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Hyperbolic maps in p-adic dynamics

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Abstract. In this paper we study the dynamics of a rational function $\phi \in K(z)$ defined over some finite extension K of $\mathbb{Q}_p$. After proving some basic results, we define a notion of 'components' of the Fatou set, analogous to the topological components of a complex Fatou set. We define hyperbolic p-adic maps and, in our main theorem, characterize hyperbolicity by the location of the critical set. We use this theorem and our notion of components to state and prove an analogue of Sullivan's No Wandering Domains Theorem for hyperbolic maps.

0. *Introduction*

Let $p \in \mathbb{Z}$ be a fixed prime number. Let $\mathbb{Q}_p$ denote the field of p-adic rational numbers, $\overline{\mathbb{Q}}_p$ its algebraic closure, and $\mathbb{C}_p$ the topological completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic absolute value $|\cdot|$. Let $\phi \in \overline{\mathbb{Q}}_p(z)$ be a morphism from the projective line $\mathbb{P}^1(\mathbb{C}_p)$ to itself; thus, ϕ is in fact a rational function defined over some finite extension K of $\mathbb{Q}_p$. Denote by ϕ^n the n-fold composition of ϕ with itself. We consider the dynamics of ϕ acting on $\mathbb{P}^1(\mathbb{C}_p)$; that is, we apply ϕ repeatedly and study the phenomena that occur. As in the complex theory of dynamical systems, we define the Fatou set $\mathcal{F}$ of ϕ to be the set of all points in the projective line having a neighborhood on which the family of iterates $\{\phi^n\}$ is equicontinuous (see, for example, [**5, 14**]). Here, the notions of neighborhood and equicontinuity are defined by the spherical metric on $\mathbb{P}^1(\mathbb{C}_p)$ determined by the non-archimedean metric on $\mathbb{C}_p$. The Fatou set is clearly open; its complement, the Julia set $\mathcal{J}$, is therefore closed. Both sets are preserved under the application of ϕ.

We begin our study of such dynamical systems by determining whether a given periodic point of ϕ lies in the Fatou or Julia set; we also show that the Fatou set is always non-empty. We then define D-components (of the Fatou set), which play the same role that connected components do in complex dynamics; in fact, we will see that the D-components of an open subset of $\mathbb{C}$ are the same as the connected components. After these fundamentals, we define hyperbolic maps, using a definition analogous to the standard definition from

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complex dynamics. In our main theorem, we prove that a p-adic map is hyperbolic if and only if there are no critical points in its Julia set; note that this equivalence is stronger than that of the complex setting. Finally, we use the main theorem to prove a p-adic version of Sullivan's No Wandering Domains Theorem [**21**] for hyperbolic maps.

The study of p-adic dynamical systems arises in Diophantine geometry in the construction of canonical local heights, used for counting rational points on algebraic varieties over a number field, as in [**4**]. Hsia [**8, 9**] has proven some basic properties of non-archimedean Julia sets and described their relation to weak Néron models. In addition, p-adic fields have arisen in physics in the theory of superstrings, prompting questions about the dynamics of p-adic functions (see, for example, [**1, 20, 22, 23**]). Other studies of non-archimedean dynamics in the neighborhood of a periodic point and of the counting of periodic points over global fields using local (non-archimedean) fields appear in [**7, 10, 11, 12, 15, 16, 18, 19, 24**].

The results of this paper constitute a portion of the author's PhD thesis [**2**]. The author would like to thank Joseph Silverman for his guidance and for suggesting these problems.

1. *Initial results*

We begin by recalling some basic terminology from the theory of dynamical systems. Let X be a set, and let ϕ be a function mapping X to itself. We say $x \in X$ is *fixed* if $\phi(x) = x$; more generally, we say x is *periodic* (of period n) if $\phi^n(x) = x$ for some $n \geq 1$. If n is the smallest positive integer such that x has period n, we say x has *minimal period* n. We say x is *pre-periodic* if $\phi^m(x)$ is periodic for some $m \geq 0$. Since our set X will be either $\mathbb{P}^1(\mathbb{C})$ or $\mathbb{P}^1(\mathbb{C}_p)$, and ϕ will be a rational function, then given a coordinate function on X, it makes sense to discuss the derivative of ϕ at x. If x is a periodic point of minimal period n, we define the *multiplier* λ of x to be $(\phi^n)'(x)$; the multiplier of a periodic point is invariant under change of coordinates. We say x is *repelling* if $|\lambda| > 1$, *attracting* if $|\lambda| < 1$, and *neutral* if $|\lambda| = 1$.

We recall that a *non-archimedean* metric space X is a metric space satisfying the ultrametric (strengthened) triangle inequality

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} \quad \text{for all } x, y, z \in X.$$

We will study dynamics of rational functions defined over p-adic fields (i.e. $\mathbb{Q}_p$ and its extensions), which are non-archimedean with respect to the p-adic absolute value. First, however, we note the following basic proposition, which applies to more general non-archimedean fields.

PROPOSITION 1.1. *Let K be an algebraically closed field which is complete with respect to a non-archimedean absolute value $|\cdot|$. Let $\phi \in K(z)$ be a rational function with Fatou set $\mathcal{F} \subset \mathbb{P}^1(K)$ and Julia set $\mathcal{J} = \mathbb{P}^1(K) \setminus \mathcal{F}$. Let $a \in \mathbb{P}^1(K)$ be a periodic point of ϕ. Then $a \in \mathcal{J}$ if and only if a is repelling. In particular, all neutral points are Fatou.*

The proof is relatively straightforward; we give a sketch but omit the details. For repelling points, the proof is similar to that of the analogous result in the complex setting; see [**5, 14**]. For non-repelling points, we may assume that the point is fixed (by considering the map ϕ^n instead of ϕ; they have the same Julia and Fatou sets, as shown in [**8**]). We

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expand ϕ as a power series in a small neighborhood U of a. Using ultrametricity (and an appropriate choice of U), it is easy to show that $\phi(U) \subset U$ and that the distance between two points of U cannot increase under application of ϕ. Thus, $\{\phi^n\}$ is equicontinuous on U, and a is Fatou.

Although non-archimedean Fatou and Julia sets are roughly similar to their archimedean counterparts, they differ in several important aspects. As we just saw, all non-archimedean neutral periodic points are Fatou; however, in the complex case, although attracting points are Fatou and repelling points are Julia, a neutral point could be either, depending on the actual value of the multiplier. Another contrast appears in the presence of the Julia set itself. Many p-adic rational functions of arbitrary degree have empty Julia set; see, for example, [**8**] and [**16**]. On the other hand, it is well known that in the complex case every rational function of degree greater than one has a non-empty Julia set (see [**5, 14**]).

In addition, it is possible for a complex rational function to have an empty Fatou set. As we shall see, this never occurs over non-archimedean fields, at least in characteristic zero; to demonstrate, we need the following proposition.

PROPOSITION 1.2. *Let K be an algebraically closed field of characteristic zero. Let $\phi \in K(z)$ be a rational function of degree d. Suppose that none of its fixed points has multiplier* 1. *If we let $\lambda_0, \ldots, \lambda_d$ denote the multipliers of the $d+1$ fixed points of ϕ in $\mathbb{P}^1(K)$, then*

$$\sum_{i=0}^{d} \frac{1}{1-\lambda_i} = 1.$$

Proof. We may assume without loss of generality that ∞ is not a fixed point by conjugating, if necessary, by an appropriate linear fractional map. Now our desired equality is known to hold in the complex case, by the use of contour integration; see, for example, [**14**]. To transfer the result to K, fix the degree of ϕ and note that $f = \sum(1-\lambda)^{-1}$ is a rational function, symmetric in the $\{\lambda_i\}$. However, any rational function symmetric in the roots of a polynomial is equal to some rational function of the coefficients of the polynomial. Since the $\{\lambda_i\}$ are the roots of a polynomial whose coefficients are determined arithmetically by those of ϕ, we see that f is a rational function in the coefficients of ϕ. However, we know from the complex case that if the coefficients of ϕ are in $\mathbb{Q}$, then $f = 1$. Since $\mathbb{Q} \subset K$ is infinite, it follows that f must be identically equal to 1. □

COROLLARY 1.3. *Let K be an algebraically closed field of characteristic zero which is complete with respect to a non-archimedean absolute value $|\cdot|$. Let $\phi \in K(z)$ be a rational function with Fatou set $\mathcal{F} \subset \mathbb{P}^1(K)$. Then $\mathcal{F}$ is non-empty.*

Proof. If any fixed point multiplier is equal to 1, then we have a neutral fixed point which, by Proposition 1.1, must be in $\mathcal{F}$. If not, we apply Proposition 1.2 to get $\sum(1-\lambda)^{-1} = 1$. Suppose that every fixed point were repelling; then all the $|\lambda|$ are greater than 1, and so every term of the sum has absolute value less than 1. But then, by the non-archimedean property, there is no way the terms could add up to 1. Thus, at least one fixed point must be non-repelling and therefore Fatou. □

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The above differences between the complex and non-archimedean Fatou sets are symptomatic of the general principle that equicontinuity is, in some sense, more common in the non-archimedean universe than in the archimedean. Another important example of this phenomenon is Hsia's theorem [**9**], which is the p-adic analogue of a classical theorem of Montel. Montel's theorem states that if a family of analytic maps from the disk to the Riemann sphere omits at least three points, then the family is equicontinuous on the disk. However, Hsia's theorem states that, if K is a complete, algebraically closed non-archimedean field, and if a family of power series in $K[[z]]$ which converge on the disk omits at least two points of $\mathbb{P}^1(K)$, then the family is equicontinuous.

2. *Disks and D-components*

In complex dynamics, it is convenient to break the Fatou set into its connected components and study the action of ϕ on the set of components. We would like to carry out a similar study in the p-adic setting. However, we immediately run into the obstacle that $\mathbb{P}^1(\mathbb{C}_p)$ is totally disconnected, and therefore the connected components are singletons. Instead, we will state a new definition of 'component' which makes sense in the non-archimedean setting and yet, in the complex setting, is equivalent to the topological definition.

Let U be any open subset of $\mathbb{C}$. Because $\mathbb{C}$ is locally path-connected, the topological components of U are exactly its path components. Thus, given any two points x and y in the same component of U, there is a path $\gamma \subset U$ connecting them. Since U is open and γ is compact, we may cover γ with a finite number of open disks contained in U. In particular, we have a finite sequence of disks

$$D_1, \ldots, D_n \subset U,$$

with $x \in D_1$ and $y \in D_n$, such that for any $i = 1, \ldots, n-1$, $D_i \cap D_{i+1} \neq \emptyset$. (By a disk we mean either a closed or open disk. We denote the closed disk of radius $r > 0$ about x by $\overline{D}_r(x)$ and the open disk by $D_r(x)$.) It is therefore reasonable to state the following definition.

Definition 2.1. Let X be a topological space with a set $\mathcal{D}$ of distinguished subsets. Let $U \subset X$ be an open subset, and let x be any point of U. We define the *D-component* of U containing x to be the set of all $y \in U$ with the following property: there exists a finite sequence of distinguished subsets of U

$$D_1, \ldots, D_n \in \mathcal{D},$$

with $x \in D_1$ and $y \in D_n$, such that for any $i = 1, \ldots, n-1$,

$$D_i \cap D_{i+1} \neq \emptyset.$$

If X is a metric space, we will choose our distinguished subsets to be disks. In light of the preceding discussion, it is easy to verify that the D-components of an open subset of $\mathbb{C}$ are the same as the topological components. Fortunately, the situation is different for non-archimedean metric spaces. It should be noted that, in a non-archimedean space, all disks (whether open or closed as disks) are both open and closed as topological sets. Furthermore, any point in a disk is a center; and if two disks intersect, then one contains

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the other. In light of these facts, we have the following proposition, whose proof is straightforward.

PROPOSITION 2.1. *Let K be an algebraically closed complete non-archimedean field, let $U \subset K$ be open, and let $x \in U$. If $U = K$, then the D-component of U containing x is K; otherwise, the D-component is the largest disk centered at x and contained in U.*

We now have a notion of connected components of open subsets of $\mathbb{C}_p$. To extend the notion to the projective line, we consider $\mathbb{P}^1(\mathbb{C}_p)$ as the union $\mathbb{C}_p \cup \{\infty\}$. A 'disk' in $\mathbb{P}^1(\mathbb{C}_p)$, then, should be either any disk in $\mathbb{C}_p$, or any image of such a disk under an automorphism of $\mathbb{P}^1(\mathbb{C}_p)$. We are led to the following definition.

Definition 2.2. A *closed $\mathbb{P}^1(\mathbb{C}_p)$-disk* is a closed disk $\overline{D}_r(x)$, *or* the complement of an open disk, $\mathbb{P}^1(\mathbb{C}_p) \setminus D_r(x)$, for some $x \in \mathbb{C}_p$ and $r > 0$. Similarly, an *open $\mathbb{P}^1(\mathbb{C}_p)$-disk* is an open disk $D_r(x)$, *or* the complement of a closed disk, $\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_r(x)$, for some $x \in \mathbb{C}_p$ and $r > 0$.

It is easy to verify that the set of $\mathbb{P}^1(\mathbb{C}_p)$-disks is, in fact, the set of all images of $\mathbb{C}_p$ disks under automorphisms of $\mathbb{P}^1(\mathbb{C}_p)$; see, for example, [**6**]. We may now define D-components of subsets of $\mathbb{P}^1(\mathbb{C}_p)$ by declaring $\mathcal{D}$ to be the set of $\mathbb{P}^1(\mathbb{C}_p)$-disks. Note that if $U \subset \mathbb{C}_p \subset \mathbb{P}^1(\mathbb{C}_p)$ is open, then its D-components are the same when U is viewed as a subset of $\mathbb{C}_p$ as they are when U is viewed as a subset of $\mathbb{P}^1(\mathbb{C}_p)$. In addition, because the union of two intersecting $\mathbb{P}^1(\mathbb{C}_p)$-disks is either all of $\mathbb{P}^1(\mathbb{C}_p)$ or one of the two original $\mathbb{P}^1(\mathbb{C}_p)$-disks, we have the following characterization of D-components in $\mathbb{P}^1(\mathbb{C}_p)$.

PROPOSITION 2.2. *Let U be an open subset of $\mathbb{P}^1(\mathbb{C}_p)$, and let $x \in U$. If $U = \mathbb{P}^1(\mathbb{C}_p)$, or if U is the complement of a single point of $\mathbb{P}^1(\mathbb{C}_p)$, then the D-component of U containing x is U. Otherwise, the D-component is the largest $\mathbb{P}^1(\mathbb{C}_p)$-disk containing x and contained in U (i.e. it is the union of all such $\mathbb{P}^1(\mathbb{C}_p)$-disks, and that union is itself a $\mathbb{P}^1(\mathbb{C}_p)$-disk).*

Ultimately, we will study D-components of the Fatou set of a rational function ϕ. We will therefore be interested in the mapping properties of $\mathbb{P}^1(\mathbb{C}_p)$-disks and D-components under power series and rational functions. We will need the following proposition.

PROPOSITION 2.3. *Let $D = \overline{D}_r(a)$ be a closed disk in $\mathbb{C}_p$, and let $f \in \mathbb{C}_p[[z-a]]$ be a power series convergent on D. Then the image $f(D)$ is also a closed disk in $\mathbb{C}_p$. Furthermore, if the image disk is $\overline{D}_s(f(a))$, then $r|f'(a)| \leq s$.*

The proof is an easy exercise in non-archimedean power series. We note that since any open disk can be written as the union of concentric closed disks, then the image of an open disk under an analytic map is either an open disk or all of $\mathbb{C}_p$. In addition, since any rational function can be expressed locally by power series, we see that morphisms of positive degree on $\mathbb{P}^1(\mathbb{C}_p)$ map a $\mathbb{P}^1(\mathbb{C}_p)$-disk either to a $\mathbb{P}^1(\mathbb{C}_p)$-disk or to all of $\mathbb{P}^1(\mathbb{C}_p)$.

We also note the following proposition, which we will need later. Like Propositions 2.2 and 2.3, its proof is straightforward. The result may be viewed as a refinement of the inequality of Proposition 2.3. In practice, it shows that given any point which is not a critical point or pole of some rational function ϕ, we can find a disk about that point on which ϕ stretches distances uniformly.

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PROPOSITION 2.4. *Let $f \in \mathbb{C}_p[[z-a]]$ be a power series defined over $\mathbb{C}_p$ and convergent on $\overline{D}_r(a)$ for some $r > 0$. Suppose that $f'(a) \neq 0$. Then there exists a positive real number s with $s \leq r$ such that for any $x, y \in \overline{D}_s(a)$,*

$$|f(x) - f(y)| = |f'(a)||x - y|.$$

Having established some of the basic mapping properties of disks, we are now prepared to use them in dynamics. Let ϕ be a rational function in $\overline{\mathbb{Q}}_p(z)$ with Fatou set $\mathcal{F}$ and Julia set $\mathcal{J}$. By Proposition 2.2, if $\mathcal{J}$ is a singleton or is empty, then $\mathcal{F}$ consists of exactly one D-component, which maps onto itself under ϕ. If not, then any D-component of $\mathcal{F}$ is a $\mathbb{P}^1(\mathbb{C}_p)$-disk, and so, by Proposition 2.3, its image under ϕ is again a $\mathbb{P}^1(\mathbb{C}_p)$-disk. Because the Fatou set is preserved under application of ϕ, it follows that the image of a D-component of $\mathcal{F}$ is contained in (but not necessarily equal to) another D-component. Thus, ϕ induces an action Φ on the set of D-components of $\mathcal{F}$, by

$$\Phi(U) = \text{the D-component containing } \phi(U).$$

Sullivan's No Wandering Domains Theorem states that any component of the Fatou set of a complex rational map is pre-periodic. Thus, in light of the above definition of Φ, our analogue of Sullivan's theorem will state that every D-component of $\mathcal{F}$ is pre-periodic under the action of Φ.

Remark. In the complex case, components map onto components; however, this is not always true in the p-adic setting. For example, let p be an odd prime, and let $\phi(z) = p^{-1}(z^2 - 1)$, which has Julia set contained in $D_1(1) \cup D_1(-1)$. Then, denoting the D-component of x by U_x, we have $U_0 = D_1(0)$ and $\phi(0) = -p^{-1}$; thus, $\phi(U_0) = D_p(-p^{-1})$. On the other hand, $U_{-p^{-1}} = \mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)$, which contains but does not equal $D_p(-p^{-1})$.

The notion of disks as components may seem too naive to be useful, especially in light of the above example. However, as we have seen, D-components are a direct analogue of topologically connected components; although they may be disks in the p-adic setting, they can be arbitrary open connected sets in the complex setting. In addition, D-components turn out to have interesting dynamical properties. For instance, if the Julia set contains at least two points, then the Fatou set breaks into infinitely many D-components, but Sullivan's theorem still holds (or at least, as we shall see, it holds for a large class of maps).

It should be noted that D-components are not the only notion of components that may be used in this setting. One possible alternative would be to turn to rigid analysis and define the component of a point to be the union of all connected affinoid domains containing the point and contained in the Fatou set. Such 'analytic' components have the advantage of mapping onto each other. However, they are always at least as large as D-components, and so the statement of Sullivan's theorem would be weaker for such components. For instance, the map ϕ in the above remark has only one Fatou analytic component; on the other hand, it has infinitely many D-components, yet it still obeys Sullivan's theorem. In general, it can be shown that a wandering D-component is in fact a full analytic component (see [**2**]); thus, the No Wandering Domains Conjecture holds for D-components if and only if it holds for

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analytic components. In this paper, we restrict our attention to D-components, and leave the subject of analytic components for a future paper. The interested reader may refer to the author's thesis [**2**] for more information on both analytic and D-components.

3. *Hyperbolic rational functions*

In complex dynamics, it is often useful to restrict one's attention to hyperbolic rational functions. As defined in [**5, 14**], these are maps which are everywhere expanding on the Julia set, with respect to some reasonable metric. The set of all hyperbolic functions is conjectured to be open and dense in the moduli space of all complex rational functions [**13**]; thus, it seems that the study of hyperbolic maps could shed light on general maps. For a more detailed exposition on complex hyperbolic maps, see [**5**] or [**14**].

We would like to study an analogous class of maps in the p-adic setting. However, if we were to carry the complex definition of hyperbolicity to $\mathbb{C}_p$ verbatim ('expanding on the Julia set, with respect to some metric'), we would end up with too small a class. Many of the useful properties of hyperbolic maps are proven using the fact that the Julia set of any complex map is compact, since it is a closed subset of the Riemann sphere. Unfortunately, $\mathbb{P}^1(\mathbb{C}_p)$ is not a compact set, and there exist p-adic rational functions with non-compact Julia sets.

On the other hand, all finite extensions of $\mathbb{Q}_p$ are locally compact; thus, if L is a finite extension of $\mathbb{Q}_p$, then the intersection of the Julia set with $\mathbb{P}^1(L)$ is compact. Appropriately modifying the complex definition given in [**5**], then, we are led to the following.

Definition 3.1. Let $\phi \in K(z)$ be a rational function defined over a finite extension K of $\mathbb{Q}_p$, and let $\mathcal{J}$ and $\mathcal{F}$ be its Julia and Fatou sets, respectively. Assume (by a change of coordinates, if necessary) that $\infty \in \mathcal{F}$. We say that ϕ is *hyperbolic* if, for every finite extension L of K, there is an open set $V \subset \mathbb{C}_p$ containing $\mathcal{J} \cap L$, a continuous function $\sigma : V \to \mathbb{R}$, and positive constants $c_1, c_2 \in \mathbb{R}$ with $\sigma(V) \subset [c_1, c_2]$ such that for any $z \in V$ with $\phi(z) \in V$, we have

$$\sigma(\phi(z))|\phi'(z)| \geq \sigma(z).$$

Note. By Corollary 1.3, we can always find a coordinate z such that ∞_z is Fatou.

The definition may seem odd at first, but the key idea is contained in the inequality; as in the complex setting, σ gives a real-valued measure (on $V \cap L$) with respect to which ϕ is expanding near $\mathcal{J} \cap L$. The only serious modification from the original complex definition is that our function σ depends on the extension L.

Since the iterates of ϕ are not equicontinuous on neighborhoods of Julia points, we would expect most maps to expand near the Julia set and therefore to be hyperbolic. Obvious non-hyperbolic maps would be ones with critical points in the Julia set; for example,

$$\phi(z) = p^{-1}(z^3 - z^2) + 1$$

has a critical point at $z = 0$ mapping to a repelling fixed point at $z = 1$. In the complex case there are also non-hyperbolic maps with no Julia critical points; however, as we shall see in our main theorem, a p-adic map is hyperbolic if and only if there are no critical

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points in its Julia set. We note that this equivalence also demonstrates that the property of hyperbolicity is independent of the choice of coordinate and of the field K of definition.

MAIN THEOREM. *Let $\phi \in K(z)$ for some finite extension K of $\mathbb{Q}_p$, and let $\mathcal{J}$ and $\mathcal{F}$ denote its Julia and Fatou sets. Then the following are equivalent:*

(1) *ϕ is hyperbolic;*
(2) *for any finite extension L of K, and for any coordinate z such that $\infty_z \in \mathcal{F}$, there is a positive integer m such that $|(\phi^m)'| > 1$ on $\mathcal{J} \cap L$;*
(3) *$\mathcal{J}$ contains no critical points.*

Proof. Clearly (1) implies (3). To show that (2) implies (1), we follow the standard proof from the complex theory as follows. (For the complex version, see, for example, [**5**].) Choose any coordinate z with $\infty_z \in \mathcal{F}$ and any finite extension L of K; let m be the positive integer given in (2). By the compactness of $\mathcal{J} \cap L$, $|(\phi^n)'(z)|$ is bounded above on $\mathcal{J} \cap L$ for any $n = 1, \ldots, m$. Since $\infty_z \in \mathcal{F}$, there cannot be any poles in $\mathcal{J}$; in addition, by (2), there are no critical points in $\mathcal{J} \cap L$. Thus, we can choose a neighborhood V_1 of $\mathcal{J} \cap L$ containing no critical points or poles of $\phi, \phi^2, \ldots, \phi^m$ such that $|(\phi^m)'(z)| > 1$ on V_1. We define

$$\sigma(z) = |\phi'(\phi^{m-2}(z))|^{1/m} |\phi'(\phi^{m-3}(z))|^{2/m} \cdots |\phi'(z)|^{(m-1)/m}$$

for $z \in V_1$. Let V be the set of all $z \in V_1$ such that $\phi^n(z) \in V_1$ for all $n = 1, \ldots, m$. It is then easy to verify that, if $z, \phi(z) \in V$, we get

$$\frac{\sigma(\phi(z))}{\sigma(z)} = \frac{|(\phi^m)'(z)|^{1/m}}{|\phi'(z)|} > \frac{1}{|\phi'(z)|}$$

from which hyperbolicity follows.

The final step, to show that (3) implies (2), requires some preparation. Fix a coordinate z and extension L with the specified properties. By change of coordinates (specifically, multiplying by a constant of small absolute value), we can guarantee that the Julia set is contained in $\overline{D}_1(0)$. (Note that a change of coordinates fixing the point at infinity does not affect ϕ'.) Because ϕ' is continuous on the compact set $\mathcal{J} \cap L$ with no zeros or poles, there are positive real numbers M_1 and M_2 such that for any $a \in \mathcal{J} \cap L$, $M_1 \leq |\phi'(a)| \leq M_2$. Without loss, assume that $M_1 < 1$.

By Proposition 2.4, we can cover every point in $\mathcal{J} \cap L$ by a (closed) disk on which ϕ stretches by a constant factor; since $\mathcal{J} \cap L$ is compact, we can take a finite subcover. Let ε be the minimum of the radii of the resulting set of disks. We consider the (finite) cover of $\mathcal{J} \cap L$ by ε-disks.

Using this cover, we can show that, for any fixed $a \in \mathcal{J} \cap L$, the set

$$\{|(\phi^n)'(a)|\}_{n \geq 1}$$

is unbounded. For suppose that there were $\rho > 1$ such that $|(\phi^n)'(a)| \leq \rho$ for all $n \geq 0$. We claim that

$$\phi^n(\overline{D}_{\varepsilon/\rho}(a)) \subseteq \overline{D}_\varepsilon(\phi^n(a))$$

for any $n \geq 0$. We prove this claim by induction on n. It is clearly true for $n = 0$. Now assume we know it to be true for any $0 \leq i \leq n$. Note that ϕ^n stretches by a constant factor

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on $\overline{D}_{\varepsilon/\rho}(a)$, since each successive iteration of ϕ puts the image in a disk of radius at most ε, which is (by definition of ε) a disk on which ϕ stretches by a constant factor. Because we assumed ϕ^n stretches by at most ρ, the image under ϕ^n has radius no larger than ε, and we have proven our claim. However, the claim implies (by Hsia's theorem) that $\{\phi^n\}$ is equicontinuous on $\overline{D}_{\varepsilon/\rho}(a)$, contradicting the fact that a is Julia. Thus, the derivatives at a are in fact unbounded.

We are now prepared to prove that (3) implies (2). Fix any $C > 1$. For any $a \in \mathcal{J} \cap L$, there is an integer $m_a \geq 1$ such that $|(\phi^{m_a})'(a)| \geq C$. By continuity, there is $\delta_a > 0$ such that $|(\phi^{m_a})'(a)| \geq C$ on $\overline{D}_{\delta_a}(a)$. We cover $\mathcal{J} \cap L$ with such δ-neighborhoods and take a finite subcover. On each of these finitely many disks, we have an integer m_a such that $|(\phi^{m_a})'| \geq C$ on the disk; we would like to find a single m which works for all of them.

Let N be the maximum of the $\{m_a\}$. Choose an integer M which is a multiple of N large enough so that $C^{M/N} M_1^N > 1$. (Recall that M_1 was a lower bound for the derivative of ϕ.) We claim that any $m \geq M$ works as the exponent we are looking for (i.e. that $|(\phi^m)'| > 1$ on $\mathcal{J} \cap L$).

To prove this, choose any $m \geq M$ and $z \in \mathcal{J} \cap L$. Then there exists $1 \leq m_1 \leq N$ with $|(\phi^{m_1})'(z)| \geq C$. Because any iterate of z is also in $\mathcal{J} \cap L$, there exist $1 \leq m_2 \leq N$ with $|(\phi^{m_2})'(\phi^{m_1}(z))| \geq C$, and $1 \leq m_3 \leq N$ with $|(\phi^{m_3})'(\phi^{m_1+m_2}(z))| \geq C$, and so on. Eventually, there is a positive integer j with $m_1 + \cdots + m_j > m$, but $m_1 + \cdots + m_{j-1} \leq m$. Note that $j > m/N \geq M/N$, since $m_i \leq N$. Also, $m - m_1 - \cdots - m_{j-1} \leq N$. Since $|\phi'| \geq M_1$ on $\mathcal{J} \cap L$, we see that

$$|(\phi^{m-m_1-\cdots-m_{j-1}})'(\phi^{m_1+\cdots+m_{j-1}}(z))| \geq M_1^{m-m_1-\cdots-m_{j-1}} \geq M_1^N.$$

(The second inequality follows because we chose $M_1 < 1$.) Thus, it suffices to show that $|(\phi^{m_1+\cdots+m_{j-1}})'(z)| > M_1^{-N}$. But

$$\begin{aligned} |(\phi^{m_1+\cdots+m_{j-1}})'(z)| &= |(\phi^{m_1})'(z)||(\phi^{m_2})'(\phi^{m_1}(z))| \cdots |(\phi^{m_{j-1}})'(\phi^{m_1+\cdots+m_{j-2}}(z))| \\ &\geq C^{j-1} \geq C^{M/N} > M_1^{-N} \end{aligned}$$

and the proof is complete. □

Our main theorem is stronger than its complex analogue. In the complex case, a rational map is hyperbolic if and only if the closure of the postcritical set (i.e. the closure of the set of all critical points and all iterates of critical points) is disjoint from the Julia set. However, in the non-archimedean setting, we only need the critical set to be disjoint from the Julia set.

Our analogue of Sullivan's theorem for hyperbolic maps now follows almost immediately from the main theorem. The proof is similar in theme to Norton's proof [**17**] of the same result in the complex case; however, ultrametricity makes a much shorter proof possible. Although this result is superseded by [**3**], we include it to demonstrate both the use of D-components and the strength of the main theorem.

COROLLARY 3.1. (No wandering domains for p-adic hyperbolic maps) *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ be a hyperbolic rational map. Then ϕ has no wandering domains; that is to say, every D-component of the Fatou set of ϕ is pre-periodic under the induced action of ϕ.*

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Remark. As demonstrated in [**3**], the above hypotheses can be weakened to allow recurrent wild critical points in the Julia set. Here, a critical point is said to be *wild* if its index of ramification is divisible by p, and a point x is said to be *recurrent* if it is an accumulation point of the sequence $\{\phi^n(x)\}$.

Proof. Let $\mathcal{J}$ and $\mathcal{F}$ be the Julia and Fatou sets of ϕ. We begin by assuming (by an appropriate change of coordinates) that ∞ is a Fatou fixed point whose D-component contains $\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)$. Now suppose that $U \subset \mathcal{F}$ is a wandering D-component of $\mathcal{F}$. Then U contains some disk $\overline{D}_R(z_0)$ with $z_0 \in L$ and $R \in |L^*|$, for some finite extension L of K. Note that all iterates of this disk are closed disks contained in $\overline{D}_1(0)$. Let m be a positive integer chosen by the previous theorem so that $|(\phi^m)'| > 1$ on $\mathcal{J} \cap L$; we can assume without loss that $m = 1$ by considering the dynamics of ϕ^m instead of ϕ.

For any positive integer n, $\phi^n(\overline{D}_R(z_0))$ is a closed disk centered at $\phi^n(z_0)$; let us denote the radius of this disk by r_n. Note that the sequence $\{r_n\}$ approaches zero as n approaches infinity, since these disks are non-overlapping and contain points of $L \cap \overline{D}_1(0)$. Thus, for infinitely many n we have $r_n < r_{n+1}$; by Proposition 2.3,

$$|\phi'(\phi^n(z_0))| < 1.$$

Consider the (infinite) set of iterates $\phi^n(z_0)$ with the property that $|\phi'(\phi^n(z_0))| < 1$. This set is contained in the compact set $L \cap \overline{D}_1(0)$ and therefore has an accumulation point $a \in L$. Now since ϕ' is continuous, $|\phi'(a)| \leq 1$; by the main theorem, a cannot be in the Julia set. Let V denote the D-component of the Fatou set containing a; V must contain infinitely many forward iterates of z_0. But then U must map into V after finitely many iterations, and V must also map into itself after finitely many iterations. Thus, U is not be wandering, and we have a contradiction. □

Sullivan proved that all complex rational functions, not just hyperbolic ones, have no wandering domains. His proof uses the theory of quasiconformal maps and other techniques which are not currently available in the non-archimedean setting. However, it is possible to prove the No Wandering Domains Theorem for a larger class of p-adic rational functions; see [**2, 3**].

As in the complex setting, it appears that the set of p-adic hyperbolic functions is dense in the set of all rational functions. Because the characterization of hyperbolic maps given by the main theorem is simpler than the corresponding complex characterization, it may be easier to solve the hyperbolic density conjecture for the p-adics than for the complexes. In general, the complex and non-archimedean theories of dynamics are similar but definitely distinct; but in spite of the differences, the properties of the one may shed light on those of the other.

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