{\tau}_0]$ for all $x \in \mathbb{R}$. Finally, $\mathcal{I}_0 = \mathcal{I}_\Psi$, and the rate function J in Theorem 1 is simply the rate function related to Ψ (see equation (3.3) below). Thus, (1.6) is equivalent to

$$\lim_{n\to\infty} \frac{1}{n} \log m_\Phi\left(\left\{x \in R : \frac{\Psi^n(x)}{n} \in (p - \delta_n,\ p + \delta_n)\right\}\right) = -J(p) - \alpha_0$$
$$\text{for all } p \in \mathrm{Int}(\mathcal{I}_\Psi). \tag{1.9}$$

(b) It is well known that for any essentially continuous function Ψ on R there exists an essentially continuous function G on Λ for which the second equality in (1.8) holds. Indeed, fix a constant $A \in \mathbb{R}$ and a smooth function $\lambda : [0, 1] \longrightarrow [0, 1]$ such that $\lambda(0) = 0$, $\lambda(1) = 1$ and $\lambda'(0) = \lambda'(1) = 0$, and set

$$G(\varphi_t(x)) = A + \left(\frac{\Psi(x)}{\tau(x)} - A\right)\lambda'(t/\tau(x)), \quad x \in R,\ 0 \le t \le \tau(x).$$

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One checks that G satisfies (1.8) and $G(x) = G(\mathcal{P}(x)) = A$ for all $x \in R$, which makes G continuous on every block B_{ij}. Moreover, if τ is essentially Lipschitz (which is the case under our assumptions; see §2), G is essentially Lipschitz (Hölder) whenever Ψ is essentially Lipschitz (Hölder), and G is constant on stable leaves whenever Ψ is.

(c) Given an essentially Lipschitz function Ψ on R, let $\mathrm{Lip}_e(\Psi)$ be the smallest constant $L \geq 0$ so that for any $x, y \in R_i \cap \mathcal{P}^{-1}(R_j)$ we have $|\Psi(x) - \Psi(y)| \leq Ld(x, y)$. (In a similar way we define $\mathrm{Lip}_e(G)$ for an essentially Lipschitz function G on Λ.) Clearly, for any such Ψ and any $\delta > 0$, adding a sufficiently large constant $C > 0$ to Ψ gives a function $\Psi + C$ satisfying $\mathrm{Lip}_e(\Psi + C)/\min(\Psi + C) \leq \delta$.

However, it is not clear whether one can find G satisfying (1.8) with $\mathrm{Lip}_e(G)/\min G \leq \mu_0$, assuming $\mathrm{Lip}_e(\Psi)/\min \Psi \leq \delta$ for some sufficiently small $\delta > 0$.

(d) It is easy to see that there is a non-trivial open set of essentially Lipschitz functions Ψ on R for which Theorem 1 applies. More precisely, for any constant $c > 0$ there is an open neighbourhood $V(c\tau)$ of $c\tau$ in the space $C_e^{\mathrm{Lip}}(U)$ of essentially Lipschitz functions on R constant on stable leaves such that for any $\Psi \in V(c\tau)$ Theorem 1 applies. Indeed, given $c > 0$, take $\delta = \delta(c) > 0$ small and let $V(c\tau)$ be the set of those $\Psi = c\tau + \omega$, where $\mathrm{Lip}_e(\omega) + \|\omega\|_\infty < \delta$. Given such $\Psi = c\tau + \omega$, define G by

$$G(\varphi_t(x)) = c + \frac{\omega(x)}{\tau(x)}\lambda'(t/\tau(x)), \quad x \in R,\ 0 \leq t \leq \tau(x),$$

where λ is as in 1.2.2(b) above. Then (1.8) holds, and if δ is chosen sufficiently small, we have $\min G \geq c/2$ and $\mathrm{Lip}_e(G) \leq C\delta$ for some constant $C > 0$ (depending on τ and λ only), so $\mathrm{Lip}_e(G)/\min G \leq 2C\delta/c$, which can be made arbitrarily small choosing δ appropriately.

In comparison, one should remark that the Diophantine condition used in [**PoS2**] for the function Ψ related to three periodic orbits is not an 'open condition', although the functions satisfying this condition form a dense space.

It is natural to conjecture that a result similar to Theorem 1 holds for

$$\lim_{n\to\infty} \frac{1}{n} \log(m_\Phi\{x \in R :\ G^{\tau^n(x)} - \tau^n(x)p \in (-\delta_n, \delta_n)\}).$$

However, this cannot be proved by using Theorem 2, since to apply the argument in §4 below, we need to know that the Ruelle operator $\mathcal{L}_{\Phi+(\xi+iu)(\Psi-p\tau)}$ is eventually contracting (see Theorem 2 below). Since the function $\Psi(x) - p\tau(x)$ on R is generated by $G(x) - p$, to use Theorem 2 we would need to have the condition $\mathrm{Lip}_e\ (G)/\min(G - p) \leq \mu_0$ which could not be satisfied. Whether the latter condition can be omitted in the assumptions of Theorem 2 is another open problem.

2. *Preliminaries*

2.1. *Hyperbolic flows on basic sets.* Throughout this paper M denotes a C^2 complete (not necessarily compact) Riemannian manifold, and $\varphi_t : M \longrightarrow M$ $(t \in \mathbb{R})$ a C^2 flow on M. A φ_t-invariant closed subset Λ of M is called *hyperbolic* if Λ contains no fixed points and there exist constants $C > 0$ and $0 < \lambda < 1$ such that there exists a $d\varphi_t$-invariant decomposition $T_xM = E^0(x) \oplus E^u(x) \oplus E^s(x)$ of T_xM $(x \in \Lambda)$ into a direct sum of

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non-zero linear subspaces, where $E^0(x)$ is the one-dimensional subspace determined by the direction of the flow at x, $\|d\varphi_t(u)\| \le C\,\lambda^t\,\|u\|$ for all $u \in E^s(x)$ and $t \ge 0$, and $\|d\varphi_t(u)\| \le C\,\lambda^{-t}\,\|u\|$ for all $u \in E^u(x)$ and $t \le 0$. A non-empty compact φ_t-invariant hyperbolic subset Λ of M which is not a single closed orbit is called a *basic set* for φ_t if φ_t is transitive on Λ and Λ is locally maximal, i.e. there exists an open neighbourhood V of Λ in M such that $\Lambda = \bigcap_{t\in\mathbb{R}} \varphi_t(V)$. When M is compact and M itself is a basic set, φ_t is called an *Anosov flow*.

For $x \in \Lambda$ and a sufficiently small $\epsilon > 0$ let

$$W^s_\epsilon(x) = \{y \in M : d(\varphi_t(x), \varphi_t(y)) \le \epsilon \text{ for all } t \ge 0,\ d(\varphi_t(x), \varphi_t(y)) \to_{t\to\infty} 0\},$$
$$W^u_\epsilon(x) = \{y \in M : d(\varphi_t(x), \varphi_t(y)) \le \epsilon \text{ for all } t \le 0,\ d(\varphi_t(x), \varphi_t(y)) \to_{t\to-\infty} 0\}$$

be the (strong) *stable* and *unstable manifolds* of size ϵ. Then $E^u(x) = T_x W^u_\epsilon(x)$ and $E^s(x) = T_x W^s_\epsilon(x)$. Given $\delta > 0$, set $E^u(x;\,\delta) = \{u \in E^u(x) : \|u\| \le \delta\}$; $E^s(x;\,\delta)$ is defined similarly. For any $A \subset M$ and $I \subset \mathbb{R}$ denote $\varphi_I(A) = \{\varphi_t(y) : y \in A,\, t \in I\}$.

From now on we will assume that Λ is a basic set for φ_t.

It follows from the hyperbolicity of Λ that if $\epsilon_0 > 0$ is sufficiently small, there exists $\epsilon_1 > 0$ such that if $x,\, y \in \Lambda$ and $d(x,\, y) < \epsilon_1$, then $W^s_{\epsilon_0}(x)$ and $\varphi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y))$ intersect at exactly one point $[x,\, y] \in \Lambda$ (cf. [**KH**]). That is, there exists a unique $t \in [-\epsilon_0,\, \epsilon_0]$ such that $\varphi_t([x,\, y]) \in W^u_{\epsilon_0}(y)$. Setting $\Delta(x,\, y) = t$ defines the so-called *temporal distance function* [**D, KB**]. For $x,\, y \in \Lambda$ with $d(x,\, y) < \epsilon_1$, define $\pi_y(x) = [x,\, y] = W^s_\epsilon(x) \cap \varphi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y))$. Thus, for a fixed $y \in \Lambda$, $\pi_y : W \longrightarrow \varphi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y))$ is the *projection* along local stable manifolds defined on a small open neighbourhood W of y in Λ. Choosing $\epsilon_1 \in (0,\, \epsilon_0)$ sufficiently small, the restriction

$$\pi_y : \varphi_{[-\epsilon_1,\epsilon_1]}(W^u_{\epsilon_1}(x)) \longrightarrow \varphi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y))$$

is called a *local stable holonomy map*†. Combining such a map with a shift along the flow we get another local stable holonomy map

$$\mathcal{H}^y_x : W^u_{\epsilon_1}(x) \cap \Lambda \longrightarrow W^u_{\epsilon_0}(y) \cap \Lambda.$$

In a similar way one defines local holonomy maps along unstable laminations. It is well known that the local holonomy maps are uniformly Hölder. Below we will assume that they are actually uniformly Lipschitz, i.e. choosing $\epsilon_1 > 0$ sufficiently small, there exists a constant $C_0 \ge 1$ such that

$$d(\mathcal{H}^y_x(u),\, \mathcal{H}^y_x(v)) \le C_0\, d(u,\, v) \tag{2.1}$$

for all $x,\, y \in \Lambda$ with $d(x,\, y) \le \epsilon_1$, and all $u,\, v \in W^u_{\epsilon_1}(x) \cap \Lambda$.

Given $A \subset \Lambda$, denote by $\operatorname{diam}(A)$ the *diameter* of A. We will say that A is an *admissible subset* of $W^u_\epsilon(z) \cap \Lambda$ ($z \in \Lambda$) if A coincides with the closure of its interior in $W^u_\epsilon(z) \cap \Lambda$. Admissible subsets of $W^s_\epsilon(z) \cap \Lambda$ are defined similarly. Following [**D, R**], a subset R_0 of Λ will be called a *rectangle* if it has the form $R_0 = [U,\, S] = \{[x,\, y] : x \in U,\, y \in S\}$, where U and S are admissible subsets of $W^u_\epsilon(z) \cap \Lambda$ and $W^s_\epsilon(z) \cap \Lambda$, respectively, for

† In a similar way one can define holonomy maps between any two sufficiently close local transversals to stable laminations.

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some $z \in \Lambda$. In what follows we will denote by $\mathrm{Int}^u(U)$ the *interior* of U in the set $W^u_\epsilon(z) \cap \Lambda$. In a similar way we define $\mathrm{Int}^s(S)$, and then set $\mathrm{Int}(R_0) = [\mathrm{Int}^u(U), \mathrm{Int}^s(S)]$. Given $\xi = [x, y] \in R_0$, set $W^u_{R_0}(\xi) = [U, y] = \{[x', y] : x' \in U\}$ and $W^s_{R_0}(\xi) = [x, S] = \{[x, y'] : y' \in S\} \subset W^s_{\epsilon_0}(x)$. The *interiors* of these sets in the corresponding leaves are defined by $\mathrm{Int}^u(W^u_{R_0}(\xi)) = [\mathrm{Int}^u(U), y]$ and $\mathrm{Int}^s(W^s_{R_0}(\xi)) = [x, \mathrm{Int}^s(S)]$.

Let $R = \bigcup_{i=1}^k R_i$ be a union of rectangles with $R_i = [U_i, S_i]$, $U_i \subset W^u_\epsilon(z_i) \cap \Lambda$ and $S_i \subset W^s_\epsilon(z_i) \cap \Lambda$, respectively, for some $z_i \in \Lambda$. The family $\mathcal{R} = \{R_i\}_{i=1}^k$ is called *complete* if there exists $T > 0$ such that for every $x \in \Lambda$, $\varphi_t(x) \in R$ for some $t \in (0, T]$. The *Poincaré map* $\mathcal{P} : R \longrightarrow R$ related to a complete family $\mathcal{R}$ is defined by $\mathcal{P}(x) = \varphi_{\tau(x)}(x) \in R$, where $\tau(x) > 0$ is the smallest positive time with $\varphi_{\tau(x)}(x) \in R$. The function τ is called the *first return time* associated with R. A complete family $\mathcal{R}$ of rectangles in Λ is called a *Markov family* of size $\chi > 0$ for the flow φ_t if $\mathrm{diam}(R_i) < \chi$ for all i and: (a) for any $i \neq j$ and any $x \in \mathrm{Int}(R_i) \cap \mathcal{P}^{-1}(\mathrm{Int}(R_j))$ we have $\mathcal{P}(\mathrm{Int}(W^s_{R_i}(x))) \subset \mathrm{Int}(W^s_{R_j}(\mathcal{P}(x)))$ and $\mathcal{P}(\mathrm{Int}(W^u_{R_i}(x))) \supset \mathrm{Int}(W^u_{R_j}(\mathcal{P}(x)))$; (b) for any $i \neq j$ at least one of the sets $R_i \cap \varphi_{[0,\chi]}(R_j)$ and $R_j \cap \varphi_{[0,\chi]}(R_i)$ is empty.

The existence of a Markov family $\mathcal{R} = \{R_i\}_{i=1}^k$ of an arbitrarily small size $\chi > 0$ for ϕ_t follows from the construction of Bowen [**B**] (cf. also Ratner [**Ra**]).

From now on we will assume that $\mathcal{R} = \{R_i\}_{i=1}^k$ is a fixed Markov family for φ_t of size $\chi < \epsilon_0/2 < 1$. Set $U = \bigcup_{i=1}^k U_i$. The *shift map* $\sigma : U \longrightarrow U$ is given by $\sigma = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \longrightarrow U$ is the *projection* along stable leaves. Notice that τ is constant on each stable leaf $W^s_{R_i}(x) = W^s_{\epsilon_0}(x) \cap R_i$. For any integer $m \geq 1$ and any function $h : U \longrightarrow \mathbb{C}$ define $h^m : U \longrightarrow \mathbb{C}$ by $h^m(u) = h(u) + h(\sigma(u)) + \cdots + h(\sigma^{m-1}(u))$. Then

$$\tau^m(x) = \tau(x) + \tau(\mathcal{P}(x)) + \cdots + \tau(\mathcal{P}^{m-1}(x)).$$

Denote by $\widehat{R}$ the *core* of R, i.e. the set of those $x \in R$ such that $\mathcal{P}^m(x) \in \mathrm{Int}_\Lambda(R)$ for all $m \in \mathbb{Z}$. It is well known (see [**B**]) that $\widehat{R}$ is a residual subset of R and has full measure with respect to any Gibbs measure on R. The set $\widehat{U} = U \cap \widehat{R}$ has similar properties. Clearly, in general τ is not continuous on U, however, under the standing assumption (A), τ is *essentially Lipschitz* on U in the sense that there exists a constant $L > 0$ such that if $x, y \in U_i \cap \sigma^{-1}(U_j)$ for some i, j, then $|\tau(x) - \tau(y)| \leq L\, d(x, y)$. The same applies to $\sigma : \widehat{U} \longrightarrow \widehat{U}$. Throughout we will mainly work with the restrictions of τ and σ to $\widehat{U}$. Set $\widehat{U}_i = U_i \cap \widehat{U}$.

Given $z \in \Lambda$, let $\exp^u_z : E^u(z; \epsilon_0) \longrightarrow W^u_{\epsilon_0}(z)$ and $\exp^s_z : E^s(z; \epsilon_0) \longrightarrow W^s_{\epsilon_0}(z)$ be the corresponding *exponential maps*. A vector $b \in E^u(z)\backslash\{0\}$ will be called *tangent to* Λ at z if there exist infinite sequences $\{v^{(m)}\} \subset E^u(z)$ and $\{t_m\} \subset \mathbb{R}\backslash\{0\}$ such that $\exp^u_z(t_m\, v^{(m)}) \in \Lambda \cap W^u_\epsilon(z)$ for all m, $v^{(m)} \to b$ and $t_m \to 0$ as $m \to \infty$. It is easy to see that a vector $b \in E^u(z)\backslash\{0\}$ is tangent to Λ at z if there exists a C^1 curve $z(t)$ $(0 \leq t \leq a)$ in $W^u_\epsilon(z)$ for some $a > 0$ with $z(0) = z$ and $\dot{z}(0) = b$ such that $z(t) \in \Lambda$ for arbitrarily small $t > 0$.

The following *local non-integrability condition* for φ_t and Λ was introduced in [**St2**].

(LNIC): *There exist* $z_0 \in \Lambda$, $\epsilon_0 > 0$ *and* $\theta_0 > 0$ *such that for any* $\epsilon \in (0, \epsilon_0]$, *any* $\hat{z} \in \Lambda \cap W^u_\epsilon(z_0)$ *and any tangent vector* $\eta \in E^u(\hat{z})$ *to* Λ *at* $\hat{z}$ *with* $\|\eta\| = 1$ *there exist* $\tilde{z} \in \Lambda \cap W^u_\epsilon(\hat{z})$, $\tilde{y}_1, \tilde{y}_2 \in \Lambda \cap W^s_\epsilon(\tilde{z})$ *with* $\tilde{y}_1 \neq \tilde{y}_2$, $\delta = \delta(\tilde{z}, \tilde{y}_1, \tilde{y}_2) > 0$ *and* $\epsilon' = \epsilon'(\tilde{z}, \tilde{y}_1, \tilde{y}_2) \in (0, \epsilon]$ *such that*

$$|\Delta(\exp^u_z(v), \pi_{\tilde{y}_1}(z)) - \Delta(\exp^u_z(v), \pi_{\tilde{y}_2}(z))| \geq \delta\, \|v\|$$

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for all $z \in W^u_{\epsilon'}(\tilde{z}) \cap \Lambda$ *and* $v \in E^u(z; \epsilon')$ *with* $\exp^u_z(v) \in \Lambda$ *and* $\langle v/\|v\|, \eta_z\rangle \geq \theta_0$, *where* η_z *is the parallel translate of* η *along the geodesic in* $W^u_{\epsilon_0}(z_0)$ *from* $\hat{z}$ *to* z.

Set

$$B^u_T(x, \epsilon) = \{y \in W^u_\epsilon(x) \cap \Lambda : d(\varphi_t(x), \varphi_t(y)) \leq \epsilon, 0 \leq t \leq T\}.$$

Following [**St2**], we will say that ϕ_t has a *regular distortion along unstable manifolds* over the basic set Λ if there exists a constant $\epsilon_0 > 0$ with the following properties.

(R_1) For any $0 < \delta \leq \epsilon \leq \epsilon_0$ there exists a constant $C = C(\delta, \epsilon) > 0$ such that

$$\operatorname{diam}(\Lambda \cap B^u_T(z, \epsilon)) \leq C \operatorname{diam}(\Lambda \cap B^u_T(z, \delta))$$

for any $z \in \Lambda$ and any $T > 0$.

(R_2) For any $\epsilon \in (0, \epsilon_0]$ and any $\rho \in (0, 1)$ there exists $\delta \in (0, \epsilon]$ such that for any $z \in \Lambda$ and any $T > 0$ we have $\operatorname{diam}(\Lambda \cap B^u_T(z, \delta)) \leq \rho \operatorname{diam}(\Lambda \cap B^u_T(z, \epsilon))$.

Let $B(\widehat{U})$ be the *space of bounded functions* $g : \widehat{U} \longrightarrow \mathbb{C}$ with its standard norm $\|g\|_\infty = \sup_{x \in \widehat{U}} |g(x)|$. Given a function $g \in B(\widehat{U})$, the *Ruelle transfer operator* $\mathcal{L}_g : B(\widehat{U}) \longrightarrow B(\widehat{U})$ is defined by $(\mathcal{L}_g h)(u) = \sum_{\sigma(v)=u} e^{g(v)} h(v)$. If $g \in B(\widehat{U})$ is Lipschitz on $\widehat{U}$, then $\mathcal{L}_g$ preserves the space $C^{\mathrm{Lip}}(\widehat{U})$ of *Lipschitz functions* $g : \widehat{U} \longrightarrow \mathbb{C}$. Given $\alpha > 0$, let $C^\alpha(U)$ be the space of Hölder continuous functions $G : U \longrightarrow \mathbb{C}$ with

$$|h|_\alpha = \sup\left\{\frac{|h(x) - h(y)|}{(d(x, y))^\alpha} : x, y \in U_i, x \neq y, i = 1, \ldots, k\right\} < \infty.$$

We will consider $C^\alpha(U)$ with the norm $\|h\|_\alpha = \|h\|_\infty + |h|_\alpha$.

The hyperbolicity of the flow on Λ and the standing assumption (A) imply the existence of constants $c_0 \in (0, 1]$ and $\gamma_1 > \gamma > 1$ such that

$$c_0 \gamma^m \, d(u_1, u_2) \leq d(\sigma^m(u_1), \sigma^m(u_2)) \leq \frac{\gamma_1^m}{c_0} d(u_1, u_2), \tag{2.2}$$

whenever $\sigma^j(u_1)$ and $\sigma^j(u_2)$ belong to the same U_{i_j} for all $j = 0, 1, \ldots, m$.

2.2. *Non-lattice condition.* First, it is easy to see that $G^{\tau^n(x)} = \Psi^n(x)$. Indeed,

$$\int_0^{\sum_{j=-1}^{n-1} \tau(\mathcal{P}^j(x))} G(\varphi_t(x)) \, dt = \sum_{k=-1}^{n-2} \int_{\sum_{j=-1}^{k} \tau(\mathcal{P}^j(x))}^{\sum_{j=-1}^{k+1} \tau(\mathcal{P}^j(x))} G(\varphi_t(x)) \, dt,$$

where by convention $\tau(\mathcal{P}^{-1}(x)) = 0$. On the other hand, setting

$$t = \sum_{j=-1}^{k} \tau(\mathcal{P}^j(x)) + s,$$

we get

$$\begin{aligned}\int_{\sum_{j=-1}^{k} \tau(\mathcal{P}^j(x))}^{\sum_{j=-1}^{k+1} \tau(\mathcal{P}^j(x))} G(\varphi_t(x)) \, dt &= \int_0^{\tau(\mathcal{P}^{k+1}(x))} G(\varphi_{\sum_{j=-1}^{k} \tau(\mathcal{P}^j(x))+s}(x)) \, ds \\ &= \int_0^{\tau(\mathcal{P}^{k+1}(x))} G(\varphi_s(\mathcal{P}^{k+1}(x)) \, ds\end{aligned}$$

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and

$$\int_0^{\sum_{j=-1}^{n-1}\tau(\mathcal{P}^j(x))} G(\varphi_t(x))\,dt = \sum_{k=-1}^{n-2}\int_0^{\tau(\mathcal{P}^{k+1}(x))} G(\varphi_s(\mathcal{P}^{k+1}(x))\,ds$$
$$= \Psi(x) + \cdots + \Psi(\mathcal{P}^{n-1}(x)) = \Psi(x) + \cdots + \Psi(\sigma^{n-1}(x)) = \Psi^n(x).$$

A Hölder continuous function $g(x)$ on R is called *non-lattice* if there do not exist a constant a, a Hölder continuous function h on R and a bounded integer-valued function Z on R so that

$$g(x) = (h \circ \mathcal{P})(x) - h(x) + a + Z(x), \quad x \in R.$$

Notice that such a function Z can only have a finite range. If the function $G^{\tau(x)}$ is lattice, for every periodic orbit γ of φ_t issued from $x \in R$ such that $\mathcal{P}^n(x) = x$ with some constant a we have

$$G^{\tau^n(x)} - an = \Psi^n(x) - an = \int_\gamma G(\varphi_t)\,dt - an \in \mathbb{Z}. \tag{2.3}$$

The above condition on Ψ is the same as the lattice condition introduced in [**PoS2**]. We show below that the condition (2.3) is equivalent to another one.

Recall that two (essentially) continuous functions F and G on Λ are called *cohomologuous* ($F \sim G$) if there exists an (essentially) continuously differentiable function H on Λ with $F - G = H'$ (see [**La**]). H is called (essentially) *continuously differentiable* if there exists an (essentially) continuous function H' on Λ such that

$$H'(x) = \lim_{t\to 0} \frac{H(\varphi_t(x)) - H(x)}{t}.$$

Notice that if Φ and Ψ are defined by (1.8) and $F \sim G$, so $F - G = H'$ for some H' as above, then

$$\Phi(x) - \Psi(x) = \int_0^{\tau(x)} H'(\varphi_t(x))\,dt = \int_0^{\tau(x)} \frac{d}{dt} H(\varphi_t(x))\,dt$$
$$= H(\varphi_{\tau(x)}(x)) - H(x) = H(\mathcal{P}(x)) - H(x)$$

for all $x \in R$, so $\Phi \sim \Psi$ on R, which by definition means that

$$\Phi(x) - \Psi(x) = H(\mathcal{P}(x)) - H(x) \quad \text{for all } x \in R$$

with function $H(x)$.

Conversely (see [**La**]), if $\Phi \sim \Psi$ on R, then $F \sim G$ on Λ. We can now express the non-lattice condition on Ψ in terms of the function G. Assume for a moment that Ψ is lattice on R, i.e. there exist a constant $a \in \mathbb{R}$ and a bounded integer-valued function Z on R so that $\Psi \sim a + Z$. Let G_0 be a fixed essentially Lipschitz function on Λ such that $G_0^{\tau(x)}(x) = 1$ for all $x \in R$, e.g. define

$$G_0(\varphi_t(x)) = \frac{1}{\tau(x)}\lambda'(t/\tau(x)), \quad x \in R,\ 0 \le t \le \tau(x),$$

where λ is as in §1.2.2(b). Setting

$$M(\varphi_t(x)) = Z(x) \quad \text{for all } x \in R,\ 0 \le t < \tau(x), \tag{2.4}$$

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we get a bounded integer-valued function on Λ, and $a + Z(x) = (aG_0 + MG_0)^{\tau(x)}(x)$ for all $x \in R$. Thus, $\Psi \sim a + Z$ means that $G \sim aG_0 + MG_0 = (a + M)G_0$. Therefore, a *sufficient condition* for Ψ to be non-lattice is that there do not exist a constant $a \in \mathbb{R}$ and a bounded integer-valued function M on Λ satisfying (2.4) so that $G \sim (a + M)G_0$. This can be used to define a non-lattice condition for G, although it is a bit stronger than what is necessary to make it equivalent to Ψ being non-lattice on R.

3. *Eventually contracting Ruelle operators*

3.1. *Topological pressure and rate functions.* Let φ_t be a C^2 flow on a Riemann manifold M, let Λ be a mixing basic set for ϕ_t, and $\mathcal{R} = \{R_i\}_{i=1}^{\kappa}$ be a fixed Markov family for φ_t of size $\chi < \epsilon_0/2 < 1$.

Denote by $\mathrm{Pr}_{\mathcal{P}}(h)$ the *topological pressure* of a continuous function h on R with respect to the map $\mathcal{P}$ on R. We will often write just $\mathrm{Pr}(h)$ instead of $\mathrm{Pr}_{\mathcal{P}}(h)$. Recall that for any continuous function $\Psi : R \longrightarrow \mathbb{R}$,

$$\mathrm{Pr}_{\mathcal{P}}(\Psi) = \sup_{m \in \mathcal{M}_{\mathcal{P}}} \left(h_{\mathcal{P}}(m) + \int_R \Psi \, dm \right), \tag{3.1}$$

where $\mathcal{M}_{\mathcal{P}}$ is the set of all $\mathcal{P}$-invariant probability measures on R and $h_{\mathcal{P}}(m)$ is the measure theoretic entropy of m with respect to $\mathcal{P}$.

Given a continuous function H on Λ, set

$$H^{\tau}(x) = \int_0^{\tau(x)} H(\varphi_t(x)) \, dt, \quad x \in R.$$

Let m be a $\mathcal{P}$-invariant probability measure on R. There exists a unique φ_t-invariant probability measure μ on Λ such that

$$\int_\Lambda H \, d\mu = \frac{\int_R \left(\int_0^{\tau(x)} H(\varphi_t(x)) \, dt \right) dm(x)}{\int_R \tau \, dm} = \frac{\int_R H^{\tau}(x) \, dm(x)}{\int_R \tau \, dm} \tag{3.2}$$

for any continuous function H on Λ (see, e.g., [**PP**, Ch. 6]). The map $m \mapsto \Omega(m) = \mu$ is onto. Moreover, we have $\mathrm{Pr}_{\mathcal{P}}(H^{\tau} - \mathrm{Pr}_{\varphi}(H)\,\tau) = 0$. If $m = m_{H^{\tau} - \mathrm{Pr}_{\varphi}(H)\,\tau}$ is the *equilibrium state* of $H^{\tau} - \mathrm{Pr}_{\varphi}(H)\,\tau$ on R, then $\mu = \Omega(m)$ coincides with the equilibrium state μ_H of the function H on Λ (see [**PP**, Ch. 6]).

For a continuous function Ψ on R set

$$\mathcal{I}_{\Psi} = \left\{ \int_R \Psi \, dm : m \in \mathcal{M}_{\mathcal{P}} \right\}.$$

Let $F, G :\longrightarrow \mathbb{R}$ be Hölder continuous functions, and let $\Phi = F^{\tau}$ and $\Psi = G^{\tau}$, i.e. (1.8) holds for all $x \in R$. As remarked in §1, Φ and Ψ are essentially Hölder on R.

It follows from the large deviation theorem in [**Kif**] that if m_{Ψ} is not the measure of maximal entropy for $\mathcal{P}$, then there exists a real analytic function $J : \mathrm{Int}(\mathcal{I}_{\Psi}) \longrightarrow [0, \infty)$ such that $J(p) = 0$ if and only if $p = \int_R \Psi \, dm_{\Phi}$ for which (1.2) holds. More precisely, we have

$$-J(p) = \inf\{\mathrm{Pr}_{\mathcal{P}}(\Phi + q\Psi) - \mathrm{Pr}_{\mathcal{P}}(\Phi) - qp : \, q \in \mathbb{R}\}. \tag{3.3}$$

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It is also known that

$$\left[\frac{d}{dq}\,\mathrm{Pr}_{\mathcal{P}}(\Phi+q\Psi)\right]_{q=\eta}=\int_R \Psi\,dm_{\Phi+\eta\Psi}. \tag{3.4}$$

Moreover, for any $p\in\mathcal{I}_\Psi$ there exists a unique number $\xi=\xi_p$ such that

$$-J(p)=\mathrm{Pr}_{\mathcal{P}}(\Phi+\xi\Psi)-\mathrm{Pr}_{\mathcal{P}}(\Phi)-\xi p,\quad p=\int_R \Psi\,dm_{\Phi+\xi\Psi}. \tag{3.5}$$

Under the standing assumptions, define the functions Φ, $\Psi: R\longrightarrow\mathbb{R}$ by

$$\Phi(x)=\int_0^{\tau(x)} F(\varphi_t(x))\,dt,\quad \Psi(x)=\int_0^{\tau(x)} G(\varphi_t(x))\,dt. \tag{3.6}$$

Then Ψ is *essentially Lipschitz on* R, i.e. for each $i\neq j$ it is Lipschitz on $R_i\cap\mathcal{P}^{-1}(R_j)$. Similarly, Φ is *essentially Hölder continuous* on R. This implies (see, e.g., [**PP**]) that there is a well-defined *equilibrium state* m_Φ of the function Φ on R, while for Ψ the consequence is that the Ruelle transfer operators of the form $\mathcal{L}_{f+s\Psi}$ with $s\in\mathbb{C}$ and f Hölder continuous on R and constant on stable leaves, are well defined in appropriately chosen spaces of Hölder continuous functions (see §2 for the main definitions).

As mentioned in §1.2, the limit (1.6) in Theorem 1 it is equivalent to (1.9). So, we must prove (1.9) and we are going to do this in the next section.

Remark 1. Notice that for the proof of (1.9) it is enough to consider the case

$$\mathrm{Pr}_{\mathcal{P}}(\Phi)=0, \tag{3.7}$$

since we can change Φ by adding a constant and this preserves the measure m_Φ. Moreover, it is enough to prove (1.9) in the case when Φ is constant on stable leaves in R, and

$$\int_R \Psi\,dm_\Phi=0. \tag{3.8}$$

Indeed, it follows from Sinai's lemma (cf., e.g., [**PP**, Ch. 1]) that Φ is cohomologous to a Hölder continuous function $\widetilde{\Phi}$ on R which is constant on stable leaves. Then $m_\Phi=m_{\widetilde{\Phi}}$, so we can replace Φ by $\widetilde{\Phi}$. For the second statement above, given functions Φ and Ψ as in Theorem 1, set $\hat{\Psi}=\Psi-c$, where $c=\int_R\Psi\,dm_\Phi$. Then $\int_R\hat{\Psi}\,dm_\Phi=0$. Recall that the function $J:\mathrm{Int}(\mathcal{I}_\Psi)\longrightarrow\mathbb{R}$ is given by (3.3).

Let $\hat{J}:\mathrm{Int}(\mathcal{I}_{\hat{\Psi}})\longrightarrow[0,\infty)$ be the corresponding functional for $\hat{\Psi}$, i.e.

$$-\hat{J}(\hat{p})=\inf\{\mathrm{Pr}_{\mathcal{P}}(\Phi+q\hat{\Psi})-q\hat{p}:\ q\in\mathbb{R}\},\quad \hat{p}\in\mathcal{I}_{\hat{\Psi}}.$$

One checks immediately that $\mathcal{I}_{\hat{\Psi}}=\mathcal{I}_\Psi-c$ and for any $\hat{p}=p-c\in\mathcal{I}_{\hat{\Psi}}$, using the properties of pressure, for any $q\in\mathbb{R}$ we get

$$\mathrm{Pr}_{\mathcal{P}}(\Phi+q\hat{\Psi})-q\hat{p}=\mathrm{Pr}_{\mathcal{P}}(\Phi+q(\Psi-c))-q(p-c)=\mathrm{Pr}_{\mathcal{P}}(\Phi+q\Psi)-qp.$$

Thus, $\hat{J}(\hat{p})=J(p)$. Hence if (1.9) holds with Ψ, J and $p\in\mathcal{I}_\Psi$ replaced by $\hat{\Psi}$, $\hat{J}$ and $\hat{p}\in\mathcal{I}_{\hat{\Psi}}$, then (1.9) holds in its present form, as well.

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3.2. *Ruelle transfer operator.* Given a Lipschitz real-valued function f on $\widehat{U}$, set $\tilde{f} = f - P\Psi$, where $P = P_f \in \mathbb{R}$ is the unique number such that the topological pressure $\mathrm{Pr}_\sigma(\tilde{f})$ of $\tilde{f}$ with respect to σ is zero (cf., e.g., [**PP**]). For $a, b \in \mathbb{R}$, one defines the *Ruelle transfer operator*

$$\mathcal{L}_{\tilde{f}-(a+ib)\Psi} : C^{\mathrm{Lip}}(\widehat{U}) \longrightarrow C^{\mathrm{Lip}}(\widehat{U})$$

in the usual way (cf. §2 above).

We will say that the *Ruelle transfer operators related to* Ψ *and the function* f *on* $\widehat{U}$ *are eventually contracting* if for every $\epsilon > 0$ there exist constants $0 < \rho < 1$, $a_0 > 0$ and $C > 0$ such that if $a, b \in \mathbb{R}$ satisfy $|a| \le a_0$ and $|b| \ge 1/a_0$, then for every integer $m > 0$ and every $h \in C^{\mathrm{Lip}}(\widehat{U})$ we have

$$\|\mathcal{L}^m_{f-(P_f+a+ib)\Psi} h\|_{\mathrm{Lip},b} \le C\ \rho^m\ |b|^\epsilon\ \|h\|_{\mathrm{Lip},b},$$

where the norm $\|\cdot\|_{\mathrm{Lip},b}$ on $C^{\mathrm{Lip}}(\widehat{U})$ is defined by

$$\|h\|_{\mathrm{Lip},b} = \|h\|_\infty + \frac{\mathrm{Lip}(h)}{|b|}.$$

This implies, in particular, that the spectral radius of $\mathcal{L}_{f-(P_f+a+ib)\Psi}$ on $C^{\mathrm{Lip}}(\widehat{U})$ does not exceed ρ.

The following theorem is one of the main ingredients in the proof of Theorem 1.

THEOREM 2. *Under the standing assumptions, let* $\Psi : R \longrightarrow \mathbb{R}$ *be defined by (1.8). Then there exists a constant* $\mu_0 > 0$ *such that if* $\mathrm{Lip}(G)/\min G \le \mu_0$, *then for any Hölder continuous real-valued function* f *on* $\widehat{U}$ *the Ruelle transfer operators related to* Ψ *and* f *are eventually contracting.*

Theorem 1 is derived in §4, while Theorem 2 is proved in §5 using some ideas from [**St2**].

4. *Proof of Theorem 1*

We will use the notation and the assumptions in §§2 and 3. Let Φ and Ψ be as in (3.6). As in the beginning of §3, we may assume (3.7) and (3.8). In this section we will always consider pressure with respect to $\mathcal{P}$, so for brevity we will write Pr instead of $\mathrm{Pr}_{\mathcal{P}}$. Let J be defined by (3.3). Consider a sequence $\{\delta_n\}_{n\in\mathbb{N}}$, $\delta_n > 0$, $\delta_n \to 0$, such that (1.5) holds and let $\epsilon_n = n\delta_n$.

Fix an arbitrary $p \in \mathrm{Int}(\mathcal{I}_\Psi)$ and set $\Psi_p = \Psi - p$. As in [**PoS2**], it is enough to prove a modified result concerning a sequence of the form

$$\rho(n) = \int_U \chi_n(\Psi_p^n(x))\, dm_\Phi, \tag{4.1}$$

where $\chi \in C_0^k(\mathbb{R} : \mathbb{R})$ is a *fixed cut-off function* and

$$\chi_n(x) = \chi(\epsilon_n^{-1} x), \quad x \in \mathbb{R}. \tag{4.2}$$

PROPOSITION 1. *Under the assumptions of Theorem 2, we have*

$$\lim_{n\to\infty} \frac{1}{n}\ \log \rho(n) = -J(p) - \alpha_0.$$

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Theorem 1 follows immediately from Proposition 1 as shown in [**PoS2**], choosing two functions $\chi_-(x)$, $\chi_+(x) \in C_0^k(\mathbb{R}:\mathbb{R})$ so that $\chi_-(x) \le \mathbb{1}_{(-1,1)}(x) \le \chi_+(x)$ and

$$2-\eta \le \int_{\mathbb{R}} \chi_-(x)\,dx \le \int_{\mathbb{R}} \chi_+(x)\,dx \le 2+\eta, \quad \eta > 0.$$

The rest of this section is devoted to the proof of Proposition 1.

For the given functions Φ and Ψ, Theorem 2 implies the following.

COROLLARY 1. *For any $\delta > 0$ and any $a > 0$ there exist $\rho \in (0,1)$, $a_0 > 0$ and $A_\delta > 0$ (depending on a as well) such that*

$$\|\mathcal{L}^n_{\Phi+(\xi+iu)\Psi} 1\|_\infty \le A_\delta\, \rho^n |u|^\delta e^{n\mathrm{Pr}(\Phi+\xi\Psi)}, \tag{4.3}$$

for all integers $n \ge 1$, all $u, \xi \in \mathbb{R}$ with $|u| \ge 1/a_0$ and $|\xi| \le a$.

Let ξ_p be the unique real number such that

$$-J(p) = \inf_{q\in\mathbb{R}} \{\mathrm{Pr}(\Phi + q\Psi_p)\} = \mathrm{Pr}(\Phi + \xi_p\Psi_p).$$

Therefore,

$$\left.\frac{d\mathrm{Pr}(\Phi+q\Psi_p)}{dq}\right|_{q=\xi_p} = \int \Psi_p\, d\mu_{\Phi+\xi_p\Psi_p} = 0.$$

Moreover, since Ψ and Ψ_p are non-lattice, we have

$$\left.\frac{d^2\mathrm{Pr}(\Phi+(\xi+iu)\Psi_p)}{du^2}\right|_{u=0} = -\sigma^2 < 0.$$

The operator $\mathcal{L}_{\Phi+\xi_p\Psi_p}$ has a simple eigenvalue $\lambda_\xi = e^{\mathrm{Pr}(\Phi+\xi_p\Psi_p)}$, and so for all sufficiently small $u \in \mathbb{C}$ the operator $\mathcal{L}_{\Phi+(\xi_p+iu)\Psi}$ has a simple eigenvalue $e^{\mathrm{Pr}(\Phi+(\xi_p+iu)\Psi)}$ and the rest of the spectrum of $\mathcal{L}_{\Phi+(\xi_p+iu)\Psi}$ is contained in a disk of radius $\theta\lambda_\xi$ with some $0 < \theta < 1$. Here we use the fact that for sufficiently small u we can define the pressure $\mathrm{Pr}(\Phi + (\xi_p + iu)\Psi)$ (see [**PP**]).

Clearly, for the Fourier transform $\hat\chi$ of χ we get $\hat\chi_n(u) = \epsilon_n \hat\chi(\epsilon_n u)$. Therefore,

$$\left\{x:\ \frac{\Psi^n(x)}{n} \in (p-\delta_n,\ p+\delta_n)\right\} = \{x:\ \Psi_p^n(x) \in (-\epsilon_n, \epsilon_n)\}.$$

Set $\xi = \xi_p$ and $\omega_n(y) = e^{-\xi y}\chi_n(y)$. We need the following lemma established in [**PoS2**].

LEMMA 1. *If Φ is normalized so that $\mathrm{Pr}(\Phi) = 0$, then*

$$\int e^{(\xi+iu)\Psi_p^n(x)} dm_\Phi(x) = \int \mathcal{L}^n_{\Phi+(\xi+iu)\Psi_p} 1(x)\, dm_\Phi(x).$$

Using this lemma and applying the Fourier transform, we have

$$\begin{aligned}
\rho(n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int e^{iu\Psi_p^n(x)} dm_\Phi(x)\Big)\hat\chi_n(u)\,du \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int \mathcal{L}^n_{\Phi+(\xi+iu)\Psi_p} 1(x) dm_\Phi(x)\Big)\hat\omega_n(u)\,du \\
&= \frac{\epsilon_n}{2\pi}\int_{-\infty}^{\infty}\Big(\int \mathcal{L}^n_{\Phi+(\xi+iu)\Psi_p} 1(x)\, d\mu_\Phi(x)\Big)\hat\chi(\epsilon_n(u-i\xi))\,du.
\end{aligned}$$

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We choose $a > 0$ sufficiently small and changing the coordinates on $(-a, a)$ to $v = v(u)$, we write

$$e^{\Pr(\Phi+(\xi+i\,u)\Psi_p)} = \lambda_\xi(1 - v^2 + i\,Q(v)),$$

where $Q(v)$ is real valued and $Q(v) = \mathcal{O}(|v|^3)$. The analysis in [**PoS2**, §4.1] yields

$$I_1 = \frac{1}{2\pi}\int_{-a}^{a}\Big(\int \mathcal{L}^n_{\Phi+(\xi+i\,u)\Psi_p}1(x)\,dm_\phi(x)\Big)\hat{\omega}_n(u)\,du = \frac{\hat{\chi}(0)}{\sigma}\frac{\epsilon_n}{\sqrt{2\pi n}}\lambda_\xi^n + \mathcal{O}\Big(\frac{\epsilon_n\lambda_\xi^n}{n}\Big).$$

Next we consider the integral

$$I_2 = \frac{\epsilon_n}{2\pi}\int_{a<|u|\le c}\Big(\int \mathcal{L}^n_{\Phi+(\xi+i\,u)\Psi_p}1(x)\,dm_\Phi(x)\Big)\hat{\chi}(\epsilon_n(u - i\xi))\,du$$

with $c \gg 1$ sufficiently large. Since Ψ_p is non-lattice, for $0 < a \le |u| \le c$ the operator $\mathcal{L}_{\Phi+(\xi+iu)\Psi_p}$ has no eigenvalues μ, $|\mu| = \lambda_\xi$ (see, for instance, [**PP**]) and the spectral radius of $\mathcal{L}_{\Phi+(\xi+iu)\Psi_p}$ is strictly less than λ_ξ. Thus, there exists $\beta = \beta(a, c)$, $0 < \beta < 1$, such that for $n \ge N(a, c)$ we have

$$\|\mathcal{L}^n_{\Phi+(\xi+iu)\Psi_p}\| \le \beta^n\lambda_\xi^n. \tag{4.4}$$

On the other hand,

$$|\hat{\chi}(\epsilon_n(u - i\xi))| \le C_j\frac{e^{c_0|\epsilon_n\xi|}}{\epsilon_n^j|u|^j}, \quad |u| \ge a,\ 0 \le j < k, \tag{4.5}$$

with $c_0 > 0$ depending on the support of χ. Applying (4.4) and (4.5) with $j = 0$, for large n we get

$$I_2 = \mathcal{O}\Big(\frac{\epsilon_n\lambda_\xi^n}{n}\Big).$$

Now consider

$$I_3 = \frac{\epsilon_n}{2\pi}\int_{|u|>c}\Big(\int \mathcal{L}^n_{\Phi+(\xi+i\,u)\Psi_p}1(x)\,dm_\Phi(x)\Big)\hat{\chi}(\epsilon_n(u - i\xi))\,du.$$

We are going to use the spectral estimate (4.3). Fix $0 < \delta \le 1/2$ and apply the estimate (4.5) with $j = 2$ and (4.3) for δ. This gives

$$|I_3| \le \epsilon_n\lambda_\xi^n A_\delta e^{c_0|\xi|}\frac{\rho^n}{\epsilon_n^2}\int_{|u|>c}|u|^{\delta-2}\,du = B\epsilon_n\lambda_\xi^n\Big(\frac{\rho^n}{\epsilon_n^2}\Big).$$

According to the condition (1.5), we can arrange for $n \ge n_0$ and $0 \le \alpha_0 < -(\log\rho)/2$ the inequality

$$-\frac{\log n}{n} + \frac{2\log\epsilon_n}{n} = \frac{\log n}{n} + \frac{2\log\delta_n}{n} \ge \log\rho,$$

which leads to

$$\frac{\rho^n}{\epsilon_n^2} \le \frac{1}{n}, \quad n \ge n_0.$$

Thus, we conclude that

$$I_3 = \mathcal{O}\Big(\frac{\epsilon_n\lambda_\xi^n}{n}\Big).$$

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Consequently,

$$\rho(n) = I_1 + I_2 + I_3 = \frac{\hat{\chi}(0)}{\sigma}\frac{\epsilon_n}{\sqrt{2\pi n}}\lambda_\xi^n\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right),$$

and taking into account (1.5), we deduce

$$\lim_{n\to\infty}\frac{1}{n}\log\rho(n) = \log\lambda_\xi + \lim_{n\to\infty}\frac{\log\epsilon_n}{n} = \Pr(\Phi + \xi_p\Psi_p) - \alpha_0 = -J(p) - \alpha_0.$$

This proves Proposition 1. □

Remark 2. By using the argument in [**PeS2, PoS1**], under the assumptions of Theorem 1 we can show that as $n \to \infty$ we have

$$m\left(\left\{x \in R : \frac{G^{\tau^n(x)}(x)}{n} \in (p - \delta_n,\ p + \delta_n)\right\}\right) \sim \frac{2\sqrt{n}\delta_n}{\sqrt{2\pi}\sigma}\exp(-n(J(p))) \quad \text{for all } p \in \mathrm{Int}(\mathcal{I}_0). \tag{4.6}$$

5. *Proof of Theorem 2*

5.1. *Temporal functions.* Let $\varphi_t : M \longrightarrow M$ be a C^2 Axiom A flow on a C^2 complete (not necessarily compact) Riemannian manifold M and let Λ be a basic set for φ_t. Let $\mathcal{R} = \{R_i\}_{i=1}^\kappa$ be a Markov family for φ_t over Λ as in §2. We will assume that the size $\chi > 0$ of the Markov family $\mathcal{R}$ is less than ϵ_1, so that $[x, y]$ is well defined for $x, y \in R_i$ for any i. Denote by $\widehat{\Lambda}$ the set of all $x \in \Lambda$ whose orbits do not have common points with the boundary of any rectangle R_i.

Next, assume that $G > 0$ on Λ and set $A = \min_{x\in\Lambda} G(x) > 0$.

Define the *temporal function*† Δ_Ψ by

$$\Delta_\Psi(x, y) = \int_0^{\Delta(x,y)} G(\varphi_t([x, y]))\, dt \tag{5.1}$$

for $x, y \in \widehat{\Lambda}$, $d(x, y) < \epsilon_1$. Just like Δ, this function is constant on stable leaves with respect to the first variable and constant on unstable leaves with respect to the second. That is, $\Delta_\Psi(x', y') = \Delta_\Psi(x, y)$ for any $x' \in W^s_{\epsilon_0}(x) \cap \Lambda$, $y' \in W^u_{\epsilon_0}(y) \cap \Lambda$ with $d(x', y') < \epsilon_1$.

An important relationship between between Ψ and Δ_Ψ is the following.

LEMMA 2. *Assume that $G \in C^\alpha(U)$ for some $\alpha > 0$. There exists a constant $C_1 \geq 1$ (depending on α) such that for any $i = 1, \ldots, k$, any $x, y \in \widehat{U}_i$ and any integer $m \geq 1$, if x, y belong to the same cylinder of length m (i.e. $\sigma^j(x), \sigma^j(y)$ belong to the same U_{i_j} for all $j = 0, 1, \ldots, m-1$), then*

$$|\Psi^m(x) - \Psi^m(y) + \Delta_\Psi(\mathcal{P}^m(x), \mathcal{P}^m(y))| \leq C_1\,|G|_\alpha\,(d(\mathcal{P}^m(x), \mathcal{P}^m(y)))^\alpha. \tag{5.2}$$

Proof. Let $x, y \in \widehat{U}_i$ and $m \geq 1$ be such that x, y belong to the same cylinder of length m. Assume $\tau^m(x) \geq \tau^m(y)$ (the other case is similar). Since $\varphi_{\tau^m(y)}(x) \in W^u_\epsilon(\mathcal{P}^m(y))$, it follows that $[\mathcal{P}^m(x), \mathcal{P}^m(y)] = \mathcal{P}^m(x)$ and $\Delta(\mathcal{P}^m(x), \mathcal{P}^m(y)) = -(\tau^m(x) - \tau^m(y))$.

† It might be called the *temporal Ψ-function*, since it relates in a natural way to Ψ.

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Thus, changing the variable t to $s = \tau^m(x) - t$ in the second integral in the right-hand side below, we get

$$\begin{aligned}\int_0^{\tau^m(x)} G(\varphi_t(x))\,dt &= \int_0^{\tau^m(y)} G(\varphi_t(x))\,dt + \int_{\tau^m(y)}^{\tau^m(x)} G(\varphi_t(x))\,dt\\ &= \int_0^{\tau^m(y)} G(\varphi_t(x))\,dt - \int_{\tau^m(x)-\tau^m(y)}^{0} G(\varphi_s(\mathcal{P}^m(x)))\,ds\\ &= \int_0^{\tau^m(y)} G(\varphi_t(x))\,dt\\ &\quad - \int_0^{\Delta(\mathcal{P}^m(x),\mathcal{P}^m(y))} G(\varphi_s([\mathcal{P}^m(x),\mathcal{P}^m(y)]))\,ds\\ &= \int_0^{\tau^m(y)} G(\varphi_t(x))\,dt - \Delta_\Psi(\mathcal{P}^m(x),\mathcal{P}^m(y)).\end{aligned}$$

This implies

$$|\Psi^m(x) - \Psi^m(y) + \Delta_\Psi(\mathcal{P}^m(x),\mathcal{P}^m(y))| = \left|\int_0^{\tau^m(y)} [G(\varphi_t(x)) - G(\varphi_t(y))]\,dt\right|.$$

For $D = d(\mathcal{P}^m(x),\mathcal{P}^m(y))$ and $0 \le t \le \tau^m(y)$ we have

$$|G(\varphi_t(x)) - G(\varphi_t(y))| \le C\,|G|_\alpha\,(d(\varphi_t(x),\varphi_t(y)))^\alpha \le C\,|G|_\alpha\,\frac{1}{c_0\,\gamma^{\alpha(\tau^m(y)-t)}}\,D^\alpha$$

for some constants $c_0 > 0$, $\gamma > 1$ (see (2.2) for the case $\alpha = 1$), so it follows that

$$\begin{aligned}\left|\int_0^{\tau^m(y)} [G(\varphi_t(x)) - G(\varphi_t(y))]\,dt\right| &\le \frac{C\,|G|_\alpha\,D^\alpha}{c_0}\int_0^{\tau^m(y)} \gamma^{\alpha(t-\tau^m(y))}\,dt\\ &= \frac{C\,|G|_\alpha\,D^\alpha}{c_0\,\alpha\,\log\gamma}(1 - 1/\gamma^{\alpha\tau^m(y)})\\ &\le C_1\,|G|_\alpha\,(d(\mathcal{P}^m(x),\mathcal{P}^m(y)))^\alpha\end{aligned}$$

for some constant $C_1 > 0$. This proves (5.2). □

5.2. *Spectral estimates for Ruelle transfer operators.* Throughout we assume that φ_t and Λ satisfy the standing assumptions stated in §1.

Given a Lipschitz real-valued function f on $\widehat{U}$, set $\tilde{f} = f - P\Psi$, where $P = P_f \in \mathbb{R}$ is the unique number such that the topological pressure $\mathrm{Pr}_\sigma(\tilde{f})$ of $\tilde{f}$ with respect to σ is zero (cf., e.g., [**PP**]). For $a, b \in \mathbb{R}$, one defines the *Ruelle transfer operator*

$$\mathcal{L}_{\tilde{f}-(a+ib)\Psi} : C^{\mathrm{Lip}}(\widehat{U}) \longrightarrow C^{\mathrm{Lip}}(\widehat{U})$$

in the usual way (cf. §2 above).

To prove Theorem 2 we apply the arguments from [**St2**, §§3 and 5] and a modification of the arguments in [**St2**, §4]. The main step is to prove the analogue of [**St2**, Lemma 4.3] for the roof function τ replaced by Ψ. Below we sketch the arguments required to achieve this following the reasoning in [**St2**, §4] with some modifications.

We will need the following lemma, the first part of which is [**St2**, Lemma 4.1].

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LEMMA 3. *Let $i = 1, \dots, k$, $\tilde{z} \in U_i$ and $\delta > 0$. Then there exists $\epsilon' > 0$ such that for any $y_1 \in W^s_\epsilon(\tilde{z}) \cap \Lambda$ and any $y_2 \in W^s_\epsilon(\tilde{z}) \cap \Lambda$ sufficiently close to y_1, we have:*

(a) $|\Delta(z', \pi_{y_2}(z)) - \Delta(z', \pi_{y_1}(z))| < \delta\, d(z, z')$ *for all* $z, z' \in \Lambda \cap W^u_{\epsilon'}(\tilde{z})$;

(b) $|\Delta_\Psi(z', \pi_{y_2}(z)) - \Delta_\Psi(z', \pi_{y_1}(z))| \le \|G\|_0\, |\Delta(z', \pi_{y_2}(z)) - \Delta(z', \pi_{y_1}(z))|$.

For the proof of part (a) see [**St2**].

Proof of part (b). Let $y_1, y_2 \in W^s_\epsilon(\tilde{z}) \cap \Lambda$ and $z, z' \in \Lambda \cap W^u_\epsilon(\tilde{z})$. By definition, $\pi_{y_j}(z) = [z, y_j] \in W^s_\epsilon(z)$ and $[z', \pi_{y_j}(z)] \in W^s_\epsilon(z')$. Since G is constant on any stable leaf in any rectangle R_i, it follows that $G([z', \pi_{y_j}(z)]) = G(z')$ and moreover $G(\varphi_t([z', \pi_{y_j}(z)])) = G(\varphi_t(z'))$ for all $t \in [0, \tau(z')]$. Thus,

$$\begin{aligned}
\Delta_\Psi(z', \pi_{y_2}(z)) - \Delta_\Psi(z', \pi_{y_1}(z)) &= \int_0^{\Delta(z', \pi_{y_2}(z))} G(\varphi_t([z', \pi_{y_2}(z)]))\, dt \\
&\quad - \int_0^{\Delta(z', \pi_{y_1}(z))} G(\varphi_t([z', \pi_{y_1}(z)]))\, dt \\
&= \int_0^{\Delta(z', \pi_{y_2}(z))} G(\varphi_t(z'))\, dt \\
&\quad - \int_0^{\Delta(z', \pi_{y_1}(z))} G(\varphi_t(z'))\, dt \\
&= \pm \int_0^{\Delta(z', \pi_{y_2}(z)) - \Delta(z', \pi_{y_1}(z))} G(\varphi_t(z'))\, dt.
\end{aligned}$$

Hence

$$|\Delta_\Psi(z', \pi_{y_2}(z)) - \Delta_\Psi(z', \pi_{y_1}(z))| \le |\Delta(z', \pi_{y_2}(z)) - \Delta(z', \pi_{y_1}(z))| \|G\|_0.$$

This proves the lemma. □

Following [**St2**, §4], fix an arbitrary point $z_0 \in \Lambda$ and constants $\epsilon_0 > 0$ and $\theta_0 \in (0, 1)$ with the properties described in (LNIC). Assume that $z_0 \in \mathrm{Int}_\Lambda(U_1)$, $U_1 \subset \Lambda \cap W^u_{\epsilon_0}(z_0)$ and $S_1 \subset \Lambda \cap W^s_{\epsilon_0}(z_0)$. Fix an arbitrary constant θ_1 such that

$$0 < \theta_0 < \theta_1 < 1.$$

Next, fix an arbitrary orthonormal basis $e_1, \dots, e_n$ in $E^u(z_0)$ and a C^1 parametrization $r(s) = \exp^u_{z_0}(s)$, $s \in V_0'$, of a small neighbourhood W_0 of z_0 in $W^u_{\epsilon_0}(z_0)$ such that V_0' is a convex compact neighbourhood of 0 in $\mathbb{R}^n \approx \mathrm{span}(e_1, \dots, e_n) = E^u(z_0)$. Then $r(0) = z_0$ and $(\partial/\partial s_i) r(s)_{|s=0} = e_i$ for all $i = 1, \dots, n$. Set $U_0' = W_0 \cap \Lambda$. Shrinking W_0 (and therefore V_0' as well) if necessary, we may assume that $\overline{U_0'} \subset \mathrm{Int}_\Lambda(U_1)$ and

$$\left| \left\langle \frac{\partial r}{\partial s_i}(s), \frac{\partial r}{\partial s_j}(s) \right\rangle - \delta_{ij} \right|$$

is uniformly small for all $i, j = 1, \dots, n$ and $s \in V_0'$, so that

$$\tfrac{1}{2} \langle \xi, \eta \rangle \le \langle\, dr(s) \cdot \xi, dr(s) \cdot \eta \,\rangle \le 2\, \langle \xi, \eta \rangle, \quad \xi, \eta \in E^u(z_0),\ s \in V_0', \tag{5.3}$$

and

$$\tfrac{1}{2}\, \|s - s'\| \le d(r(s), r(s')) \le 2\, \|s - s'\|, \quad s, s' \in V_0'. \tag{5.4}$$

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Definitions [**St2**].

(a) For a cylinder $\mathcal{C} \subset U_0'$ and a unit vector $\xi \in E^u(z_0)$ we will say that a *separation by a ξ-plane occurs* in $\mathcal{C}$ if there exist $u, v \in \mathcal{C}$ with $d(u, v) \geq \frac{1}{2}\operatorname{diam}(\mathcal{C})$ such that

$$\left\langle \frac{r^{-1}(v) - r^{-1}(u)}{\|r^{-1}(v) - r^{-1}(u)\|}, \xi \right\rangle \geq \theta_1.$$

Let $\mathcal{S}_\xi$ be the *family of all cylinders* $\mathcal{C}$ contained in U_0' such that a separation by an ξ-plane occurs in $\mathcal{C}$.

(b) Given an open subset V of U_0' which is a finite union of open cylinders and $\delta > 0$, let $\mathcal{C}_1, \ldots, \mathcal{C}_p$ ($p = p(\delta) \geq 1$) be the family of maximal closed cylinders in $\overline{V}$ with $\operatorname{diam}(\mathcal{C}_j) \leq \delta$. For any unit vector $\xi \in E^u(z_0)$ set $M_\xi^{(\delta)}(V) = \bigcup\{\mathcal{C}_j : \mathcal{C}_j \in \mathcal{S}_\xi, 1 \leq j \leq p\}$.

In what follows we will construct, amongst other things, a sequence of unit vectors $\xi_1, \xi_2, \ldots, \xi_{j_0} \in E^u(z_0)$. For each $\ell = 1, \ldots, j_0$ set $B_\ell = \{\eta \in \mathbf{S}^{n-1} : \langle \eta, \xi_\ell \rangle \geq \theta_0\}$. For $t \in \mathbb{R}$ and $s \in E^u(z_0)$ set $I_{\eta,t}g(s) = (g(s + t\eta) - g(s))/t$, $t \neq 0$ (*increment* of g in the direction of η).

As in [**St2**], where we dealt with the case $\Psi = \tau$, the main aim will be to prove the following.

LEMMA 4. *There exist integers $1 \leq n_1 \leq N_0$ and $\ell_0 \geq 1$, a sequence of unit vectors $\eta_1, \eta_2, \ldots, \eta_{\ell_0} \in E^u(z_0)$ and a non-empty open subset U_0 of U_0' which is a finite union of open cylinders of length n_1 such that setting $\mathcal{U} = \sigma^{n_1}(U_0)$ we have:*

(a) *for any integer $N \geq N_0$ there exist Lipschitz maps $v_1^{(\ell)}, v_2^{(\ell)} : U \longrightarrow U$ ($\ell = 1, \ldots, \ell_0$) such that $\sigma^N(v_i^{(\ell)}(x)) = x$ for all $x \in \mathcal{U}$ and $v_i^{(\ell)}(\mathcal{U})$ is a finite union of open cylinders of length N ($i = 1, 2$; $\ell = 1, 2, \ldots, \ell_0$);*

(b) *there exists a constant $\hat{\delta} > 0$ such that for all $\ell = 1, \ldots, \ell_0$, $s \in r^{-1}(U_0)$, $0 < |h| \leq \hat{\delta}$ and $\eta \in B_\ell$ so that $s + h\eta \in r^{-1}(U_0 \cap \Lambda)$ we have*

$$\left[I_{\eta,h}\left(\Psi^N(v_2^{(\ell)}(\tilde{r}(\cdot))) - \Psi^N(v_1^{(\ell)}(\tilde{r}(\cdot)))\right)\right](s) \geq \frac{A\hat{\delta}}{4};$$

(c) *we have $\overline{v_i^{(\ell)}(U)} \cap \overline{v_{i'}^{(\ell')}(U)} = \emptyset$ whenever $(i, \ell) \neq (i', \ell')$;*

(d) *for any open cylinder V in U_0 there exists a constant $\delta' = \delta'(V) > 0$ such that*

$$V \subset M_{\eta_1}^{(\delta)}(V) \cup M_{\eta_2}^{(\delta)}(V) \cup \cdots \cup M_{\eta_{\ell_0}}^{(\delta)}(V)$$

for all $\delta \in (0, \delta']$.

Clearly, if U_0 and $\mathcal{U}$ are as in the lemma, then we must have $\overline{\mathcal{U}} = U$.

The following lemma is proved in [**St2**, §4].

LEMMA 5. [**St2**, Lemma 4.4] *There exist an integer $\ell_0 \geq 1$, open cylinders $U_0^{(\ell_0)} \subset \cdots \subset U_0^{(1)}$ contained in U_0', and for each $\ell = 1, \ldots, \ell_0$, an integer $m_\ell \geq 1$ such that the following hold.*

(i) *For each $i = 1, 2$, there exists a contracting map $w_i^{(\ell)} : U_0^{(\ell)} \longrightarrow U_1$ such that $\sigma^{m_\ell}(w_i^{(\ell)}(x)) = x$ for all $x \in U_0^{(\ell)}$, $w_i^{(\ell)} : U_0^{(\ell)} \longrightarrow w_i^{(\ell)}(U_0^{(\ell)})$ is a homeomorphism, $w_i^{(\ell)}(U_0^{(\ell)})$ is an open cylinder of length $\geq m_\ell$ in U_1, and the sets $\overline{w_i^{(\ell)}(U_0^{(\ell)})}$ ($\ell = 1, \ldots, \ell_0$, $i = 1, 2$) are disjoint.*

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(ii) *For each* $\ell = 1, \dots, \ell_0$ *there exist a number* $\hat{\delta}_\ell \in (0, \delta_0)$ *and a vector* $\eta_\ell \in \mathbf{S}^{n-1}$ *such that* $|\langle \eta_\ell, \eta_{\ell'} \rangle| \le \theta_1$ *whenever* $\ell' \ne \ell$, *and*

$$\inf_{0<|h|\le\hat{\delta}_\ell} \left| \left[I_{\eta,h} \Big(\tau^{m_\ell}(w_2^{(\ell)}(r(\cdot))) - \tau^{m_\ell}(w_1^{(\ell)}(r(\cdot))) \Big) \right](s) \right| \ge \hat{\delta}_\ell, \quad r(s) \in U_0^{(\ell)}, \ \eta \in B_\ell,$$

for all $\ell = 1, \dots, \ell_0$, *where the* inf *is taken over* h *with* $r(s + h\eta) \in U_0^{(\ell)}$.

(iii) *For any open cylinder* V *in* $U_0^{(\ell_0)}$ *there exists a constant* $\delta' = \delta'(V) \in (0, \delta_0)$ *such that* $V \subset M_{\eta_1}^{(\delta)}(V) \cup M_{\eta_2}^{(\delta)}(V) \cup \cdots \cup M_{\eta_{\ell_0}}^{(\delta)}(V)$ *for all* $\delta \in (0, \delta']$.

In what follows we use the objects constructed in Lemma 5. Set $\hat{\delta} = \min_{1\le\ell\le\ell_0} \hat{\delta}_\ell$, $n_0 = \max_{1\le\ell\le\ell_0} m_\ell$, and fix an arbitrary point $\hat{z}_0 \in U_0^{(\ell_0)} \cap \widehat{U}$. Set

$$L = \mathrm{Lip}(G), \quad D = \mathrm{diam}(\Lambda),$$

and recall that $A = \min_{x\in\Lambda} G(x) > 0$. Without loss of generality we may assume that $D \ge 1$. We will also assume that

$$\mu_0 \le \frac{c_0\, \hat{\delta}}{128\, C_0\, C_1\, D}. \tag{5.5}$$

Next, essentially repeating some arguments from the proof of Lemma 5 we get an analogue of property (ii) there with τ replaced by Ψ.

LEMMA 6. *Assume that (5.5) holds and* $L \le \mu_0\, A$. *Then for every* $\ell = 1, \dots, j_0$ *we have*

$$\inf_{0<|h|\le\hat{\delta}_\ell} \left| I_{\eta,h} \Big[\Psi^{m_\ell}(w_2^{(\ell)}(r(s))) - \Psi^{m_\ell}(w_1^{(\ell)}(r(s))) \Big] \right| \ge \frac{A\, \hat{\delta}_\ell}{4}, \quad r(s) \in U_0^{(\ell)}, \ \eta \in B_\ell,$$

where the inf *is taken over* h *with* $r(s + h\eta) \in U_0^{(\ell)}$.

Sketch of proof of Lemma 6. Clearly, $\|G\|_0 \le A + LD$, so assuming $L \le \mu_0\, A$, we have $\|G\|_0 \le A(1 + \mu_0\, D)$.

We will follow the inductive steps in [**St2**, proof of Lemma 4.4]. In fact, we will treat the case $\ell = 1$ in detail; in the general case the argument is very similar.

Step 1. As in [**St2**], consider a unit vector $\xi_1 \in E^u(z_0)$ tangent to Λ at z_0, and then using the condition (LNIC) and the choice of z_0 we find $\tilde{z} = r(\tilde{\sigma}) \in U_0'$, $y_1', y_2' \in W_{R_1}^s(\tilde{z})$ with $y_1' \ne y_2'$, $\delta_1' > 0$, $\epsilon_1' > 0$, and an open cylinder $U_0^{(1)}$ contained in $B_U(\tilde{z}, \epsilon_1') \subset U_0'$ with $\tilde{z} \in U_0^{(1)}$ such that

$$|\Delta(r(s + h\,\eta), \pi_{y_1'}(r(s))) - \Delta(r(s + h\,\eta), \pi_{y_1''}(r(s)))| \ge \hat{\delta}_1\, |h|$$

whenever $r(s) \in U_0^{(1)}$, $\eta \in B_1$, $|h| \le \hat{\delta}_1$ and $r(s + h\eta) \in U_0^{(1)}$, where $\hat{\delta}_1 = \min\{\delta_1'/2, \epsilon_1'\}$. The above and Lemma 3(b) imply

$$|\Delta_\Psi(r(s + h\,\eta), \pi_{y_1'}(r(s))) - \Delta_\Psi(r(s + h\,\eta), \pi_{y_1''}(r(s)))| \ge A\hat{\delta}_1\, |h|. \tag{5.6}$$

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Now for $z = r(s) \in U_0^{(1)}$, $\eta \in B_1$ and $h \in \mathbb{R}$ with $r(s + h\,\eta) \in U_0^{(1)}$ using (5.2) we get

$$\begin{aligned}
&|[\Psi^{m_1}(w_2^{(1)}(r(s+h\,\eta))) - \Psi^{m_1}(w_1^{(1)}(r(s+h\,\eta)))] \\
&\quad - [\Psi^{m_1}(w_2^{(1)}(r(s))) - \Psi^{m_1}(w_1^{(1)}(r(s)))]| \\
&= |[\Psi^{m_1}(w_1^{(1)}(r(s))) - \Psi^{m_1}(w_1^{(1)}(r(s+h\,\eta)))] \\
&\quad - [\Psi^{m_1}(w_2^{(1)}(r(s))) - \Psi^{m_1}(w_2^{(1)}(r(s+h\,\eta)))]| \\
&\geq |\Delta_\Psi(\mathcal{P}^{m_1}(w_1^{(1)}(r(s+h\,\eta))), \mathcal{P}^{m_1}(w_1^{(1)}(r(s)))) \\
&\quad - \Delta_\Psi(\mathcal{P}^{m_1}(w_2^{(1)}(r(s+h\,\eta))), \mathcal{P}^{m_1}(w_2^{(1)}(r(s))))| \\
&\quad - C_1\,L\,\sum_{i=1}^{2} d(\mathcal{P}^{m_1}(w_i^{(1)}(r(s+h\,a))), \mathcal{P}^{m_1}(w_i^{(1)}(r(s)))) \\
&= |\Delta_\Psi(\pi_{y_1'}(r(s+h\,\eta)), \pi_{y_1'}(r(s))) - \Delta_\Psi(\pi_{y_1''}(r(s+h\,\eta)), \pi_{y_1''}(r(s)))| \\
&\quad - C_1\,L\,\sum_{i=1}^{2} d(\mathcal{P}^{m_1}(w_i^{(1)}(r(s+h\,\eta))), \mathcal{P}^{m_1}(w_i^{(1)}(r(s)))). \qquad (5.7)
\end{aligned}$$

Since $\mathcal{P}^{m_1}(w_1^{(1)}(x)) = \pi_{y_1'}(x)$ and $\mathcal{P}^{m_1}(w_2^{(1)}(x)) = \pi_{y_1''}(x)$, using (2.1) and (5.4) we get

$$\begin{aligned}
d\Big(\mathcal{P}^{m_1}(w_1^{(1)}(r(s+h\,\eta))), \mathcal{P}^{m_1}(w_1^{(1)}(r(s)))\Big) &= d\Big(\pi_{y_1'}(r(s+h\,\eta)), \pi_{y_1'}(r(s))\Big) \\
&\leq C_0\,d\Big((r(s+h\,\eta)), r(s)\Big) \leq 2C_0\,|h|.
\end{aligned}$$

Similarly, $d(\mathcal{P}^{m_1}(w_2^{(1)}(r(s+h\,\eta))), \mathcal{P}^{m_1}(w_2^{(1)}(r(s)))) \leq 2C_0\,|h|$. This, (5.7), (5.6) and (5.5) imply

$$\begin{aligned}
&\Big|I_{\eta,h}\Big[\Psi^{m_1}(w_2^{(1)}(r(s))) - \Psi^{m_1}(w_1^{(1)}(r(s)))\Big]\Big| \\
&\quad \geq \Big|\Delta_\Psi\Big(r(s+h\,\eta), \pi_{y_1'}(r(s))\Big) - \Delta_\Psi\Big(r(s+h\,\eta), \pi_{y_1''}(r(s))\Big)\Big| - 4C_0C_1L\,|h| \\
&\quad \geq A\,\hat{\delta}_1\,|h| - 4\mu_0 A\,C_0\,C_1\,|h| \geq \frac{A\,|h|\,\hat{\delta}_1}{2}
\end{aligned}$$

whenever $r(s) \in U_0^{(1)}$, $\eta \in B_1$, $0 < |h| \leq \epsilon_1'$ and $r(s + h\,\eta) \in U_0^{(1)}$.

This proves the statement for $\ell = 1$. As we mentioned above, in the general case the argument is the same. □

Next, as in [**St2**] we fix $n_1 > 0$ and an open neighbourhood U_0 of $\hat{z}_0$ in $U_0^{(j_0)}$ such that U_0 is a finite union of open cylinders of length n_1, $\mathcal{U} = \sigma^{n_1}(U_0) = \mathrm{Int}_\Lambda(U)$ and $\sigma^{n_1} : U_0 \longrightarrow \mathcal{U}$ is a homeomorphism. The inverse homeomorphism $\psi : \mathcal{U} \longrightarrow U_0$ is Lipschitz, so it has a Lipschitz extension

$$\psi : U \longrightarrow \overline{U_0} \quad \text{such that } \sigma^{n_1}(\psi(x)) = x, \quad x \in \mathcal{U}. \qquad (5.8)$$

Then $\tilde{r}(s) = \sigma^{n_1}(r(s))$, $s \in V_0$, where $V_0 = r^{-1}(U_0) \subset V_0'$, gives a Lipschitz parametrization of $\mathcal{U}$ with $\psi(\tilde{r}(s)) = r(s)$ for all $s \in V_0$. Finally, set

$$V_i^{(j)} = w_i^{(j)}(U_0) \subset \widetilde{V}_i^{(j)}, \quad i = 1, 2;\ j = 1, \ldots, j_0. \qquad (5.9)$$

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It follows from the choice of U_0 and the properties of $w_i^{(j)}$ (see [**St2**, Lemma 5(i) and Proposition 3.1]) that $V_i^{(j)}$ is a finite union of open cylinders of lengths $n_1 + m_j$.

The following lemma is proved by using arguments from [**D**] (see also [**St2**]). Since the proof of the analogous lemma in [**D, St2**] is very sketchy (or absent), we provide some details here.

LEMMA 7. *Let $\delta'' > 0$ and assume that $\mu_0 \leq \delta''/4C_0C_1$. Then there exists an integer $n_2 > 0$ such that for any $m \geq n_0 + n_2$, any $j = 1, \ldots, j_0$ and $i = 1, 2$ there exist contracting maps $\tilde{v}_i^{(j)} : V_i^{(j)} \longrightarrow U$ with $\sigma^{m-m_j}(\tilde{v}_i^{(j)}(w)) = w$ for all $w \in V_i^{(j)}$ such that*

$$\mathrm{Lip}(\Psi^{m-m_j} \circ \tilde{v}_i^{(j)}) \leq A\,\delta'' \quad \text{on } V_i^{(j)}, \tag{5.10}$$

$\tilde{v}_i^{(j)}(V_i^{(j)})$ is a finite union of open cylinders of length $n_1 + m$ and

$$\overline{\tilde{v}_i^{(j)}(V_i^{(j)})} \cap \overline{\tilde{v}_{i'}^{(j')}(V_{i'}^{(j')})} = \emptyset$$

whenever $(i, j) \neq (i', j')$.

Proof of Lemma 7. It follows from Lemma 3 and $\|\Phi\|_0 \leq A + LD \leq A(1 + \mu_0 D)$ that there exists $\delta > 0$ such that for any $y \in R_1$ with $d(y, \pi^{(U)}(y)) < \delta$ we have

$$|\Delta_\Psi(x', \pi_y(x))| \leq \frac{A\,\delta''}{2}\, d(x, x')$$

for all $x, x' \in U_1$.

Fix arbitrary non-empty open subsets (e.g. cylinders) $Q_1, \ldots, Q_{j_0}$ in U_1 such that $\overline{Q_j} \cap \overline{Q_{j'}} = \emptyset$ whenever $j \neq j'$. Now take n_2 so large that for any $m \geq n_2$ and any $j = 1, \ldots, j_0$, $\mathcal{P}^m(Q_j)$ contains a whole leaf $W^u_{R_1}(z)$ for some $z \in R_1$ such that $d(y, \pi^{(U)}(y)) < \delta$ for all $y \in W^u_{R_1}(z)$. This is possible, since $\mathcal{P}^m(Q_j)$ fills in R_1 densely as $m \to \infty$.

Let $m \geq n_2 + n_0$. Given $j = 1, \ldots, j_0$, we have $m - m_j \geq n_2$, so we can choose $\zeta_j \in R_1$ such that $W^u_{R_1}(\zeta_j) \subset \mathcal{P}^{m-m_j}(Q_j)$ and $d(y, \pi^{(U)}(y)) < \delta$ for all $y \in W^u_{R_1}(\zeta_j)$. There exists a (closed) cylinder L_j in $Q_j \subset U_1$ such that the Lipschitz extension of the restriction of $\mathcal{P}^{m-m_j}$ to $\overset{\circ}{L}_j$ defines a Lipschitz homeomorphism between L_j and $W^u_{R_1}(z_j)$, so in particular σ^{m-m_j} can be extended from $\overset{\circ}{L}_j$ to a Lipschitz homeomorphism $\sigma^{m-m_j} : L_j \longrightarrow U_1$. Then for any $w \in U_1$ there exists a unique element $\tilde{v}_i^{(j)}(w)$ of L_j with $\pi^{(U)}(\mathcal{P}^{m-m_j}(\tilde{v}_i^{(j)}(w))) = w$, i.e. $\sigma^{m-m_j}(\tilde{v}_i^{(j)}(w)) = w$. It is clear from the definition that $\tilde{v}_i^{(j)} : U_1 \longrightarrow L_j \subset U_1$ is contracting (assuming that n_2 is sufficiently large). For $w, w' \in U_1$, as in the the proof of (5.7) we have

$$\begin{aligned}
&|\Psi^{m-m_j}(\tilde{v}_i^{(j)}(w')) - \Psi^{m-m_j}(\tilde{v}_i^{(j)}(w))| \\
&\quad \leq |\Delta_\Psi(\mathcal{P}^{m-m_j}(\tilde{v}_i^{(j)}(w)), \mathcal{P}^{m-m_j}(\tilde{v}_i^{(j)}(w')))| \\
&\qquad + C_1 L\, d(\mathcal{P}^{m-m_j}(\tilde{v}_i^{(j)}(w)), \mathcal{P}^{m-m_j}(\tilde{v}_i^{(j)}(w'))) \\
&\quad \leq |\Delta_\Psi(w, \pi_{z_j}(w'))| + 2C_0C_1 L\, d(w, w') \\
&\quad \leq A\,[\delta''/2 + 2C_0C_1\,\mu_0]\, d(w, w') \leq A\,\delta''\, d(w, w').
\end{aligned}$$

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In particular, $\mathrm{Lip}(\Psi_{m-m_j} \circ \tilde{v}_i^{(j)}) \le A\,\delta''$. To conclude the proof, consider the restriction of the map $\tilde{v}_i^{(j)}$ to $V_i^{(j)}$; then by [**St2**, Proposition 3.1], $\tilde{v}_i^{(j)}(V_i^{(j)})$ is a finite union of open cylinders of lengths $n_1 + m_j + (m - m_j) = n_1 + m$.

It remains to show that the cylinders $\tilde{v}_i^{(j)}(V_i^{(j)})$ are disjoint. Assume that $\overline{\tilde{v}_i^{(j)}(V_i^{(j)})}$ and $\overline{\tilde{v}_{i'}^{(j')}(V_{i'}^{(j')})}$ have a common point for some $i, i' = 1, 2$ and $j, j' = 1, 2, \ldots, j_0$. Since $\tilde{v}_i^{(j)}(V_i^{(j)}) \subset L_j$ and $\tilde{v}_{i'}^{(j')}(V_{i'}^{(j')}) \subset L_{j'}$, it follows that $L_j \cap L_{j'} \neq \emptyset$, so $Q_j \cap Q_{j'} \neq \emptyset$. This implies $j = j'$ and therefore $m_j = m_{j'}$. It now follows that $V_i^{(j)} = \sigma^{m-m_j}(\tilde{v}_i^{(j)}(V_i^{(j)}))$ and $V_{i'}^{(j)} = \sigma^{m-m_j}(\tilde{v}_{i'}^{(j)}(V_{i'}^{(j)}))$ have a common point, which implies $i = i'$. □

Set $\delta'' = c_0\,\hat{\delta}/32$ and notice that (5.5) implies $\mu_0 \le \delta''/4C_0C_1$, so the assumptions of Lemma 7 are satisfied. Fix $n_2 = n_2(\delta'') > 0$ with the properties listed in Lemma 7, and denote $N_0 = n_0 + n_1 + n_2$.

Proof of Lemma 4. Parts (a), (c) and (d) follow from [**St2**, Lemma 4.2] and its proof. It remains to prove (b). We will follow [**St2**, proof of Lemma 4.2].

Let $N \ge N_0$. Then $m = N - n_1 \ge n_0 + n_2$, so by Lemma 6 for any $j = 1, \ldots, j_0$ and any $i = 1, 2$ there exists a contracting homeomorphism

$$\tilde{v}_i^{(j)} : V_i^{(j)} \longrightarrow \tilde{v}_i^{(j)}(V_i^{(j)}) \subset U \quad \text{with } \sigma^{N-m_j-n_1}(\tilde{v}_i^{(j)}(w)) = w, \quad w \in V_i^{(j)}, \tag{5.11}$$

such that (5.11) holds with $m = N - n_1$ and δ'' as above. Moreover, we can choose the maps $\tilde{v}_i^{(j)}$ so that $\overline{\tilde{v}_i^{(j)}(V_i^{(j)})} \cap \overline{\tilde{v}_{i'}^{(j')}(V_{i'}^{(j')})} = \emptyset$ whenever $(i, j) \neq (i', j')$. Now define Lipschitz maps

$$v_i^{(j)} : U \longrightarrow U \quad \text{such that } v_i^{(j)}(x) = \tilde{v}_i^{(j)}(w_i^{(j)}(\psi(x))), \quad x \in \mathcal{U}. \tag{5.12}$$

It follows immediately from the above that $v_i^{(j)}(U) \cap v_{i'}^{(j')}(U) = \emptyset$ whenever $(i, j) \neq (i', j')$, while [**St2**, Proposition 3.1] shows that each $v_i^{(j)}(\mathcal{U})$ is a finite union of open cylinders of length N.

Moreover, for any $x \in \mathcal{U}$, according to part (i) in Lemma 5, (5.12) and (5.11), we have

$$\sigma^{N-n_1}(v_i^{(j)}(x)) = \sigma^{m_j}(\sigma^{N-m_j-n_1}(v_i^{(j)}(x))) = \sigma^{m_j}(w_i^{(j)}(\psi(x))) = \psi(x),$$

which is the same for all j and i. Consequently, $\sigma^p(v_1^{(j)}(x)) = \sigma^p(v_2^{(j)}(x))$ for all $p \ge N - n_1$ and $x \in \mathcal{U}$. Thus,

$$\Psi^N(v_2^{(j)}(x)) - \Psi^N(v_1^{(j)}(x)) = \Psi^{N-n_1}(v_2^{(j)}(x)) - \Psi^{N-n_1}(v_1^{(j)}(x))$$

for $x \in \mathcal{U}$, and given $\eta \in B_j$ and $h > 0$, we have

$$\begin{aligned} &I_{\eta,h}\,[\Psi^N(v_2^{(j)}(\tilde{r}(s))) - \Psi^N(v_1^{(j)}(\tilde{r}(s)))] \\ &\quad = I_{\eta,h}\,[\Psi^{N-n_1}(\tilde{v}_2^{(j)}(w_2^{(j)}(\psi(\tilde{r}(s))))) - \Psi^{N-n_1}(\tilde{v}_1^{(j)}(w_1^{(j)}(\psi(\tilde{r}(s)))))] \\ &\quad = I_{\eta,h}\,[\Psi^{N-n_1}(\tilde{v}_2^{(j)}(w_2^{(j)}(r(s)))) - \Psi^{N-n_1}(\tilde{v}_1^{(j)}(w_1^{(j)}(r(s))))]. \end{aligned}$$

Since

$$\Psi^{N-n_1}(z) = [\Psi(z) + \cdots + \Psi(\sigma^{N-m_j-n_1-1}(z))] + [\Psi(y) + \cdots + \Psi(\sigma^{m_j-1}(y))],$$

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where $y = \sigma^{N-m_j-n_1}(z)$, using (5.11) we get

$$\begin{aligned}\Psi^{N-n_1}(\tilde{v}_i^{(j)}(w_i^{(j)}(x))) &= \Psi^{N-m_j-n_1}(\tilde{v}_i^{(j)}(w_i^{(j)}(x))) + \Psi^{m_j}(\sigma^{N-m_j-n_1}(\tilde{v}_i^{(j)}(w_i^{(j)}(x))))\\ &= \Psi^{N-m_j-n_1}(\tilde{v}_i^{(j)}(w_i^{(j)}(x))) + \Psi^{m_j}(w_i^{(j)}(x)).\end{aligned}$$

Therefore,

$$\begin{aligned}&I_{\eta,h}\,[\Psi^N(v_2^{(j)}(\tilde{r}(s))) - \Psi^N(v_1^{(j)}(\tilde{r}(s)))]\\ &\quad= I_{\eta,h}\,[\Psi^{N-m_j-n_1}(\tilde{v}_2^{(j)}(w_2^{(j)}(r(s)))) - \Psi^{N-m_j-n_1}(\tilde{v}_1^{(j)}(w_1^{(j)}(r(s))))]\\ &\qquad+ I_{\eta,h}\,[\Psi^{m_j}(w_2^{(j)}(r(s))) - \Psi^{m_j}(w_1^{(j)}(r(s)))].\end{aligned}$$

Let $\eta \in B_j$, and let s and $s + h\eta$ (for some h with $0 < |h| \le \hat{\delta}$) belong to V_0. Since $\mathrm{Lip}(w_i^{(j)}(x)) \le 1/(c_0\gamma^{m_j}) < 1/c_0$, using (5.4) and (5.10) with $m = N - n_1$, it follows that

$$\begin{aligned}|I_{\eta,h}[\Psi^{N-m_j-n_1}(\tilde{v}_i^{(j)}(w_i^{(j)}(r(s))))]| &\le \mathrm{Lip}[\Psi^{N-m_j-n_1}(\tilde{v}_i^{(j)})]\cdot\mathrm{Lip}(w_i^{(j)})\cdot\mathrm{Lip}(r)\\ &\le A\,\delta''\,\frac{2}{c_0} \le \frac{A\,\hat{\delta}}{8}.\end{aligned}$$

Combining this with the above and Lemma 6 gives

$$|I_{\eta,h}[\Psi^N(v_2^{(j)}(\tilde{r}(s))) - \Psi^N(v_1^{(j)}(\tilde{r}(s)))]| \ge \frac{A\,\hat{\delta}}{2} - 2\frac{A\,\hat{\delta}}{8} = \frac{A\,\hat{\delta}}{4}.$$

This proves the assertion. □

Proof of Theorem 2. For Lipschitz functions f, this is just a repetition of [**St2**, §5] without any changes. For Hölder continuous f, one just needs to combine this with the approximation procedure in [**D**] (see also [**St1**, §3]). Since Ψ is Lipschitz, the approximation procedure can be carried out in the same way as in [**D**]. □

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