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Δ^1_1 EFFECTIVIZATION IN BOREL COMBINATORICS

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Abstract. We develop a flexible method for showing that Borel witnesses to some combinatorial property of Δ^1_1 objects yield Δ^1_1 witnesses. We use a modification the Gandy–Harrington forcing method of proving dichotomies, and we can recover the complexity consequences of many known dichotomies with short and simple proofs. Using our methods, we give a simplified proof that smooth Δ^1_1 equivalence relations are Δ^1_1-reducible to equality; we prove effective versions of the Lusin–Novikov and Feldman–Moore theorems; we prove new effectivization results related to dichotomy theorems due to Hjorth and Miller (originally proven using "forceless, ineffective, and powerless" methods); and we prove a new upper bound on the complexity of the set of Schreier graphs for $\mathbb{Z}^2$ actions. We also prove an equivariant version of the G_0 dichotomy that implies some of these new results and a dichotomy for graphs induced by Borel actions of $\mathbb{Z}^2$.

§1. Introduction. Recent research has focused on the projective complexity of various Borel combinatorial properties of Borel graphs, equivalence relations, etc. For instance, the main result of [12] is that the set of (codes for) Borel 3-colorable Borel graphs is $\mathbf{\Sigma}^1_2$ complete. Such a complexity lower bound rules out dichotomy theorems like the G_0 dichotomy characterizing countably Borel colorable graphs [6]. It also implies the existence of Δ^1_1 graphs which are Borel 3-colorable but not Δ^1_1 3-colorable. In this note, we explain a modification of the Gandy–Harrington forcing arguments for dichotomy theorems which yields effectivization results. That is, we show how Borel witnesses to combinatorial properties of Δ^1_1 objects imply the existence of Δ^1_1 witnesses. This kind of effectivization implies a strong upper bound on projective complexity, and can be construed as a sort of weak dichotomy theorem. Our methods give short and simple proofs of these weak dichotomies even for properties where an actual dichotomy is unwieldy, and they allow us to make a fairly sweeping generalization of many known results.

The basic tool that we use is the observation that, for Φ among a large class of properties including what we will call independence properties, if B is a Borel set, A is a Gandy–Harrington condition, $\Phi(B)$, and $A \Vdash \dot{x} \in B$, then $\Phi(A)$. So, for instance, if a space can be covered with countably many Borel independent sets, then in any nonempty Σ^1_1 set we can use the forcing relation to find an independent set, so by a reflection argument we have a cover by independent Δ^1_1 sets. This is made precise in Lemma 3.4 and Theorem 3.5.

In the next section we give three illustrative examples, including a short proof that smooth Δ^1_1 equivalence relations are Δ^1_1-reducible to equality. In Section 3 we

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define a large class of properties Φ where this method works and prove some general results. In Section 4 we give several applications. We prove effective versions of the Lusin–Novikov and Feldman–Moore theorems (Theorem 4.1). We show that any Δ^1_1 graph generated by a single Borel function, a countable family of Borel functions, or a Borel free action of $\mathbb{Z}^2$ must be generated by a Δ^1_1 function, countable family of functions, or $\mathbb{Z}^2$-action (Theorems 4.5, 4.3, and 4.11, respectively). We show that if G is locally countable, Δ^1_1, and admits a Borel end selection, then it admits a Δ^1_1 end selection (Theorem 4.10). And, we prove effectivization for Borel local colorings in the sense of Miller's $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy (Theorem 4.17). In the last section we prove a handful of new dichotomy theorems related to some of the effectivization results in Section 4. In particular, we prove an equivariant version of the G_0 dichotomy which implies the results about generating graphs with functions (Theorem 5.2). And, we prove a dichotomy for Schreier graphs of Borel actions of $\mathbb{Z}^2$ which generalizes Miller's characterization of undirectable forests of lines (Theorem 5.16).

1.1. Conventions and notation. Throughout, graphs are simple and undirected. Formally, a graph on a vertex set X is a symmetric, irreflexive subset of X^2. An edge in a graph is some ordered pair in the graph (so our edges are directed, and our graphs contain both possible directions).

For an edge e, we typically write e_0 and e_1 for its left and right coordinate (its tail and head), so $e = (e_0, e_1)$. And, we write $-e$ for the edge with the opposite direction, i.e., $-(e_0, e_1) = (e_1, e_0)$. For a set of edges A, let $-A = \{-e : e \in A\}$.

We write $\mathcal{N}$ for Baire space, i.e. ω^ω with the product topology. The map $\pi_i : \mathcal{N}^n \to \mathcal{N}$ is projection onto the i^{th} coordinate. A box is a subset of $\mathcal{N}^n$ of the form $A_1 \times \cdots \times A_n$ for some $A_1, \ldots, A_n \subseteq \mathcal{N}$.

We use lightface capital Greek letters to refer to the effective projective hierarchy. So first, $A \in \Delta^1_1(x)$ means that A is Borel and there is some code for A which is computable in x. We will remain agnostic about the exact details of the coding, but one could code a Borel by a labelled well-founded tree whose leaves correspond to open sets and whose other nodes are labelled with union, intersection, and complement. And, $A \in \Sigma^1_1(x)$ means that for some $B \in \Delta^1_1(x)$,

$$y \in A \Leftrightarrow (\exists z)\ (y, z) \in B,$$

and $A \in \Pi^1_1(x)$ means that, for some $B \in \Delta^1_1$,

$$y \in A \Leftrightarrow (\forall z)\ (y, z) \in A.$$

In general $\Sigma^1_{n+1}(x)$ sets are projections of $\Pi^1_n(x)$, and $\Pi^1_n(x)$ sets are complements of $\Sigma^1_n(x)$ sets. We will mostly be concerned with $n = 1$ or 2 here.

Let us stress that the set of codes for Δ^1_1 sets is Π^1_1, but that if we write $C(x)$ for the set coded by x, then there is an arithmetic property Φ so that for any code x, $y \in C(x) \Leftrightarrow \Phi(x, y)$.

We say that a property Φ of subsets of X is Π^1_1 on Σ^1_1 if, whenever $A \subseteq Y \times X$ is Σ^1_1

$$\{y \in Y : \Phi(A_x)\} \in \Pi^1_1.$$

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That is, Φ is Π^1_1 on Σ^1_1 if the set of sections with property Φ in any Σ^1_1 set is Π^1_1. For instance, being empty is Π^1_1 on Σ^1_1:

$$\Phi(A) \Leftrightarrow (\forall y)\ y \notin A.$$

Σ^1_1 on Σ^1_1 and similar notions are defined mutatis mutandis. The most important fact about these kinds of properties are the reflection theorems:

THEOREM 1.1 (Reflection theorems, [2, Lemma 1.4], [7, Theorem 2.27]). *Say a property is monotone if it passes to supersets and continuous if it passes to decreasing intersections.*

(1) *If Φ is Π^1_1 on Σ^1_1, $A \in \Sigma^1_1$, and $\Phi(A)$, then there is some $B \supseteq A$ so that $B \in \Delta^1_1$ and $\Phi(B)$.*
(2) *For Φ a property of two sets, Φ is Π^1_1 on Π^1_1, monotone in both coordinates, and continuous downward in the second coordinate, and if $\Phi(A, \mathcal{N} \setminus A)$ for some $A \in \Pi^1_1$, then there is $B \subseteq A$ so that $B \in \Delta^1_1$ and $\Phi(B, \mathcal{N} \setminus B)$.*

We say that a fact about Σ^1_n or Π^1_n relativizes if it remains true if we replace Σ^1_n with $\Sigma^1_n(x)$ for some oracle x (and likewise for Π). The reflection theorems relative.

Bold-faced Greek letters are the classical projective hierarchy, which we can equivalently define as

$$\mathbf{\Sigma}^1_n = \bigcup_{x \in \mathbb{R}} \Sigma^1_n(x), \quad \mathbf{\Pi}^1_n = \bigcup_{x \in \mathbb{R}} \Pi^1_n(x).$$

So, facts about the effective Projective hierarchy which relativize are also true about the classical hierarchy. We will be interested to the extent to which the converse is true.

We will work with Gandy–Harrington forcing on $\mathcal{N}^n$, i.e., the poset of non-empty Σ^1_1 subsets of $\mathcal{N}^n$ ordered by inclusion. We write for $\mathbb{P}_n$ this forcing, and we write $\dot{x}$ for the standard name for the real coded by a $\mathbb{P}_1$-generic filter, $(\dot{x}, \dot{y})$ for the name for the reals coded by a $\mathbb{P}_2$-generic, and $\dot{\mathbf{x}} = (\dot{x}_1, \dots, \dot{x}_n)$ for the $\mathbb{P}_n$-generic reals.

For a Borel set B, when we write $\Vdash \dot{x} \in B$ or consider B in some generic extension, we mean the set in coded in the extension by some ground model code for B. In particular, $\Phi(B)$ is absolute whenever Φ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Delta}^1_1$.

When we say a Δ^1_1 sequence of Δ^1_1 sets $\langle A_\alpha : \alpha \in \beta \rangle$ (for a computable ordinal β), we mean that there is a Δ^1_1 order $\prec$ on ω and Δ^1_1 set of pairs (c_α, i_α) so that $i_\alpha \in \omega$ has order type α in $\prec$ and c_α is a code for A_α. A family of Borel sets is Π^1_1 in the codes if the set of Borel codes for Borel sets in this family is Π^1_1. Our references for effective descriptive set theory are Marks's lecture notes [7] and the paper by Harrington, Marker, and Shelah [2].

When $s_1, s_2, s_3, \dots$ are ordinal length sequences, say $s_i \in A^{\alpha_i}$ for some A, then their concatenation, $s = s_1^\frown s_2^\frown s_3^\frown \dots$ is the sequences of length $\sum_i \alpha_i$ defined by

$$s(\beta) = s_i(\gamma) :\Leftrightarrow \beta = (\sum_{j<i} \alpha_j) + \gamma.$$

§2. First examples. We begin with three examples. First we reprove the effectivization consequences of the G_0 dichotomy:

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THEOREM 2.1. *If g is a code for a Borel graph G on $\mathcal{N}$, and G admits a Borel countable coloring, then G admits a $\Delta^1_1(g)$ countable coloring.*

PROOF. By relativization, we may assume $g \in \Delta^1_1$. We first show that G admits a Δ^1_1 coloring if and only if $\mathcal{N}$ is a union of Δ^1_1 G-independent sets. If $f : \mathcal{N} \to \omega$ is a coloring, then $\mathcal{N} = \bigcup_i f^{-1}(i)$ and each of the fibers is G-independent and Δ^1_1. For the converse, suppose every x is in some Δ^1_1 independent set. Then

$$\{(x, i) : i \text{ codes a set } A \in \Delta^1_1,\ x \in A,\ A \text{ is } G \text{ independent}\}$$

is a Π^1_1 total relation, so admits a Δ^1_1 uniformization by [7, Theorem 2.15]. The uniformizing function is a countable coloring.

Let $X = \mathcal{N} \setminus \bigcup\{A \in \Delta^1_1 : A \text{ is } G \text{ independent}\}$. Note that X is Σ^1_1. Suppose $X \neq \emptyset$. (If we were to try to prove a dichotomy result here we would start trying to build some generic obstruction to colorability.) Suppose toward contradiction f is a Borel countable coloring of G.

Recall that $\mathbb{P}_i$ is Gandy–Harrington forcing on $\mathcal{N}^i$. By absoluteness, the interpretation of the code for f remains a Borel coloring in any extension by a $\mathbb{P}_i$-generic. We can find some $A \in \mathbb{P}_1$ below X so that $A \Vdash f(\dot{x}) = i$, where $\dot{x}$ is a name for the generic real. If A is not independent, then $B = \{(x, y) : (x, y) \in G, x, y \in A\}$ is a condition in $\mathbb{P}_2$ and

$$B \Vdash f(\dot{x}) = f(\dot{y}) \ \wedge\ (\dot{x}, \dot{y}) \in G,$$

which is a contradiction. Since being independent is Π^1_1 on Σ^1_1, A is contained in some Δ^1_1 independent set. But this means $A \cap X = \emptyset$, contradicting our choice of A. ⊣

COROLLARY 2.2. *The set of Borel countably colorable graphs is Π^1_1 in the codes.*

PROOF. Write B_c for the set coded by a Borel code c, and write ϕ^x_n for the partial function computed by the nth oracle machine with oracle x.

For any $g \in \mathcal{N}$, g codes a graph with a countable coloring if and only if g codes a graph G, and

$$(\exists f \in \Delta^1_1(g))\ f \text{ is a countable coloring of } G.$$

In detail, this second clause says

$$(\exists n \in \omega)\ \phi^g_n \text{ is a Borel code, and } B_{\phi^g_n} \text{ is a countable coloring of } G.$$

And f is a countable coloring of G if and only if

$$\begin{aligned} f \subseteq \mathcal{N} \times \omega \wedge (\forall x, i, j)\ (x, i), (x, j) \in f \to i \neq j \\ \wedge\ (\forall x, y, i)\ (x, i), (y, i) \in f \to (x, y) \notin G. \end{aligned}$$

This all Π^1_1. ⊣

We will only state results for Δ^1_1 sets below, but every statement relativizes and gives a complexity bound as above.

The general outline of the method is as follows: We first reduce whatever problem is at hand to the problem of covering some set X by countably many Borel sets satisfying some property Φ (this is usually the combinatorial heart of the problem).

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Then we suppose that some $X \in \Delta^1_1$ can be covered by Borel sets $\langle A_i : i \in \omega \rangle$ satisfying Φ, but not by Δ^1_1 sets. We get a contradiction by forcing below

$$\widetilde{X} := X \setminus \bigcup \{A \in \Delta^1_1 : \Phi(A)\}$$

to find some p and i with

$$p \Vdash \dot{x} \in A_i.$$

We then show by contradiction that in fact $\Phi(p)$, so by a reflection argument $p \cap X = \emptyset$. In the next section, we formalize this proof sketch and offer some technical variations on the idea.

Our second example involves a case where we want our covering sets to satisfy a notion of independence and a kind of closure. We can give a much simplified proof of the effectivization consequences of Harrington–Kechris–Louveau [10, Theorem 5.2.7] by considering boxes in $\mathcal{N}^2$ which avoid a given equivalence relation.

THEOREM 2.3. *If E is a Δ^1_1 equivalence relation and Borel reducible to $(=_{2^\omega})$, then E is Δ^1_1 reducible to $(=_{2^\omega})$.*

PROOF. First, say that a subset of $\mathcal{N}^2$ is a box if it is a product of two subsets of $\mathcal{N}$, and note that E is Δ^1_1 (or Borel) reducible to $(=_{2^\omega})$ if and only if E^c is covered by a countable Δ^1_1 (or Borel) family of boxes which all avoid E. Indeed, if f is a reduction, then we can set

$$A_i = \{x : f(x)(i) = 0\} \text{ and } B_i = \{x : f(x)(i) = 1\}$$

and consider the boxes $A_i \times B_i$. Conversely, suppose E^c can be covered by Δ^1_1 boxes. Setting $\mathcal{C} = \{n \in \omega : n \text{ is a code for a } \Delta^1_1 \text{ box avoiding } E\}$, we have

$$(\forall (x, y) \notin E)(\exists n \in \omega)\ n \in \mathcal{C} \wedge (x, y) \text{ is in the set coded by } n.$$

This is Π^1_1 on Π^1_1 as a property of $\mathcal{C}$, so by the first reflection theorem (applied to the complement) there is a Δ^1_1 subset of $\mathcal{C}$, say $\mathcal{D}$ with the same property. We can enumerate $\mathcal{D}$ to get a Δ^1_1 covering sequence $\langle A_i \times B_i : i \in \omega \rangle$. By the second reflection theorem applied to

$$\Phi(Z, Y) :\Leftrightarrow (\forall (x, y) \in E)\ x \in Z \vee (y \in Y \wedge x \notin B_i),$$

we can assume each A_i is E-invariant and avoids B_i, and we can define a reduction by $f(x)(i) = 0$ if and only if $x \in A_i$.

Suppose $E \in \Delta^1_1$, $X = E^c \setminus \bigcup \{A \times B : A, B \in \Delta^1_1, (A \times B) \cap E = \emptyset\}$ is nonempty, and $E^c = \bigcup_i A_i \times B_i$ with A_i, B_i Borel. A set $Y \subseteq \mathcal{N}^2$ is a box if and only if

$$\big(\forall (x, y), (a, b) \in Y\big)\ (x, b) \in Y.$$

So again by the second reflection theorem applied to

$$\Phi(Z, Y) :\Leftrightarrow \big(\forall (x, y), (a, b) \notin Z\big)\ \ (x, b) \in Y \text{ and } (x, y) \notin E,$$

X does not meet any Σ^1_1 boxes avoiding E. Also note that X is Σ^1_1.

Since $X \Vdash (\dot{x}, \dot{y}) \notin E$, there is some $p \subseteq X$ so that

$$p \Vdash \dot{x} \in A_i \ \wedge \ \dot{y} \in B_i.$$

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And, since any $\mathbb{P}_2$ generic (x, y) has x and y separately $\mathbb{P}_1$-generic, it must be that $A = \pi_1(p) \Vdash \dot{x} \in A_i$ and $B = \pi_2(p) \Vdash \dot{x} \in B_i$. So, $A \times B$ must avoid E or else $(A \times B) \cap E \Vdash (x, y) \in E \cap (A_i \times B_i)$.

But then since X doesn't meet any boxes which avoid E, $p \subseteq A \times B \cap X = \emptyset$, which is a contradiction. ⊣

COROLLARY 2.4. *The set of smooth relations is Δ^1_1 in the codes.*

PROOF. A real c codes a smooth relation E if and only if

$$(\exists f \in \Delta^1_1)(\forall x, y) f(x) = f(y) \leftrightarrow xEy.$$

As above, $(\exists f \in \Delta^1_1)$ is equivalent to a universal quantifier over $\mathcal{N}$. ⊣

Our last example involves an ordinal length construction. It turns out we can also effectivize the ordinals length of the sequence to lie below ω_1^{CK}. This gives another proof of the effectivization consequences of a dichotomy due independently to Kanovei and Louveau [10, Theorem 5.2.3].

DEFINITION 2.5. For an ordinal α, $\leq^{\alpha}_{lex}$ is the lexicographic order on 2^{α}, i.e.,

$$x \leq^{\alpha}_{lex} y :\Leftrightarrow x = y \text{ or } (\exists \beta < \alpha)\left[(x \restriction \beta) = (y \restriction \beta) \text{ and } x(\beta) < y(\beta)\right].$$

THEOREM 2.6. *Suppose R is a quasi-order on $\mathcal{N}$ with a Borel homomorphism to $\leq^{\alpha}_{lex}$ for some $\alpha < \omega_1$ which induces an injection on $\mathcal{N}/\equiv_R$. Then there is a Δ^1_1 homomorphism of R to $\leq^{\alpha'}_{lex}$ for some $\alpha' < \omega_1^{CK}$ which induces an injection on $\mathcal{N}/\equiv_R$.*

PROOF. Given a family of functions $\mathcal{F}$ on $\mathcal{N}$, define an equivalence relation by

$$x \equiv_{\mathcal{F}} y \;:\Leftrightarrow\; (\forall f \in \mathcal{F})\; f(x) = f(y).$$

And for a function f, let $(\equiv_f) = (\equiv_{\{f\}})$. We want to find some Δ^1_1 homomorphism, say f, from R to a lexicographic ordering so that $(\equiv_R) = (\equiv_f)$.

Consider

$$\mathcal{F} = \{f : (\exists \alpha < \omega_1^{CK})\; f \text{ is a } \Delta^1_1 \text{ homomorphism from } R \text{ to } \leq^{\alpha}_{lex}\}.$$

Note that $\mathcal{F}$ is Π^1_1 and $(\equiv_R) \subseteq (\equiv_{\mathcal{F}})$. If $(\equiv_R) = (\equiv_{\mathcal{F}})$, then since this is a Π^1_1 on Π^1_1 statement about subsets of $\mathcal{F}$, by reflection there is a Δ^1_1 set $\mathcal{G} \subseteq \mathcal{F}$ so that $(\equiv_{\mathcal{G}}) = (\equiv_R)$. We can then enumerate $\mathcal{G}$ as $\langle f_i : i \in \omega\rangle$ with a Δ^1_1 enumeration. So each f_i is a homomorphism from R to some $\leq^{\alpha_i}_{lex}$ with $\alpha_i < \omega_1^{CK}$. Then $(\equiv_R) = (\equiv_f)$ where

$$f(x) = f_0(x)^\frown f_1(x)^\frown f_2(x)^\frown \ldots.$$

Now suppose towards contradiction that $X = (\equiv_{\mathcal{F}}) \setminus (\equiv_R)$ is nonempty, and fix a Borel homomorphism g from R to some $\leq^{\tilde{\alpha}}_{lex}$. To get our contradiction we use a reflection argument to find a Δ^1_1 homomorphism which refines $\equiv_{\mathcal{F}}$.

We have that

$$X \Vdash g(\dot{x})(\alpha) \neq g(\dot{y})(\alpha) \text{ for some } \alpha.$$

So we can define

$$\alpha_0 = \min\{\alpha : (\exists p \subseteq X)\; p \Vdash g(\dot{x})(\alpha) \neq g(\dot{y})(\alpha)\}.$$

Note that $X \Vdash g(\dot{x}) \restriction \alpha_0 = g(\dot{y}) \restriction \alpha_0$.

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Choose $p \subseteq X$ witnessing the above formula. Without loss of generality, we may assume $p \Vdash g(\dot{x})(\alpha_0) = 1 \ \wedge \ g(\dot{y})(\alpha_0) = 0$. Let $A = \pi_1(p)$ and $B = \pi_2(p)$. Then, if $q = (A \times B) \cap R \cap (\equiv_{\mathcal{F}}) \neq \emptyset$,

$$q \Vdash g(\dot{x})(\alpha_0) > g(\dot{y})(\alpha_0) \ \wedge \ g(\dot{x}) \restriction \alpha_0 = g(\dot{y}) \restriction \alpha_0 \ \wedge \ xRy$$

which contradicts the fact that g is a homomorphism.

So, $(A \times B) \cap R \cap (\equiv_{\mathcal{F}}) = \emptyset$. By the second reflection theorem we may assume A is Δ^1_1 and closed upwards under $R \cap (\equiv_{\mathcal{F}})$ and avoids B. By reflection there is a Δ^1_1 sequence of functions $\langle f_i : i \in \omega \rangle$ so that each f_i is in $\mathcal{F}$ and

$$(A \times B) \cap R \cap (\equiv_{\{f_i : i \in \omega\}}) = \emptyset.$$

Suppose f_i is a homomorphism into the lexicographic order on α_i. We can define a homomorphism into the lexicographic order on $\gamma := (\sum_i \alpha_i) + 1$ as follows. Set $f_\infty(x) = 1$ if and only if $x \in A$ and

$$f(x) := \big(f_0(x)^\frown f_1(x)^\frown \ldots\big)^\frown f_\infty(x).$$

To check this is a homomorphism, suppose xRy. If $f_i(x) <^{\alpha_i}_{lex} f_i(y)$ for some i, then $f(x) \leq^{\gamma}_{lex} f(y)$. Otherwise $x \equiv_{\mathcal{F}} y$, and so if $x \in A$ then $y \in A$ by our closure assumption. This means $f_\infty(x) \leq f_\infty(y)$. In any case $f(x) \leq^{\gamma}_{lex} f(y)$ and f is a homomorphism.

This means $f \in \mathcal{F}$, so $f(x) = f(y)$ for $(x, y) \in X$, but for $(x, y) \in p \subseteq X$ we have $f(x) \neq f(y)$ which is a contradiction. ⊣

§3. General results. Our first goal in this section is to prove a general result on effectivizing countable covers by Borel sets satisfying some property, Φ. We can do this when Φ is conjunction of what we call independence properties and closure properties. These notions are defined below, but an informal description is as follows: Φ is an independence property if $\Phi(A)$ says all points in A satisfy a combinatorial relation (e.g., A is G-independent for a graph G), and Ψ is a closure property if $\Psi(A)$ says that A contains all points which stand in a combinatorial relation with points from A (e.g., A is a box).

DEFINITION 3.1. We say that a property $\Phi(A)$ of a subset of $\mathcal{N}$ is an *independence property* if there is some Δ^1_1 property ϕ such that

$$\Phi(A) \Leftrightarrow \neg(\exists \mathbf{x}, y) \ \mathbf{x} \in A^k \text{ and } \phi(x_1, \ldots, x_k, y).$$

Note that if Φ is an independence property then Φ is Π^1_1 on Σ^1_1. In practice, many Π^1_1 on Σ^1_1 properties are independence properties. But the two notions are not equivalent, and it is unclear how far beyond independence properties the results below generalize.

DEFINITION 3.2. A property $\Psi(A)$ of sets is a *closure property* if there is a Δ^1_1 property ψ so that $\Psi(A)$ if and only if

$$(\forall \mathbf{x} \in A^k)(\forall y, z) \ \psi(\mathbf{x}, y, z) \to z \in A.$$

For a closure property Ψ and set A, the Ψ-closure of A is $A^\Psi := \bigcup_m f^m(A)$, where

$$f(A) := A \cup \{z : (\exists \mathbf{x} \in A^k)(\exists y)\psi(x, y, z)\}.$$

Of course, the Ψ-closure of any set satisfies Ψ. And, if A is Σ^1_1, then so is A^Ψ.

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The key point about an independence property, closure property, or a conjunction thereof, say Φ, is that any time a Gandy–Harrington condition p forces the generic to be in set with property Φ, p is contained in a Δ^1_1 set with property Φ.

DEFINITION 3.3. A property $\Phi(A)$ is *reflectable* if it is Π^1_1 in the codes (i.e., $\{c : c \text{ codes } A, \Phi(A)\}$ is Π^1_1), $\Phi(B)$ is absolute between V and $V[G]$ for any $\mathbb{P}_i$-generic G and Borel set B, and the following condition holds: whenever B is Borel, $A \in \mathbb{P}_1$, $A \Vdash \dot{x} \in B$, and $\Phi(B)$, there is some $\tilde{A} \in \Delta^1_1$ so that $A \subseteq \tilde{A}$ and $\Phi(\tilde{A})$.

One subtlety to point is out that by property we mean first-order property in the language of set theory with parameters. So there should never be any confusion about how to interpret Φ in different models (provided all the relevant parameters are present). In practice we may be ambiguous about exactly what property we mean, but there should always be a Δ^1_1 translation between any reasonable interpretations. Also, most natural properties which are Π^1_1 in the codes are uniformly $\Pi^1_1(x)$ in the codes for $\Delta^1_1(x)$ sets, so this absoluteness assumption is usually automatic.

LEMMA 3.4. *The following properties are all reflectable*:

(1) $\bigvee_{i\in\omega} \Phi_{i\in\omega}(A)$, *where* $\langle \Phi_i : i \in \omega\rangle$ *is a sequence of reflectable properties which are uniformly* Π^1_1 *in the codes.*
(2) $\Phi(f[A])$, *where* Φ *is reflectable and* $f : \mathcal{N} \to \mathcal{N}$ *is a* Δ^1_1 *bijection.*
(3) $\Phi \wedge \Psi$ *where* Φ *is an independence property and* Ψ *is a closure property.*

PROOF. Checking (1) and (2) is routine. For (3), suppose $A \Vdash \dot{x} \in B$ and $(\Phi \wedge \Psi)(B)$. We first show by induction on m that $f^m(A) \Vdash \dot{x} \in B$, with f as in the definition of A^Ψ. The base case $m = 0$ is our assumption that

$$f^0(A) = A \Vdash \dot{x} \in B.$$

Suppose toward contradiction that $f^m(A) \Vdash \dot{x} \in B$, but that there is some $p \subseteq f^{m+1}(A)$ so that $p \Vdash \dot{x} \notin B$. We have a nonempty condition

$$q := \{(\mathbf{x}, y, z) : \mathbf{x} \in (f^m(A))^k,\ z \in p, \text{ and } \psi(\mathbf{x}, y, z)\} \in \mathbb{P}_{k+2}.$$

The first k projections of q are contained in $f^m(A)$, so by the induction hypothesis $q \Vdash \dot{\mathbf{x}} \in B^k$. Since the projection of q onto the z-coordinate is contained in p, our assumption about p gives $q \Vdash \dot{z} \notin B$. And the definition of q ensures $q \Vdash \psi(\dot{\mathbf{x}}, y, z)$. So,

$$q \Vdash \dot{\mathbf{x}} \in B^k \wedge \dot{z} \notin B \wedge (\exists w)\psi(\dot{x}, w, \dot{z}).$$

Thus we can find generics in the extension witnessing that $\neg\Psi(B)$, but in the ground model $\Psi(B)$ contradicting Schoenfield absoluteness.

Now suppose that $A \Vdash \dot{x} \in B$ and $(\Phi \wedge \Psi)(B)$. From the above, we may assume $A^\Psi = A$. Suppose toward contradiction that $\neg\Phi(A)$. Then

$$C = \{(\mathbf{x}, y) : \mathbf{x} \in A^k \text{ and } \phi(\mathbf{x}, y)\} \in \mathbb{P}_{n\times k}$$

and since $A \Vdash \dot{x} \in B$,

$$C \Vdash \dot{\mathbf{x}} \in B^k \ \wedge\ \phi(\dot{\mathbf{x}}, \dot{y})$$

which, just as above, contradicts the fact that $\Phi(B)$.

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So, $(\Phi \wedge \Psi)(A)$, and by the second reflection theorem applied to

$$\Theta(X, Y) := (\forall \mathbf{x} \in (\mathcal{N} \setminus X)^k, y, z)\ \big(\psi(\mathbf{x}, y, z) \to z \in Y\big) \text{ and } \neg\phi(\mathbf{x}, y)$$

there is a Δ^1_1 set $\tilde{A} \supseteq A$ so that $(\Phi \wedge \Psi)(\tilde{A})$. ⊣

With the lemma above in hand, we can quickly prove a general effectivization result for countable covers by independent sets.

THEOREM 3.5. *Fix a reflectable property Φ. Suppose $X \in \Delta^1_1$ and $X \subseteq \bigcup_{i\in\omega} B_i$, where each B_i is a Borel set so that $\Phi(B_i)$. Then there is a Δ^1_1 sequence of Δ^1_1 sets $\langle A_i : i \in \omega\rangle$ so that $\Phi(A_i)$ for all i and $X \subseteq \bigcup_{i\in\omega} A_i$.*

PROOF. Suppose toward contradiction X admits no Δ^1_1 sequence as described. Consider the following set:

$$X' = X \setminus \bigcup\{A \in \Delta^1_1 : \Phi(A)\}.$$

We want to check that X' is Σ^1_1. The set of codes for sets with property Φ is Π^1_1 by the definition of reflectable, so

$$x \in \bigcup\{A \in \Delta^1_1 : \Phi(A)\} \Leftrightarrow (\exists n \in \omega)\ n \text{ codes a set } A \wedge x \in A.$$

Since Π^1_1 sets are closed under number quantifiers, this is Π^1_1.

By assumption, X' is nonempty, so it is in $\mathbb{P}_1$. We have

$$X' \Vdash (\exists i)\ \dot{x} \in B_i$$

so for some $i \in \omega$ and $A \subseteq X'$,

$$A \Vdash \dot{x} \in B_i.$$

But then, $\Phi\big(\tilde{A}\big)$ for some Δ^1_1 set $\tilde{A} \supseteq A$. So A avoids X, which contradicts our choice of A. ⊣

3.1. Minor technical variations. We want two minor technical variations on this theorem. The first of these variations says that we can effectivize countable Borel families whose projections cover a given space, provided the projection map has countable fibers.

THEOREM 3.6. *Suppose that $\Phi(A)$ is a reflectable property, $R \subseteq \mathcal{N}^2$ is Δ^1_1 with countable sections, and $\Phi(A)$ implies $A \subseteq R$. If $X \subseteq \mathcal{N}^n$ admits a countable cover by Borel sets of the form $\pi_1(A)$ with $\Phi(A)$, then X admits a cover by Δ^1_1 sets of this form.*

PROOF. By the effective Feldman–Moore theorem (see Theorem 4.1 and the preceding comments in Section 4), there is a Δ^1_1 sequence of involutions $\langle f_i : i \in \omega\rangle$ so that, for any $x \in \mathcal{N}$ and $(x, y) \in R$,

$$\{(x, y') : (x, y') \in R\} = \{f_i(x, y) : i \in \omega\}.$$

Then, $X = \bigcup_i \pi_1(A_i)$ if and only if

$$\{(x, y) \in R : x \in X\} = \bigcup_i \bigcup_j \{f_j(x, y) : (x, y) \in A_i\}.$$

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So, X admits a cover by Δ^1_1 sets of the form $\pi_1(A)$ with $\Phi(A)$ if and only if $\widetilde{X} = \{(x, y) \in R : x \in X\}$ admits a cover by Δ^1_1 sets so that $(\exists i)\Phi(f_i[A])$ holds. By Lemma 3.4, $(\exists i)\Phi(f_i[A])$ is reflectable, and we can apply the previous theorem. ⊣

The second variation generalizes beyond covering spaces with unions of sets to covering spaces with unions of intersections of unions of sets. One could extend this further to arbitrary finite alternations of unions and intersections with mostly notational changes.

THEOREM 3.7. *Fix a sequence of reflectable properties $\langle \Phi_{ij} : i, j \in \omega \rangle$ which are uniformly Π^1_1 in the codes. Suppose there is a family of Borel sets $\langle B_{i,j,k} : i, j, k \in \omega \rangle$ such that*

$$\widetilde{X} = \bigcup_i \bigcap_j \bigcup_k B_{i,j,k},$$

where $\widetilde{X} \subseteq \mathcal{N}$ is Δ^1_1 and for all $i, j, k \in \omega$

$$\Phi_{i,j}(B_{i,j,k}).$$

Then there is a Δ^1_1 such family.

By uniformity here we mean $\{(n, i, j) : n \text{ codes a set } A, \Phi_{ij}(A)\}$ is Π^1_1.

PROOF. Let

$$X = (\widetilde{X} \times \omega^2) \setminus \bigcup \{A \times \{(i, j)\} : A \in \Delta^1_1, (i, j) \in \omega^2, \Phi_{i,j}(A)\}.$$

If we have that

$$(\forall x)(\exists i)(\forall j)\ (x, i, j) \notin X,$$

then we can find our Δ^1_1 family by letting B_{ijk} be a Δ^1_1 enumeration of the Δ^1_1 sets with $\Phi_{ij}(B_{ijk})$. So suppose towards contradiction that

$$(\exists x)(\forall i)(\exists j)\ (x, i, j) \in X.$$

We can find a condition $p \subseteq \{x : (\forall i)(\exists j)\ (x, i, j) \in X\}$, and an index $\tilde{i}$ so that

$$p \Vdash (\forall j)(\exists k)\ \dot{x} \in B_{\tilde{i},j,k}.$$

Refining p, we can find some $\tilde{j}$ so that

$$p \subseteq \{x : (x, \tilde{i}, \tilde{j}) \in X\}.$$

But then, refining p again, we can find $\tilde{k}$ so that

$$p \Vdash \dot{x} \in B_{\tilde{i},\tilde{j},\tilde{k}}.$$

But then p is contained in a Δ^1_1 set $\tilde{p}$ so that $\Phi_{\tilde{i},\tilde{j}}(\tilde{p})$, and $\tilde{p} \times \{(\tilde{i}, \tilde{j})\} \cap X = \emptyset$, which contradicts the fact that $p \subseteq \{x : (x, \tilde{i}, \tilde{j}) \in X\}$. ⊣

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3.2. Transfinite constructions. Our last variation is somewhat more complicated and covers situations where we want to cover an object with some ordinal length sequence of sets. This generalizes Theorem 2.6.

DEFINITION 3.8. Given $\overline{A} = \langle A_\alpha : \alpha < \beta \rangle$ a countable ordinal length sequence of sets in $\mathcal{N}$ and $X \subseteq \mathcal{N}$, define

$$X_{\overline{A}} = X \setminus \bigcup_{\alpha<\beta} A_\alpha.$$

So, in particular, $\mathcal{N}_{\overline{A}} = \mathcal{N} \setminus \bigcup_{\alpha<\beta} A_\alpha$ And, given $\mathcal{F}$ a set of such sequences, let

$$X_{\mathcal{F}} = X \setminus \bigcup_{\overline{A}\in\mathcal{F}} \bigcup_{\alpha<\beta} A_\alpha = \bigcap_{\overline{A}\in\mathcal{F}} X_{\overline{A}}.$$

Abusing notation slightly, identify the ordinal product $\omega \cdot \alpha$ with the lexicographic order on $\alpha \times \omega$. For $\overline{A} = \langle A_\beta : \beta \in \omega \cdot \alpha \rangle = \langle A_{\beta,i} : (\beta, i) \in \alpha \times \omega \rangle$, set $\overline{A}_0 = \langle A_{\beta,0} : \beta \in \alpha \rangle$.

Say that a property $\Phi(\langle A_\beta : \beta \in \omega \cdot \alpha \rangle)$ of an ordinal length sequence of sets (with length of the form $\omega \cdot \alpha$ for some countable ordinal α) is a *refinement* property if for some Δ^1_1 sequence of Σ^1_1 properties $\langle \phi_i : i \in \omega \rangle$ (all with arity k, though this is just a notational convenience) and Δ^1_1 sequence of closure properties $\langle \Psi_i : i \in \omega \rangle$, $\Phi(\overline{A})$ if and only if

$$(\forall \beta, i) \left[\Psi_i(A_{\beta,i}) \wedge \left(\forall \mathbf{x} \in (\mathcal{N}_{(\overline{A}_0 \restriction \beta)})^k \right) \mathbf{x} \in A^k_{\beta,i} \to \neg \phi_i(\mathbf{x}) \right].$$

Informally, a refinement property says an independence property holds for each $A_{\alpha,i}$ if we ignore points in earlier $A_{\alpha,0}$'s, and each $A_{\alpha,i}$ is Ψ_i-closed. For any refinement property property Φ, the set $\{\overline{A} \in \Delta^1_1 : (\forall \beta)\ A_\beta \in \Delta^1_1 \text{ and } \Phi(\overline{A})\}$ is Π^1_1.

If some countable collection of sequences satisfies a refinement property, so does the sequence obtained by concatenating them together. We will often use this in conjunction with the following proposition.

PROPOSITION 3.9. *Suppose $\mathcal{F}$ is a Π^1_1 family of Δ^1_1 sequences which is closed under Δ^1_1 concatenations. For any Σ^1_1 set $p \subseteq \mathcal{N}^n$, whenever $p \cap (X_{\mathcal{F}})^n = \emptyset$ there is a single Δ^1_1 sequence $\overline{A} \in \mathcal{F}$ so that $p \cap (X_{\overline{A}})^n = 0$.*

PROOF. For any Σ^1_1 sets X and p, $p \cap (X_{\mathcal{F}})^n = \emptyset$ if and only if

$$(\forall \mathbf{x})\ \mathbf{x} \notin p \text{ or } (\exists \overline{A}_1, \ldots, \overline{A}_n \in \Delta^1_1)(\forall i) \left[\overline{A}_i \in \mathcal{F} \text{ and } (\exists \alpha_i < \omega_1^{CK})\ x_i \in (A_i)_{\alpha_i} \right]. \qquad (*)$$

As usual, existential quantifiers over Δ^1_1 are equivalent to universal quantifiers, so

$$(\exists \alpha < \omega_1^{CK})\ x \in A_\alpha \Leftrightarrow (\exists n \in \omega)\ n \text{ is a code for a well-order, } \alpha \ \&\ x \in A_\alpha$$

is Π^1_1 on Π^1_1. Likewise, $(*)$ is Π^1_1 on Π^1_1 as a property of $\mathcal{F}$. So, if $\mathcal{F}$ is a Π^1_1 collection of sequences with $p \cap X^n_{\mathcal{F}} = \emptyset$, we can find a Δ^1_1 subset $\mathcal{G} \subseteq \mathcal{F}$ so that $p \cap X^n_{\mathcal{G}} = \emptyset$. We can find a Δ^1_1 enumeration of the sequences in $\mathcal{G}$ and concatenate them together to get a single sequence $\overline{A}$ so that $p \cap X^n_{\overline{A}} = \emptyset$. ⊣

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We can prove that if there is some family of ordinal length sequence of Borel sets satisfying a refinement property that covers a Δ^1_1 space, then there is a Δ^1_1 such sequence (whose length is an effective ordinal).

THEOREM 3.10. *Suppose that Φ is a refinement property, $X \subseteq \mathcal{N}$ is Δ^1_1, and there a sequence of Borel sets $\overline{B} = \langle B_\beta : \beta \in \omega \cdot \alpha \rangle$ with $X_{\overline{B}} = \emptyset$, $\alpha < \omega_1$, and $\Phi(\overline{B})$. Then, there is a Δ^1_1 sequence of Δ^1_1 sets $\overline{A} = \langle A_\beta : \beta \in \omega \cdot \alpha' \rangle$ so that $X_{\overline{A}} = \emptyset$, $\Phi(\overline{A})$ and $\alpha' < \omega_1^{CK}$.*

Of course one can work in $\mathcal{N}^n$ for some n with only notational changes, or using the fact that $\mathcal{N}$ and $\mathcal{N}^k$ are computably isomorphic.

PROOF. First, define $\mathcal{F}$ by

$$\mathcal{F} = \left\{ \overline{A} \in \Delta^1_1 : \Phi(\overline{A}), \text{ and } (\forall \beta)\ A_\beta \in \Delta^1_1 \right\}.$$

If $X_{\mathcal{F}} = \emptyset$, then by Proposition 3.9, there is single Δ^1_1 sequence $\overline{A}$ so that $X_{\overline{A}} = \emptyset$. So, suppose toward contradiction that $X_{\mathcal{F}} \neq \emptyset$. Note that $X_{\mathcal{F}}$ is Σ^1_1. So,

$$X_{\mathcal{F}} \Vdash \dot{x} \in B_{\alpha,i} \text{ for some } \alpha.$$

Define

$$\alpha_0 := \min\{\alpha : (\exists p \subseteq X_{\mathcal{F}}, i \in \omega)\ p \Vdash \dot{x} \in B_{\alpha,i}\}$$

and fix some $i \in \omega$ and $p \subseteq X_{\mathcal{F}}$ so that $p \Vdash \dot{x} \in B_{\alpha_0,i}$. By the proof of Lemma 3.4, we may assume $\Psi_i(p)$.

Let ϕ_i be as in the definition of refinement property. So ϕ is Σ^1_1 and has arity k. If $q = \{(x_1, \dots, x_k) \in p^k : \phi_i(x)\} \cap (X_{\mathcal{F}})^k$ is nonempty, then

$$q \Vdash \dot{x}_1, \dots, \dot{x}_k \in B_{\alpha_0,i} \wedge\ \phi_i(\dot{x}_1, \dots, \dot{x}_k).$$

Since $\Phi(\overline{B})$, by absoluteness $q \Vdash (\exists j, \beta < \alpha_0)\ x_j \in B_{\beta,0}$. But then we can find a condition below p forcing $\dot{x} \in B_{\beta,0}$ with $\beta < \alpha_0$, contradicting minimality.

So, $\{(x_1, \dots, x_k) \in p^k : \phi_i(x)\} \cap (X_{\mathcal{F}})^k = \emptyset$, and by Proposition 3.9 there is a single Δ^1_1 sequence $\overline{A}$ so that $\{(x_1, \dots, x_k) \in p^k : \phi(x_1, \dots, x_k)\} \cap (X_{\overline{A}})^k = \emptyset$. But then by reflection we can find a Δ^1_1 set $A' \supseteq p$ so that $\Psi_i(A')$ and $\phi_i(\mathbf{x})$ for $\mathbf{x} \in (A')^k \cap (\mathcal{N}_{\overline{A}_0})^k$. Then, setting $A'_j = A'$ if $j = i$ and $A'_j = A_{0,j}$ otherwise, we get a Δ^1_1 sequence of sets $\overline{A}' = \langle A'_j : j \in \omega \rangle$ so that $p \subseteq A_i$ and $\overline{A}^\frown \overline{A}' \in \mathcal{F}$, contradicting our choice of p. ⊣

The only application of this theorem we give below is to local colorings of graphs equipped with quasi-orders. However, we point out that the "forceless, ineffective, and powerless" proofs of Miller typically work by marrying sequences of Borel sets satisfying a refinement property to the usual derivative process on a tree of attempts to build a homomorphism from a canonical obstruction (so that there is no homomorphism if and only if the tree is well-founded if and only if there a sequence of refining sets that cover $\mathcal{N}$). So Theorem 3.10 should usually apply whenever Miller's methods apply.

§4. Applications. In this section we give several applications of the results above to the study of Borel graphs and relations. Throughout, we assume that our graphs

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and relations are on $\mathcal{N}$. The complexity results lift to other spaces by standard universality arguments. We start by giving proofs of two folklore theorems.

The classical Lusin–Novikov uniformization theorem says that any Borel relation with countable sections is the union of countably many Borel partial functions [4, Theorem 18.10]. One important corollary is the Feldman–Moore theorem, which says that any countable Borel equivalence relation is induced by a Borel group action and in fact is a union of countably many Borel involutions [5, Theorem 1.3]. There are elegant effective proofs of these theorems based on the effective perfect set theorem, but these effective proofs do not seem to be recorded in the literature. Here we show how the effective versions of these theorems can be deduced from the classical versions.

THEOREM 4.1. *If R is Δ^1_1 relation with countable sections, then $R = \bigcup_i f_i$ where $\langle f_i : i \in \omega \rangle$ is a Δ^1_1 sequence of Δ^1_1 partial functions. Further, if R is an equivalence relation, then each f_i can be chosen to be an involution.*

PROOF. The classical Lusin–Novikov theorem tells us that R is a union of Borel partial functions. A set $f \subseteq \mathcal{N}^2$ is a partial function if and only if

$$(\forall x, y, y')\ (x, y), (x, y') \in f \to y = y'.$$

This is an independence property, so by Theorem 3.5 R is a union of a Δ^1_1 sequence of Δ^1_1 partial functions.

And if R is an equivalence relation, then the classical Feldman–Moore theorem says that R is a union of Borel involutions. A set $f \subseteq \mathcal{N}^2$ is a partial involution if and only if

$$f \text{ is a function and } (\forall x, y)\ (x, y) \in f \to (y, x) \in f.$$

This is a conjunction of an independence property and a closure property. So again by Theorem 3.5, R is a union of a Δ^1_1 sequence of Δ^1_1 partial involutions.

To finish, we need to show that any partial Δ^1_1 involution extends to an involution. The domain of any Δ^1_1 partial function is Δ^1_1, so we can then extend a partial involution f by setting $f(x) = x$ for $x \notin \mathrm{dom}(f)$. ⊣

Next, we can characterize graphs which can be (not necessarily symmetrically) generated by countable families of functions. That is, we reprove [11, Theorem 4.1].

DEFINITION 4.2. A graph G is *generated* by a list of functions $\langle f_i : i \in I \rangle$ if $G \subseteq \{(x, y) : y = f_i(x) \text{ or } x = f_i(y)\}$.

We could equivalently require $G = \{(x, y) : y = f_i(x) \text{ or } x = f_i(y)\}$ if we allow partial functions. Since any Δ^1_1 partial function extends to a Δ^1_1 total function, this equivalence holds even when we restrict to Borel or Δ^1_1 functions.

THEOREM 4.3. *If G is a Δ^1_1 graph generated by a countable list of Borel functions, then G is generated by a Δ^1_1 list of Δ^1_1 functions.*

PROOF. A graph G is generated by a countable list of functions if and only if G can be covered by countably many sets f so that f is a partial function or $-f$ is a partial function (where $-(x, y) = (y, x)$). Since being a function is an independence property and $e \mapsto -e$ is a Δ^1_1 bijection, Lemma 3.4 tells us that this is a reflectable property. So, if we have a Borel cover we have a Δ^1_1 cover by Theorem 3.6. ⊣

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COROLLARY 4.4. *The set of Borel graphs generated by a countable family of Borel functions is* Π^1_1 *in the codes.*

We can prove a similar result for graphs generated by a single function. This argument is inspired by Hjorth and Miller's dichotomy for end selection, but we need to make use of some features of acyclic graphs to get around their assumption that G is locally countable. This settles [11, Problem 4.5].

THEOREM 4.5. *If* G *is* Δ^1_1 *and generated by a single Borel function, then it is generated by a single* Δ^1_1 *function.*

PROOF. For this theorem, the reduction to a covering problem is nontrivial. First we reduce to the case where G is acyclic.

Any graph generated by a single function is a sidewalk.[1] Let

$$A = \{x \in \mathcal{N} : [x]_G \text{ contains a cycle}\}.$$

Each component contains at most one cycle, so this set is Δ^1_1, and we can select a single vertex from each component of $G \restriction A$ in a Δ^1_1 way as follows: each vertex in A can enumerate the vertices in its cycle using the effective Luzin–Novikov theorem, and then we choose a vertex if it is the smallest vertex in a cycle according to some Δ^1_1 linear order $\mathcal{N}$. From this selector, it is straightforward to construct a function which generates $G \restriction A$.

Therefore, without loss of generality we may assume that G is acyclic. This means simple paths between points are unique, so the connectedness relation is Δ^1_1.

DEFINITION 4.6. A (directed) edge $e = (e_0, e_1)$ *points* to a vertex x if x and e_0 are connected and the simple path from e_0 to x goes through e_1. We write ePx.

Write eIe' if $e \neq e'$, e_0, e'_0 are connected, and neither ePe'_0 nor $e'Pe_0$. Write $e \prec e'$ if ePe'_0 but not $e'Pe_0$.

A set of edges A is *consistent* if $(\forall e, e' \in A)\ \neg(eIe')$.

The simple path from a vertex x to itself contains only x. So, if A is consistent, then we cannot have (x, y) and (x, y') both in A unless $y = y'$. That is, any consistent set must be a partial function. For any Δ^1_1 consistent set A, we can take its downward closure under $\prec$ to get a Σ^1_1 consistent $\prec$-closed set. By the second reflection theorem, any Σ^1_1 consistent $\prec$-closed set extends to a Δ^1_1 consistent $\prec$-closed set.

LEMMA 4.7. $G = \bigcup_i (A_i \cup (-A_i))$ *where* $\langle A_i : i \in \omega \rangle$ *is a* Δ^1_1 *sequence of consistent sets if and only if* G *is generated by a* Δ^1_1 *function.*

PROOF OF LEMMA Suppose f generates. We want to show that $f \cap G$ is consistent. Otherwise, there are some $e, e' \in f$ so that eIe'. If there are n undirected edges on the simple path between e_0 and e'_0, there are $n - 1$ vertices on the path besides e_0 and e'_0. Label these vertices $v_1, \ldots, v_{n-1}$. Since $f(e_0)$ and $f(e'_0)$ point off of the path and f generates G, there can only be the $n - 1$ edges on the path of the form $\{v_i, f(v_i)\}$. This is a contradiction. So if f generates, then $G \subseteq f \cup (-f)$ gives a cover.

For the other direction we may assume each A_i is $\prec$-closed. Note that if X, Y are $\prec$-closed and consistent, then so is $Y \cup (X \setminus (-Y))$. Define f_i inductively by

[1] A sidewalk is a graph with at most one cycle in each component. These are sometimes called pseudoforests.

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$f_{i+1} = f_i \cup (A_{i+1} \setminus (-f_i))$, and set $f = \bigcup_i f_i$. Then f is a partial function so that $G = f \cup (-f)$, and G is generated by a single function. ⊣

So, G is generated by a (Borel) Δ^1_1 function if and only if G is covered by (Borel) Δ^1_1 sets A such that A is consistent and $-A$ is consistent. ⊣

COROLLARY 4.8. *The set of graphs generated by a Borel function is* Π^1_1 *in the codes.*

We can also prove an effectivization result for end selection in locally countable graphs using an argument of Hjorth and Miller [3]. Note, their dichotomy was proven without effective methods and only gives a complexity upper bound of $\mathbf{\Delta}^1_2$ for end selection. In the proof we will use Hjorth and Miller's characterization of end selection, but for completeness we include the definition of end and end selection below:

DEFINITION 4.9. For a graph G, write $[G]^{<\omega}$ for the G-connected finite subsets of G. (Note that if G is locally countable, this can be coded as a subset of $\mathcal{N}$ using the effective Lusin–Novikov theorem.) For $S \in [G]^{<\omega}$, $[S]_G$ is the connected component of G. If v is a vertex, we write $[v]_G$ for $[\{v\}]_G$. Write $G \setminus S$ for the induced subgraph on the complement of S.

If G is a locally countable connected graph, then an *end* in G is an equivalence class of infinite injective paths in G with respect to the equivalence relation

$$\langle v_i : i \in \omega \rangle \sim \langle u_i : i \in \omega \rangle :\Leftrightarrow (\forall S \in [G]^{<\omega})(\exists N \in \omega)(\forall i, j > N)\, v_i \in [u_i]_{G \setminus S}.$$

That is, an end is a class of paths which eventually land in the same complement if we cut out any finite set.

An *end selection* in a graph G is a choice of infinite paths for each vertex so that vertices in the same component choose paths which represent the same end.

THEOREM 4.10. *If G is a locally countable Δ^1_1 graph which admits a Borel end selection, then G admits a Δ^1_1 end selection.*

PROOF. Since G is locally countable, E_G is Δ^1_1. Let

$$R = \{(S, x) : S \in [G]^{<\infty}, x \in [S]_G\}.$$

Note that R has countable sections over $[G]^{<\infty}$. Following [3], $\mathcal{D} \subseteq [G]^{<\infty} \times \mathcal{N}$ is *directed* if:

(1) $\mathcal{D} \subseteq R$.
(2) For all $(S, x) \in \mathcal{D}$, $x \notin S$.
(3) If $(S, x), (S', x') \in \mathcal{D}$ and S and S' are in the same G-component, then and x', x are connected in at least one of $G \setminus S$ or $G \setminus S'$.

Note that directedness is an independence property. And, [3, Proposition 3.4] says a locally countable Borel graph G admits a Borel end selection if and only $[G]^{<\infty}$ can be covered by projections of Borel directed sets. It is straightforward to check that one can replace Borel with Δ^1_1 in their proof. We can then apply Theorem 3.6. ⊣

The theorem above implies that any Δ^1_1 Schreier graph of a free Borel action of $\mathbb{Z}$ is the Schreier graph of a free Δ^1_1 action. Or, in the language of [8], the set of directable forests of lines is Π^1_1 in the codes. We can generalize this to effectivize Borel actions of $\mathbb{Z}^2$.

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THEOREM 4.11. *If G is a Δ^1_1 graph and the Schreier graph of free Borel action of $\mathbb{Z}^2$ with the usual generating set, then G is the Schreier graph of a free Δ^1_1 action of $\mathbb{Z}^2$.*

PROOF. All components of G are isomorphic the Cayley graph of $\mathbb{Z}^2$. In particular G is locally countable.

DEFINITION 4.12. A *straight line* is a directed path so that no other path of the same or shorter length has the same endpoints. A *rectangle* is a simple directed cycle that can be divided into four straight lines.

Two directed edges e, e' are *parallel* if they lie on some straight line together or if $(-e)$ and e' are on opposite sides of a rectangle. We write $e \| e'$. We say e, e' are *anti-parallel* if $(-e) \| e$. And we say e, e' are *perpendicular* if they are in the same component and are neither parallel nor anti-parallel. We write $e \perp e'$.

Note that all of these relations are Δ^1_1.

DEFINITION 4.13. Let $\{a, b, -a, -b\}$ be the usual generating set for $\mathbb{Z}^2$. For a graph G, say that a partial function $f : G \to \{a, b, -a, -b\}$ is a partial diagram if for all e, e',

(1) if $e \| e'$, then $f(e) = f(e')$,
(2) if $e \| (-e')$ then $f(e) = -f(e')$,
(3) if $e \perp e'$ then $f(e)$ and $f(e')$ have different letters (i.e. $f(e) \neq \pm f(e')$).

A Cayley diagram is a partial diagram whose domain is all of G.

We first check that being induced by a free $\mathbb{Z}^2$ action is equivalent to having Cayley diagram. If $a : \mathbb{Z}^2 \curvearrowright \mathcal{N}$ is an action generating G, then any straight line is of the form $\{n\gamma \cdot x : n \in \mathbb{Z}\}$ for some generator γ of $\mathbb{Z}^2$. It follows that $f(x, \gamma \cdot x) = \gamma$ defines a Cayley diagram. Conversely, given a Cayley diagram f, we can define an action of a generator c by $c \cdot x = y \Leftrightarrow f(x, y) = c$. To see this extends to an action of $\mathbb{Z}^2$, note that, for any x, y, $f(x, y) = -f(y, x)$ so $(\gamma - \gamma) \cdot x = x$, and $(x, a \cdot x, (b + a) \cdot x, (-a + b + a) \cdot x, (-b - a + b + a) \cdot x)$ must be a rectangle, so $x = (-b - a + b + a) \cdot x$.

Since being a partial diagram is an independence property, it suffices to show that any sequence of partial diagrams whose domains cover G can be patched together into a Cayley diagram.

Suppose we have two Δ^1_1 partial diagrams f and g. We show how to modify and glue them together to get a partial diagram whose domain is the union of their domains. The result then follows by induction. Define h by:

(1) if $e \in \mathrm{dom}(f)$, then $h(e) = f(e)$,
(2) if $e \in \mathrm{dom}(g) \setminus [\mathrm{dom}(f)]_G$, $h(e) = g(e)$,
(3) else if $e \in \mathrm{dom}(g)$ and $e \| e'$ for some $e' \in \mathrm{dom}(f)$ (or $e \| (-e')$), then $h(e) = f(e')$ (or $-f(e')$),
(4) else if $e \perp e' \in \mathrm{dom}(f)$, $h(e)$ has the sign of $g(e)$ and the opposite letter of $f(e')$.

It is straightforward to check that h is a partial diagram. ⊣

COROLLARY 4.14. *The set of Schreier graphs of Borel $\mathbb{Z}^2$ actions is Π^1_1 in the codes.*

A similar idea works for $\mathbb{Z}^n$. We can also effectivize a recent result of Miller which we will use in the next section.

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THEOREM 4.15 (cf. [9, Theorem 1]). *If $\langle G_{i,j} : i, j \in \omega\rangle$ is a Δ^1_1 array of graphs, and there are Borel sets B_i such that $\mathcal{N} = \bigcup_i B_i$ and $G_{i,j} \restriction B_i$ has a Borel countable coloring for every j, then there are Δ^1_1 sets A_i so that $G_{i,j} \restriction A_i$ has a Δ^1_1 countable coloring for all j.*

PROOF. This is equivalent to effectivizing sequences $\langle A_{ijk} : i, j, k \in \omega\rangle$ of sets so that A_{ijk} is G_{ij}-independent and so that for all x there is an i so that for all j there is a k so that $x \in A_{ijk}$. The result then follows from Theorem 3.7. ⊣

COROLLARY 4.16. *The set of arrays of Borel graphs $G_{i,j}$ so that there are Borel sets B_i with $\chi_B(G_{i,j} \restriction B_i) \leq \aleph_0$ for all i, j and $\mathcal{N} = \bigcup_i B_i$ is Π^1_1 in the codes.*

Finally, we can effectivize Miller's dichotomy for local colorings of graphs equipped with quasi-orders [10, Theorem 5.1.2].

THEOREM 4.17. *If (G, R) is a Δ^1_1 graph and quasi-order on $\mathcal{N}$ so that R admits a Borel homomorphism, f, to some $\leq^\alpha_{lex}$ with $\alpha < \omega_1$ with $G \cap (\equiv_f)$ countably Borel colorable, then (G, R) admits a Δ^1_1 such homomorphism into some $\leq^\alpha_{lex}$ with $\alpha < \omega_1^{CK}$.*

PROOF. First we show that such a homomorphism from R is equivalent to a sequence of pairs of sets $\langle (R_\alpha, S_\alpha) : \alpha < \beta\rangle$ in $\mathcal{N}^2$ with:

(1) Each R_α is a box $A_\alpha \times B_\alpha$, and each S_α is a box $C_\alpha \times D_\alpha$.
(2) $(R_\alpha \cap R) \setminus \left(\bigcup_{\gamma<\alpha} R_\gamma\right) = \emptyset$.
(3) $\left(C^2_\alpha \cap G\right) \setminus \left(\bigcup_{\gamma<\alpha} R_\gamma\right) = \emptyset$.
(4) $C_\alpha \cap D_\alpha = \emptyset$.
(5) $G \subseteq \bigcup_{\alpha<\beta} R_\alpha \cup S_\alpha$.

Given such a sequence, we may assume each A_α is closed upwards under the quasi-order generated by $R \setminus \left(\bigcup_{\gamma<\alpha} R_\gamma\right)$. So, $f(x)(\alpha) = \chi_{A_\alpha}(x)$ defines a homomorphism. And $c(x) = \min\{\alpha : x \in C_\alpha\}$ gives a coloring of $G \cap (\equiv_f)$. Conversely, given a homomorphism and coloring we can get such a sequence by considering the color classes and the sets of the form $\{x : f(x)(\alpha) = i\}$ for $i \in \{0, 1\}$.

Since (1), (2), (3), and (4) describe a refinement property, we can apply Theorem 3.10. ⊣

COROLLARY 4.18. *The set of pairs (G, R) where G is a Borel graph and R is a Borel quasi-order with a homomorphism and coloring as above is Π^1_1 in the codes.*

§5. Some dichotomies. We end by establishing some dichotomy theorems related to some of the new effectivization results of the previous section. Our two results about generating graphs with functions can be deduced from a dichotomy for graphs equipped with group actions.

DEFINITION 5.1. For G a Borel graph on X, Γ a countable group, and a an action of Γ on X, we say that $f : \mathcal{N} \to \omega \times \Gamma$ is an *a-coloring* of (G, a) if, for all $n \in \omega$, $\{h \cdot_a x : f(x) = (n, h)\}$ is G-independent.

Fix $\langle (\gamma_n, s_n) : n \in \omega\rangle$ a sequence with $(\gamma_n, s_n) \in \Gamma \times 2^n$ such that, for each $\gamma \in \Gamma$, $\gamma = \gamma_n$ for infinitely many n, and every string in $2^{<\omega}$ is extended by some s_n. Call

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such a sequence *generic*. Then G_Γ is the graph on $\Gamma \times 2^\omega$ defined by

$$xG_\Gamma y :\Leftrightarrow (\exists z \in 2^\omega, n \in \omega)\left[x = s_n^\frown 0^\frown z \text{ and } y = s_n^\frown 1^\frown z\right].$$

We equip $\Gamma \times 2^\omega$ with the obvious Γ action, a_Γ: for any $z \in 2^\omega$

$$\gamma \cdot (\gamma', z) = (\gamma\gamma', z).$$

Note that a graph G is generated by a countably family of functions if and only if there is an a-coloring of (G', a) where G' is the graph on G given by

$$G' = \{(e, e') \in G^2 : e_0 = e'_0\}$$

and a is the action of $C_2 = (\{\pm 1\}, \times)$ given by

$$-1 \cdot (x, y) = -(x, y) = (y, x).$$

This along with the following theorem settles [11, Problem 4.4]. Similarly, a graph is generated by a single function if and only if there is an a-coloring of (G'', a), where $eG''e'$ if and only if eIe' (with I as in Definition 4.6) and a is as before.

THEOREM 5.2. *For any Borel graph G and Borel action of $a : \Gamma \curvearrowright \mathcal{N}$, either (G, a) admits a Borel a-coloring or there is an equivariant homomorphism from G_Γ to G, i.e., a map $f : \Gamma \times 2^\omega \to \mathcal{N}$ so that*:

(1) $f(h \cdot_{a_\Gamma} x) = h \cdot_a f(x)$.
(2) *If $xG_\Gamma y$ then $f(x)Gf(y)$.*

PROOF. First we show that the two options are mutually exclusive. If c is an a-coloring of (G, a) and f is an equivariant homomorphism of (G_Γ, a_Γ) into G, then $c \circ f$ is an a-coloring of G_Γ. So, it suffices to show no such a-coloring exists. Suppose $c : \Gamma \times 2^\omega \to \omega \times \Gamma$ is a Borel a-coloring. Then, for some (n, γ), $c^{-1}(n, \gamma)$ is nonmeager. Since Γ acts by homeomorphisms, $U = \gamma \cdot c^{-1}(n, \gamma)$ is also nonmeager. Suppose U is comeager in $N_{(\gamma', s)}$. By genericity of $\langle(\gamma_n, s_n) : n \in \omega\rangle$, there is some n with $\gamma' = \gamma_n$ and $s \subseteq s_n$. Let t be the homeomorphism that flips the $(n+1)$th bit of a string. Then, every $(\gamma', \sigma) \in N_{(\gamma_n, s_n)}$ has a neighbour $(\gamma', t(\sigma)) \in N_{\gamma', s}$. So, by Baire category, there is a nonmeager set of points in U with a neighbour in U. This contradicts the definition of a-colorings.

For $\gamma \in \Gamma$, define G_γ by $x\ G_\gamma\ y$ if and only if $\gamma \cdot x\ G\ \gamma \cdot y$. Then, there is an a-coloring of (G, a) if and only if there a cover of X by sets which are independent for some G_γ.

Let $G_{0,\gamma}$ be the graph on 2^ω given by

$$x\ G_{0,\gamma}\ y :\Leftrightarrow (\exists z \in 2^\omega, n \in \omega)\ \gamma_n = \gamma, x = s_n^\frown 0^\frown z, y = s_n^\frown 1^\frown z.$$

Suppose (G, a) does not admit an a-coloring. By [9] and the genericity of $\langle(\gamma_n, s_n) : n \in \omega\rangle$, there is some $g : 2^\omega \to G$ which is a homomorphism from $G_{0,\gamma}$ to G_γ for every Γ.

Now define $f(\gamma, x) = \gamma \cdot g(x)$. We show this is an equivariant homomorphism from (G_Γ, a_Γ) to (G, a). Equivariance is clear from the definition, and if $(\gamma, x)G_\Gamma(\gamma, y)$, then $x\ G_{0,\gamma}\ y$, so $g(x)\ G_\gamma\ g(y)$ and $\gamma \cdot g(x)\ G\ \gamma \cdot g(y)$. ⊣

COROLLARY 5.3. *The set of pairs of Borel graphs and actions which admit a-colorings is Π^1_1 in the codes.*

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PROOF. The proof above gives a reduction to the set of codes for sequences of graphs as in Theorem 4.15. ⊣

So, a graph G is generated by a countable family of Borel functions if and only if there is an equivariant homomorphism from (G_{C_2}, a_{C_2}) to (G', a) as defined in the comments after Definition 5.1. We can refine this analysis somewhat to eliminate reference to the auxiliary graph G'.

DEFINITION 5.4. For tuples of relations $R = (R_1, \dots, R_n)$ on X and $S = (S_1, \dots, S_n)$ on Y, a homomorphism from R to S is a function $f : X \to Y$ so that, for each i, f is a homomorphism from R_i to S_i: $R_i(x_1, \dots, x_k)$ implies $S_i(f(x_1), \dots, f(x_k))$.

As usual, $C_2 = \{1, -1\}$ acts on $C_2 \times 2^\omega$ by $a \cdot (b, x) = (ab, x)$.

THEOREM 5.5. *Define graphs R and $G^-_{C_2}$ on $C_2 \times 2^\omega$ as follows*:

$$R = \{(x, (-1) \cdot x) : x \in C_2 \times 2^\omega\}$$

and

$$G^-_{C_2} = \{((-1) \cdot x, (-1) \cdot y) : (x, y) \in G_{C_2}\}.$$

A graph G on X is generated by a countable family of Borel functions if and only if there is no homomorphism from $(G_{C_2}, G^-_{C_2}, R)$ to $(=, \neq, G)$.

PROOF. Let's abbreviate $(-1) \cdot x$ by $-x$ for $x \in C_2 \times 2^\omega$. So, a homomorphism from $(G_{C_2}, G^-_{C_2}, R)$ to $(=, \neq, G)$ amounts to a map $f : C_2 \times 2^\omega$ so that:

(1) $f(x) = f(y)$ if $(x, y) \in G_{C_2}$.
(2) $(f(x), f(-x)) \in G$.
(3) If $-x G_{C_2} - y$, then $f(x) \neq f(y)$ (or equivalently if $x G_{C_2} y$ then $f(-x) \neq f(-y)$).

We will show that such a map exists if and only if there is an equivariant homomorphism from (G_{C_2}, a_{C_2}) into the graph $\widetilde{G} = \{((x, y), (x, y')) : (x, y), (x, y') \in G\}$ on G.

Suppose f is a map as above. Then define $\tilde{f} : C_2 \times 2^\omega \to G$ by $\tilde{f}(x) = (f(x), f(-x))$. This is well-defined by property (2). If x, y are neighbours in G_{C_2}, then by property (3), $f(-x) \neq f(-y)$, and by property (1) $f(x) = f(y)$. So, $\tilde{f}(x) = (f(x), f(-x)) = (f(y), f(-x))$ and $\tilde{f}(y) = (f(y), f(-y))$ are neighbours in $\widetilde{G}$.

Suppose f is an equivariant homomorphism from (G_{C_2}, a_{C_2}) to $\widetilde{G}$. Then define $\tilde{f} : C_2 \times 2^\omega \to X$ by $\tilde{f}(x) = x_0$ where $f(x) = (x_0, x_1)$. If $f(x) = (x_0, x_1) \in G$, then $f(-x) = -(x_0, x_1) = (x_1, x_0)$, so $\tilde{f}(-x) = x_1$. This means $\tilde{f}(x)$ and $\tilde{f}(-x)$ share an edge in G, giving property (2). And, if x and y are neighbours in G_{C_2}, then $f(x)$ and $f(y)$ are neighbours in $\widetilde{G}$, so $x_0 = y_0$ and $x_1 \neq y_1$; this gives properties (1) and (3) above. ⊣

Note that (1) and (2) above say that f descends to a homomorphism to G from the graph on $(C_2 \times 2^\omega)/E_{G_{C_2}}$ with edges $([x], [-x])$.

We can also characterize graphs induced by an action of $\mathbb{Z}^2$. There are two canonical obstructions in this case, one in two dimensions and one in one dimension.

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DEFINITION 5.6. A *square lattice* is a graph isomorphic to the usual Cayley graph on $\mathbb{Z}^2$. A graph is *locally* a square lattice if each component is a square lattice.

For G locally a square lattice, let $\perp_G$ denote the graph whose vertex set is G and two vertices (meaning edges in G) are adjacent if they are perpendicular in the sense of Definition 4.12.

The proof of Theorem 4.11 gives the following:

PROPOSITION 5.7. *If G is locally a square lattice, then G is induced by a free Borel action of $\mathbb{Z}^2$ if and only if $\perp_G$ admits a Borel 2-coloring and any forest of straight lines in G is directable.*

We think of $\perp_G$ having a 2-coloring as a two-dimensional requirement, and having all forests of straight lines be directable as one-dimensional. These two requirements are independent.

PROPOSITION 5.8. *There are Borel graphs G, Γ where $\perp_G$ admits a Borel 2-coloring, and every forest of lines in Γ is directable, but where neither is induced by a Borel free action of $\mathbb{Z}^2$.*

PROOF. Let $E = \{e_0, e_1, -e_0, -e_1\}$ be the standard generating set for $\mathbb{Z}^2$, and let $\langle n, m\rangle$ be the canonical image of $(n, m) \in \mathbb{Z}^2$ in $\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}^2, E))$. Suppose $\Gamma \subseteq \mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}^2, E))$ is a subgroup so that $\gamma(0) = 0$ and $\langle \gamma(n, m)\rangle^{-1}\gamma\langle n, m\rangle \in \Gamma$ for all $\gamma \in \Gamma$. Note that such a group must be finite.

Let Fr_Γ be the free part of $[0, 1]^{\mathbb{Z}^2}$. The group $\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}^2, E))$ acts on Fr_Γ by shifting indices, $\gamma \cdot x = x \circ \gamma^{-1}$. Let $\tilde{S}_\Gamma$ be the graph on $\mathrm{Fr}_\Gamma/\Gamma$ defined by

$$\tilde{S}_\Gamma := \{([x], [\langle a\rangle \cdot x]) : x \in \mathrm{Fr}_\Gamma, a \in E\}.$$

First, we show that $\tilde{S}_\gamma$ is always locally isomorphic to $\mathrm{Cay}(\mathbb{Z}^2, E)$. In fact, for any $x \in \mathrm{Fr}_\gamma$, $(a, b) \mapsto [\langle a, b\rangle \cdot x]$ is an isomorphism from $\mathrm{Cay}(\mathbb{Z}^2, E)$ to the component of $[x]$. Clearly, this map is a homomorphism. The relation

$$\gamma \cdot (\langle n, m\rangle \cdot x) = \langle \gamma(n, m)\rangle \cdot \big(\langle \gamma(n, m)^{-1}\rangle \gamma\langle n, m\rangle \cdot x\big)$$

says that, for every $n, m \in \mathbb{Z}$ and $\gamma \in \Gamma$, there are unique $(a, b) \in \mathbb{Z}^2$ and $\delta \in \Gamma$ so that $\gamma \cdot (\langle n, m\rangle \cdot x) = \langle a, b\rangle \cdot (\delta \cdot x)$. So this map is a bijection. And since Γ acts freely and preserves E, the same relation implies that non-edges are sent to non-edges.

Second, we show that if Γ is nontrivial, then $\tilde{S}_\Gamma$ is not the Cayley graph of a Borel action of $\mathbb{Z}^2$. Suppose otherwise and for $x \in \mathrm{Fr}_\Gamma$ consider the bijection $f_x : E \to E$ defined by

$$f_x(a) = b :\Leftrightarrow [\langle a\rangle \cdot x] = b \cdot [x].$$

The same relation as above says that $f_{\gamma\cdot x}(a) = f_x(\gamma(b)^{-1})$ for $\gamma \in \Gamma$. And $f_{\langle n,m\rangle\cdot x} = f_x$ for $n, m \in \mathbb{Z}^2$ since $\langle n, m\rangle$ commutes with $\langle a\rangle$ for $a \in E$.

For some $f : E \to E$, the set $\{x : f_x = f\}$ has positive measure and is $\mathbb{Z}^2$-invariant. By ergodicity of the shift action, this set has measure one. But then $f_x = f_{\gamma\cdot x}$ for some x, which is a contradiction.

All that remains is to find suitable Γ and Δ so that $\perp_{\tilde{S}_\Gamma}$ has a Borel 2-coloring and every forest of straight lines in $\tilde{S}_\Delta$ is directable.

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Let Γ be the group generated by t_0 and t_1 where $t_0(n,m) = (-n, m)$ and $t_1(n,m) = (n, -m)$. Since $t_i(\pm e_j) \in \{e_j, -e_j\}$ for each i, j, we can define a Borel 2-coloring of $\tilde{S}_\Gamma$ by

$$f([x],[\pm e_i \cdot x]) = i.$$

Let Δ be the group generated by s where $s(n,m) = (m,n)$. Since $s(-e_i) = -s(e_i)$ for each i, we can define an orientation o of $\tilde{S}_\Delta$ by

$$o = \{([x],[y]) : x = e_0 \cdot y \text{ or } x = e_1 \cdot y\}.$$

This orientation is balanced when restricted to any straight line, so gives a direction to every forest of straight lines. ⊣

Miller's $\mathcal{L}_0$ dichotomy characterizes undirectable forests of lines [8], and a result of Carroy, Miller, Schrittesser, and Vidyanszky characterizes 2-colorable graphs [1]. In the case of $\perp_G$, we can reflect this to a characterization of G, but we need to strengthen the results in [1].

DEFINITION 5.9. Let $s_0 = \emptyset$, and for $n > 0$ let $s_n = 0^{n-1}1$. Define a graph $\mathcal{L}$ on 2^ω by $\{x, y\} \in \mathcal{L}$ if and only if

$$(\exists z \in 2^\omega, n \in \omega) x = s_n^\frown 0^\frown z, y = s_n^\frown 1^\frown z.$$

Let d_G be the graph metric for G, i.e., for vertices x, y, $d_G(x,y)$ is the minimal number of edges in a path from x to y (or infinite if x and y are not connected). For any graph G, let $G_{odd} = \{(x,y) : d_G(x,y) \text{ is odd}\}$.

If we let G_n be the graph on 2^n gotten by restricting the definition of G to strings of length n, then G is a projective limit of the sequence $\langle G_n : n \in \omega\rangle$. Each G_n is a path graph, and G_{n+1} is gotten by connecting two endpoints of two copies of G_{n+1}.

THEOREM 5.10 (Ess. [1, Theorem 1.1]). *If G is a Borel graph and not Borel 2-colorable, then there is a homomorphism of $\mathcal{L}$ into G_{odd}.*

Note that if G is locally a square lattice, then $(\perp_G)_{odd} = \perp_G$. The strengthening we need is a strong form of local injectivity for the homomorphism given by the theorem above.

THEOREM 5.11. *If G is locally a square lattice, and $\perp_G$ does not admit a Borel 2-coloring, then there is a Borel homomorphism from $\mathcal{L}$ into $\perp_G$ so that, if x, y are connected and $x \neq y$, then $f(x)$ and $f(y)$ do not lie on a straight line.*

PROOF. By relativization, we may assume G is Δ^1_1. Suppose $\perp_G$ does not admit a Δ^1_1 2-coloring. Define

$$X := G \setminus \bigcup \{A \in \Delta^1_1 : A \text{ is } \perp_G\text{-independent}\}.$$

By reflection, X is the same as G without its Σ^1_1 independent sets. If X is empty, then we can construct a Δ^1_1 2-coloring of $\perp_G$. So X is nonempty.

As usual, we will build a homomorphism by constructing trees of conditions forcing the graph relations we want. To ensure we get the kind of injectivity we want, we will make sure our generics force points in the image of our homomorphism to be far apart.

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For edges e, e' let $d_G(e, e') = d_G(e_0, e_0')$ with respect to the graph metric. Say that e, e' are n-spaced if, whenever we have intersecting paths p, p' which are perpendicular line segments such that e_0 is on p and e_0' is on p', then the length of p, p' are both of length at least n.

Say that $\langle U_\sigma : \sigma \in 2^\omega \rangle$ and $\langle R_{\sigma,\tau} : (\sigma, \tau) \in G_n \rangle$ are an n-approximation with spread d if:

(1) $U_\sigma \subseteq X$ and $R_{\sigma,\tau} \subseteq \perp_G$.
(2) If $\langle x_\sigma : \sigma \in 2^n \rangle$ satisfies $(\forall \sigma, \tau \in G_n)\, (x_\sigma, x_\tau) \in R_{\sigma,\tau}$, then for all $\sigma, \tau \in 2^n$, $d_G(x_\sigma, x_\tau) < d$.
(3) For any $x \in U_\rho$ there is $\langle x_\sigma : \sigma \in 2^n \rangle$ so that $(\forall (\sigma, \tau) \in G_n)\, (x_\sigma, x_\tau) \in R_{\sigma,\tau}$ and $x_\rho = x$.

We want to build a tree of generics which contain nice approximations. I claim that for any n-approximation with spread d, $\langle R_{\sigma,\tau} : (\sigma, \tau) \in G_n \rangle$ and $\langle U_\sigma : \sigma \in 2^n \rangle$, and any dense sets of conditions $E \subseteq \mathbb{P}_1$ and $F \subseteq \mathbb{P}_2$ there is an $(n+1)$-approximation $\langle R_{\sigma,\tau} : (\sigma, \tau) \in G_{n+1} \rangle$ and $\langle U_\sigma : \sigma \in 2^{n+1} \rangle$ so that:

a) $U_{\sigma^\frown i} \subseteq U_\sigma$ and $R_{\sigma^\frown i, \tau^\frown i} \subseteq R_{\sigma,\tau}$ for all $\sigma, \tau \in G_n$ and $i \in \{0, 1\}$.
b) For all σ, τ, $U_\sigma \subseteq E$ and $R_{\sigma,\tau} \subseteq F$.
c) Any pair of points $(x_{\sigma_n^\frown 0}, x_{\sigma_n^\frown 1}) \in R_{\sigma_n^\frown 0, \sigma_n^\frown 1}$ are at least $(3d+1)$-spaced.

To this end, define V to be the set of realizations of our n-approximation

$$V := \{x : (\exists \langle x_\sigma : \sigma \in 2^n \rangle)\, x = x_{\sigma_n}, x_\sigma \in U_\sigma, x_{\sigma,\tau} \in R_{\sigma\tau}\}.$$

Since V is a nonempty Σ^1_1 subset of X, it must not be independent. That is V contains two perpendicular edges. We need V to contain two edges which are perpendicular and $(3d+1)$-spaced. If we can't find such edges, then the intersection of V with any component is either finite or bounded by two parallel lines. But then, in either case, we can refine V to an independent set:

- if V meets a component in a finite set can choose a single point from that component (say the least according to some Δ^1_1 order,
- if V is bounded by two parallel lines we can take all edges parallel to these lines (or perpendicular if none are parallel).

This contradicts the fact that X doesn't contain any Σ^1_1 independent sets. So, we can let $R_{\sigma_n^\frown 0, \sigma_n^\frown 1}$ be any condition in F below

$$\{(x, y) \in V^2 \cap (\perp_G) : x, y \text{ are } (3d+1)\text{-spaced}\}.$$

Since G_n is acyclic, the rest of our approximation can be built by iteratively refining out from here, just as in the proof of the G_0-dichotomy.

From the claim above, working over a countable transitive model of a large enough fragment of ZFC, we can build a sequence of n-approximations with spread $d(n)$ so that if $(x, y) \in R_{\sigma_n^\frown 0, \sigma_n^\frown 1}$ then x, y are $(3d(n)+1)$-spaced, and we can build a tree of generics $\langle x_s : s \in 2^\omega \rangle$ so that:

- For all n, $x_s \in U_{s\restriction n}$.
- If $(s, t) \in \mathcal{L}$ then for all n either $s \restriction n = t \restriction n$ or $(x_s, x_t) \in R_{s\restriction n, t\restriction n}$.

In particular, $s \mapsto x_s$ is a homomorphism from $\mathcal{L}$ to $\perp_G$. Now we check that no two connected points get sent to the same straight line. Suppose $(s, t) \in \mathcal{L}$, say

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$s = \sigma^\frown 0^\frown z$ and $t = \tau^\frown 1^\frown z$ with $|\sigma| = |\tau| = n$. Then by our assumption on spread $d_G(x_s, x_{\sigma_n^\frown 0^\frown z}) < d(n)$ and likewise for x_t and $x_{\sigma_n^\frown 1^\frown z}$. And since $x_{\sigma_n^\frown 0^\frown z}$ and $x_{\sigma^\frown 1^\frown z}$ are at least $(3d(n)+1)$-spaced, we must have that x_s and x_t are on different straight lines. ⊣

With this in place, we can start characterizing graphs G where $\perp_G$ admits a Borel 2-coloring.

DEFINITION 5.12. If G and H are graphs on X, Y which are locally square lattices, then say $f : X \to Y$ is a conformal map from G to H if, for any vertices x, y of G, x, y are on a straight line if and only if $f(x), f(y)$ are on a straight line.

Conformal maps preserve perpendicularity, so we can pull a 2-coloring back from $\perp_G$ to $\perp_H$ whenever there is a conformal map from H to G.

PROPOSITION 5.13. *If there is a Borel conformal map from H to G and $\perp_G$ has a Borel 2-coloring, then $\perp_H$ has a Borel 2-coloring.*

PROOF. First note that if v_1, v_2, v_3, and v_4 are consecutive corners of a rectangle in a square lattice if and only v_i and v_j are on straight line exactly when $i = j \pm 1 \pmod 4$. So the corners of a rectangle in H are sent to the corners of a rectangle in G by any conformal map.

Now suppose that f is a Borel conformal map from H to G and g is a Borel 2-coloring of $\perp_G$. Then we have a Borel map $\tilde{g} : H \to 2$ given by $\tilde{g}(e_0, e_1) = g(e')$ for any edge e' on the straight line between $f(e_0)$ and $f(e_1)$. If $e \perp_H e'$, then possibly replacing e with $-e$, we can find a rectangle containing e, e'. This rectangle is sent by f to a rectangle with consecutive sides containing $f(e_0), f(e_1)$ and $f(e_0'), f(e_1')$. The coloring g must assign opposite colors to all edges on consecutive sides of a rectangle, so $\tilde{g}(e) \neq \tilde{g}(e')$. ⊣

Our goal is to show there is a minimal square lattice with respect Borel conformal maps among those where $\perp$ is not Borel 2-colorable.

DEFINITION 5.14. Let $X_{\mathbb{S}_0} = \{(x, y) \in 2^\omega : d_{\mathcal{L}}(x, y) \text{ is odd}\}/E$ with the quotient Borel structure, where $[(x, y)]_E = \{(x, y), (y, x)\}$. Abbreviate $[(x, y)]_E$ by $[x, y]$. Then $\mathbb{S}_0$ is the graph on $X_{\mathbb{S}_0}$ defined by

$$[x, y]\ \mathbb{S}_0\ [a, b] :\Leftrightarrow x = a, d_{\mathcal{L}}(y, b) = 2, \text{ or } y = b, d_{\mathcal{L}}(x, a) = 2.$$

Before proving our dichotomy, let us show that $\mathbb{S}_0$ is locally a square lattice.

PROPOSITION 5.15. *$\mathbb{S}_0$ is locally a square lattice.*

PROOF. Fix a component C of $\mathbb{S}_0$. The corresponding component of $\mathcal{L}$ is 2-regular and acyclic, so we can fix an isomorphism between it and the usual Cayley graph for $\mathbb{Z}$. So we can represent C as $\{(n, m) : n \in 2\mathbb{Z}, m \in 2\mathbb{Z}+1\}$. And we have that (n, m) and (ℓ, k) are adjacent if $n = \ell \pm 2$ or $m = k \pm 2$. Thus $(a, b) \mapsto (2a, 2b+1)$ defines an isomorphism between the usual Cayley graph for $\mathbb{Z}^2$ and C. ⊣

THEOREM 5.16. *If a Borel graph G is locally a square lattice, then either $\perp_G$ admits a Borel 2-coloring, or there is a Borel conformal map from $\mathbb{S}_0$ to G.*

PROOF. First we show $\mathbb{S}_0$ does not admit a conformal map into any graph G where $\perp_G$ is Borel 2-colorable. It suffices to show $\perp_{\mathbb{S}_0}$ does not admit a Borel 2-coloring.

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Note that the straight lines in $\mathbb{S}_0$ are the sets $L_a = \{[a,b] : d(a,b) \text{ is odd}\}$ for $a \in \mathcal{L}$. Suppose $f : G \to 2$ is a Borel 2-coloring of $\perp_{\mathbb{S}_0}$. Then for each a, the edges in L_a are all assigned the same color by f. This defines a Borel function $g : \mathcal{L} \to 2$. Suppose $d(a,b)$ is odd, $d(a,a') = d(b,b') = 2$. Then $([a,b],[a',b])$ and $([a,b],[a,b'])$ are perpendicular edges in L_b and L_a respectively, so edges in L_a and L_b are assigned different colors by f, and $g(a) \neq g(b)$. Thus g is a Borel 2-coloring of $\mathcal{L}$, which is a contradiction.

Now suppose G is locally a square lattice and $\perp_G$ does not admit a Borel 2-coloring. So, there is a Borel homomorphism f from $\mathcal{L}$ to $\perp_G$ which sends connected vertices to edges which are not on a straight line. Define an map g from $\mathbb{S}_0$ to G by setting $g([a,b])$ to be the unique point on both the straight line containing $f(a)$ and the straight line containing $f(b)$. This is well-defined since any two perpendicular lines in the same plane must intersect. All that remains is to check this is conformal.

Suppose $[x,y_0]$ and $[x,y_1]$ are two points on a straight line L_x. Then $g([x,y_0])$ and $g([x,y_1])$ are on the straight line through $f(x)$. Conversely, suppose $[x_0,x_1]$ and $[y_0,y_1]$ are not a straight line, and $d_{\mathcal{L}}(x_0,y_0) = 0 \pmod 2$. Then, $x_0 \neq y_0$ and $x_1 \neq y_1$. So $f(x_0)$, $f(y_0)$, $f(x_1)$, and $f(y_1)$ are all on different straight lines. So, $g([x_0,x_1])$ and $g([y_0,y_1])$ are not on a straight line. Thus g is conformal. ⊣

Combining this with Miller's $\mathcal{L}_0$ dichotomy for directable forests of lines, we get the following:

COROLLARY 5.17. *If G is a Borel graph, either G is the Schreier graph of a free Borel action of $\mathbb{Z}^2$ or at least one of the following holds*:

(1) *G is not locally a square lattice.*
(2) *There is a Borel conformal map from $\mathbb{S}_0$ to G.*
(3) *There is a Borel betweenness-preserving embedding of $\mathcal{L}_0$ into some forest of straight lines in G.*

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