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ON A GENERALIZATION OF THE STATIONARY EXCESS OPERATOR

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We show that overshoots over Erlang random variables give rise to a natural generalization of the stationary excess operator and its iterates. The new operators can be used to derive expansions for the expectation $\mathrm{E}[g(X)]$ of a non-negative random variable, similar to Taylor-like expansions encountered when studying stationary excess operators.

1. INTRODUCTION

Let X be a non-negative random variable with distribution function F and finite-positive mean. Then

$$F_0(x) = \frac{1}{\mathrm{E}[X]} \int_0^x (1 - F(u))\, du, \quad x \geq 0. \tag{1}$$

defines the so-called *stationary excess distribution* or *equilibrium distribution*. If we let $F_0^{(1)} = F_0$ and define $F_0^{(n)} = (F_0^{(n-1)})_0^{(1)}$ for $n = 1, 2, \ldots$, then we obtain a sequence of probability distributions, provided the required moments of X are finite. The operator that maps the distribution F to the distribution F_0 is called the *stationary excess operator* [20]. The idea of this paper is to generalize this operator and show that thereby the following Taylor-like expansion, which has been shown in [13], can be generalized and given a probabilistic meaning:

$$\mathrm{E}[g(X)] = \sum_{k=0}^{n-1} \mathrm{E}[X^k] \frac{g^{(k)}(0)}{k!} + \mathrm{E}[g^{(n)}(X_0^{(n)})] \frac{\mathrm{E}[X^n]}{n!}. \tag{2}$$

Here g is a sufficiently smooth function, $X_0^{(n)}$ is a random variable with distribution $F_0^{(n)}$ and all necessary moments are assumed to exist.

In this paper, we generalize this operator as follows. Let $(N_t)_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$ and event times $(T_k)_{k\in\mathbb{N}}$. For the distribution function F of a positive

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random variable X define

$$F_\lambda^{(n)}(x) := \mathrm{P}(X - T_n \le x | X > T_n).$$

We call $F \mapsto F_\lambda^{(n)}$ the *overshoot operator* (note that this is not the "iterated overshoot" as defined in [18]). It is shown in Proposition 1 that $(F_\lambda^{(n)})_\lambda^{(m)} = F_\lambda^{(n+m)}$ and the operator generalizes the stationary excess operator in the sense that $F_\lambda^{(n)}(x) \to F_0^{(n)}(x)$ as $\lambda \to 0$, as will be shown later.

Then the following expansion related to (2) can be proven:

$$\mathrm{E}[g(X)] = \sum_{k=0}^{n-1} \mathrm{P}(N_X = k) g^*(k, 0) + \mathrm{P}(X \ge T_n) \mathrm{E}[g^*(n, X_\lambda^{(n)})], \tag{3}$$

where the function g^* is given by

$$g^*(n, x) = \sum_{k=0}^{n} \binom{n}{k} \lambda^{-k} g^{(k)}(x), \quad n \ge 0, \quad x \ge 0. \tag{4}$$

In contrast to the situation in (2) no finite moments are required for X. Moreover, the expansion has an obvious probabilistic interpretation. This can be seen if one writes (3) shortly as

$$\mathrm{E}[g(X)] = \mathrm{E}\big[g^*(n \wedge N_X, (X - T_n)_+)\big].$$

Let us interpret X as a random time. Then in order to find the expectation $\mathrm{E}[g(X)]$ it is sufficient to know either N_X (or equivalently the interval $[T_k, T_{k+1}]$ into which X felt) if it happened by time T_n, or to know the remaining overshoot $X_\lambda^{(n)}$, if X did not happen by time T_n.

The paper is organized as follows. Before we introduce the overshoot operator in Section 2 we state some known results for the stationary excess operator. Section 2 is devoted to the overshoot operator and its properties. We show the expansion (2), find formulas for the moments and Laplace transform and discuss the behavior of the overshoot operator as $\lambda \to 0$ and as $n \to \infty$. See the end of the paper for an overview over the used notation.

1.1. The Stationary Excess Operator

For the sake of completeness, we give a short overview of existing results for the stationary excess operator. The stationary excess operator defined in (1) maps a distribution function F of a non-negative random variable X with finite-positive mean to the absolutely continuous distribution function F_0 of a non-negative random variable X_0. The relevance of this operator is for the most part due to the fact that F_0 is the limit distribution as time tends to infinity of both the forward and backward recurrence times in a renewal process with inter-renewal distribution function F (see, e.g., [2,4] or [6]). Investigations of the stationary excess operator and related topics can be found in [1,3,5,7–19] and [20].

The distribution function F_0 is absolutely continuous, the respective density $(1 - F(x))/\mathrm{E}[X]$ is bounded and non-increasing.

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If $\mathrm{E}[X^n]$ is finite, we may define $F_0^{(n)}$ iteratively by $F_0^{(1)} = F_0$ and $F_0^{(n)} = (F_0^{(n-1)})_0$. An induction then shows, that $F_0^{(n)}$ can be expressed explicitly by the formula

$$F_0^{(n)}(x) = 1 - \frac{\int_0^\infty (1 - F(x+u))u^{n-1}\,du}{\int_0^\infty (1 - F(u))u^{n-1}\,du}. \tag{5}$$

Let $X_0^{(n)}$ denote a random variable with distribution $F_0^{(n)}$. It has been shown in [9] that the kth moment of $X_0^{(n)}$ is given by

$$\mathrm{E}[(X_0^{(n)})^k] = \frac{k!n!\mathrm{E}[X^{n+k}]}{(k+n)!\mathrm{E}[X^n]}, \tag{6}$$

provided that $\mathrm{E}[X^{n+k}]$ is finite. One may write this suggestively as

$$\mathrm{E}[X^n] = \binom{n}{k} \mathrm{E}[(X_0^{(n)})^k]\,\mathrm{E}[X^{n-k}].$$

Note that the special case where $n = 1$ and $k = 1$ yields $\mathrm{E}[X_0^{(1)}] = \mathrm{E}[X^2]/2\mathrm{E}[X]$, which is well known and often used in applied probability. The Laplace–Stieltjes transform $\varphi_n(s) = \mathrm{E}[e^{-sX_0^{(n)}}]$ of $X_0^{(n)}$ is given by

$$\varphi_n(s) = \frac{n!}{(-s)^n \mathrm{E}[X^n]}\Big(\varphi(s) - \sum_{k=0}^{n-1} \frac{(-s)^k}{k!}\mathrm{E}[X^k]\Big), \tag{7}$$

that is, $\varphi_n(s)$ is, up to the factor, equal to the remainder term in the Taylor expansion of $\varphi(s)$ around $s = 0$.

2. THE OVERSHOOT OPERATOR

2.1. Definition of the Operator

Let as before X be a non-negative random variable for which we do not assume any integrability conditions, for example, the mean may exist or not. Let $T_1, T_2, \ldots$ denote the event times of a Poisson process $(N_t)_{t\geq 0}$ with rate λ, independent of X.

We define the overshoot operator $F_\lambda^{(n)}$ by

$$F_\lambda^{(n)}(x) = \mathrm{P}(X - T_n \leq x | X > T_n) \tag{8}$$

and denote $X_\lambda^{(n)}$ a random variable with distribution $F_\lambda^{(n)}$. We may write informally $X_\lambda^{(n)} = (X - T_n)|(X > T_n)$. The following explicit formula is immediate from (8):

$$F_\lambda^{(n)}(x) = 1 - \frac{\int_0^\infty (1 - F(u+x))u^{n-1}e^{-\lambda u}\,du}{\int_0^\infty (1 - F(u))u^{n-1}e^{-\lambda u}\,du}. \tag{9}$$

Provided that $F_0^{(n)}(x)$ is well defined, that is, $\mathrm{E}[X^n] < \infty$, we have the pointwise convergence

$$\lim_{\lambda\to 0} F_\lambda^{(n)}(x) = F_0^{(n)}(x), \quad \forall x \geq 0,$$

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which justifies to speak of a generalization of the stationary excess operator. Less obvious from (9), but clear by the definition (8) is the limit as $\lambda \to \infty$,

$$\lim_{\lambda\to\infty} F_\lambda^{(n)}(x) = F(x), \quad \forall x \geq 0.$$

The notation $F_\lambda^{(n)}$ suggests that $F_\lambda^{(n)}$ can be interpreted as the nth power of F_λ, which is indeed true as the next result shows.

PROPOSITION 1. *We have* $(F_\lambda^{(n)})_\lambda^{(m)} = F_\lambda^{(n+m)}$ *for* $n, m \in \mathbb{N}$ *and* $\lambda > 0$.

PROOF: Let T_n and T'_m be independent Erlang-n and Erlang-m distributed random variables and let $Y = X - T_n$. Then $F_\lambda^{(n+m)}(x) = \mathrm{P}(Y - T'_m \leq x | Y > T'_m) = G_\lambda^{(m)}(x)$, where G is the conditional distribution of Y given $Y > 0$, which is $F_\lambda^{(n)}(x)$. ∎

If we define the *nth exponential stop loss of* X as the random variable

$$X_+^{(n)} = (X - T_n)_+ = \begin{cases} X - T_n & ; \ X \geq T_n \\ 0 & ; \ X < T_n, \end{cases}$$

then its distribution function is given by

$$F_+^{(n)}(x) = \mathrm{P}((X - T_n)_+ \leq x) = \frac{\lambda^n}{(n-1)!} \int_0^\infty F(u+x) u^{n-1} e^{-\lambda u}\, du, \quad x \geq 0 \qquad \textbf{(10)}$$

and it is not difficult to show that $(X_+^{(n)})_+^{(m)} \overset{d}{\sim} X_+^{(n+m)}$, where $\overset{d}{\sim}$ denotes equality in distribution, and hence we have $(F_+^{(n)})_+^{(m)}(x) = F_+^{(n+m)}(x)$. With these notations we can write

$$1 - F_\lambda^{(n)}(x) = \frac{1 - F_+^{(n)}(x)}{1 - F_+^{(n)}(0)}. \qquad \textbf{(11)}$$

Using the abbreviation

$$\pi_n = \frac{(n-1)!}{\lambda^n \int_0^\infty F(u) u^{n-1} e^{-\lambda u}\, du} = \frac{1}{1 - F_+^{(n)}(0)} = \frac{1}{\mathrm{P}(X \geq T_n)} \qquad \textbf{(12)}$$

relation (11) can be written as

$$1 - F_\lambda^{(n)}(x) = \pi_n \big(1 - F_+^{(n)}(x)\big). \qquad \textbf{(13)}$$

The distribution function $F_\lambda^{(n)}(x)$ is absolutely continuous and its density function is given by

$$\begin{aligned} f_\lambda^{(n)}(x) &= \pi_n \frac{\lambda^n}{(n-1)!} \int_0^\infty f(u+x) u^{n-1} e^{-\lambda u}\, du \\ &= \pi_n \frac{\lambda^n}{(n-1)!} \int_x^\infty f(u)(u-x)^{n-1} e^{-\lambda(u-x)}\, du \\ &= \pi_n \frac{\lambda^n}{(n-1)!} \mathrm{E}[(X-x)_+^{n-1} e^{-\lambda(X-x)_+}], \end{aligned} \qquad \textbf{(14)}$$

with the understanding that $(X-x)_+^{n-1} = \mathbb{1}_{\{X>x\}}$ if $n = 1$.

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2.2. The Expansion

In [13] it is shown that the stationary excess operator appears in the remainder term of a Taylor's series type expansion for the expectation $\mathrm{E}[g(X)]$, where g is some sufficiently smooth function (c.f. [12], Lemma 2.3., [5] and [8], Section 3). More precisely, for an n-times differentiable function $g : [0,\infty) \to \mathbb{R}$, with nth derivative $g^{(n)}$, $\mathrm{E}|g^{(n)}(X_0^{(n)})| < \infty$ and $\mathrm{E}[X^n] < \infty$ it is shown that $\mathrm{E}[|g(X)|] < \infty$ and that

$$\mathrm{E}[g(X)] = \sum_{k=0}^{n-1} \mathrm{E}[X^k] \frac{g^{(k)}(0)}{k!} + \mathrm{E}[g^{(n)}(X_0^{(n)})] \frac{\mathrm{E}[X^n]}{n!}. \tag{15}$$

If instead of the stationary excess operator the overshoot operator is used, then one obtains a similar expansion with some interesting properties.

THEOREM 2. *Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ is n-times continuously differentiable and $\mathrm{E}|g^{(j)}(X_\lambda^{(n)})| < \infty$ for $j = 0, 1, \ldots, n$. Then $\mathrm{E}|g(X)| < \infty$ and*

$$\mathrm{E}[g(X)] = \mathrm{E}[g^*(n \wedge N_X, X_+^{(n)})] \tag{16}$$

with $g^* : [0,\infty) \to \mathbb{R}$ defined as

$$g^*(n,x) = \sum_{k=0}^{n} \binom{n}{k} \lambda^{-k} g^{(k)}(x), \quad n \geq 0, \quad x \geq 0.$$

Remark 1. In terms of the overshoot operator, the expansion (16) is given by

$$\mathrm{E}[g(X)] = \sum_{k=0}^{n-1} \mathrm{P}(N_X = k) g^*(k,0) + \mathrm{P}(X \geq T_n) \mathrm{E}[g^*(n, X_\lambda^{(n)})], \tag{17}$$

which looks much closer to (15). In fact, the expansion (15) is the $\lambda \to 0$ limit of (17).

PROOF: The Taylor expansion of the function $e^{\lambda x} g(x)$ with remainder term around the origin is given by

$$\begin{aligned} e^{\lambda x} g(x) &= \sum_{k=0}^{n-1} \frac{x^k}{k!} \frac{d^k}{dx^k} \left(e^{\lambda x} g(x)\right)\Big|_{x=0} + \int_0^x \frac{(x-u)^{n-1}}{(n-1)!} \frac{d^n}{dx^n} \left(e^{\lambda x} g(x)\right)\Big|_{x=u} du \\ &= \sum_{k=0}^{n-1} \frac{x^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j} g^{(j)}(0) \\ &\quad + \frac{1}{(n-1)!} \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} \int_0^x g^{(j)}(u) e^{\lambda u} (x-u)^{n-1} du. \end{aligned}$$

It follows that for $n \geq 1$ (recall that we defined $(x-u)^{n-1} = \mathbb{1}_{\{x \geq u\}}$ for $n = 1$)

$$\begin{aligned} g(x) &= \sum_{k=0}^{n-1} e^{-\lambda x} \frac{(\lambda x)^k}{k!} g^*(k,0) \\ &\quad + \frac{\lambda^n}{(n-1)!} \int_0^\infty \Big(\sum_{j=0}^{n} \binom{n}{j} \lambda^{-j} g^{(j)}(u) \Big) e^{-\lambda (x-u)_+} (x-u)_+^{n-1} du. \end{aligned}$$

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By taking expectations, applying (14) and using the fact that $X_\lambda^{(n)}$ has the same distribution as $X_+^{(n)}|(X_+^{(n)} > 0)$, we obtain

$$\begin{aligned}\mathrm{E}[g(X)] &= \sum_{k=0}^{n-1} \mathrm{P}(N_X = k) g^*(k,0) + \frac{1}{\pi_n} \mathrm{E}[g^*(n, X_\lambda^{(n)})] \\ &= \mathrm{E}[g^*(N_X,0)\mathbb{1}_{\{X<T_n\}}] + \mathrm{E}[g^*(n, X_+^{(n)})\mathbb{1}_{\{X\geq T_n\}}]\end{aligned}$$

from which both (16) and (17) follow. ∎

One might wonder if the expansion in Theorem 2 stays true as n tends to infinity. Obviously a sufficient criterion is that the remainder term $\mathrm{E}[g^*(n, X_+^{(n)})\mathbb{1}_{\{X\geq T_n\}}]$ vanishes as $n \to \infty$.

Corollary 3. *Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ is analytic at $x = 0$ and that $\mathrm{E}|g^{(j)}(X_\lambda^{(n)})| < \infty$ for $j = 0, 1, \ldots$. If $\mathrm{E}[g^*(n, X_+^{(n)})\mathbb{1}_{\{X\geq T_n\}}] \to 0$ as $n \to \infty$ then $\mathrm{E}|g(X)| < \infty$ and*

$$\mathrm{E}[g(X)] = \mathrm{E}[g^*(N_X, 0)]. \tag{18}$$

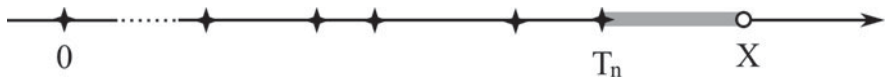
2.3. An Application

Suppose that some event happens at time X and we want to find $\mathrm{E}[g(X)]$, but an exact measurement from time zero is not possible or at least not reliable. Instead one can access a series of observations at Poisson event times $T_1, T_2, \ldots, T_n$ for some fixed n. These observation points report the following information.

(a) If the event happened before time T_n then the observation points report the value of N_X (the value is k in the figure below).



(b) Otherwise, if X happens after time T_n, the overshoot $X_\lambda^{(n)}$ is reported.



Then we estimate $g(X)$ by (16)

$$\widehat{g}(X) = \begin{cases} g^*(N_X, 0) & ; \ N_X < n \\ g^*(n, X_\lambda^{(n)}) & ; \ N_X \geq n \end{cases}.$$

It follows from Theorem 2 that $\widehat{g}(X)$ is an unbiased estimator.

In the special case where $g(x) = x$ we have $g^*(n, x) = x + \frac{n}{\lambda}$ and hence obtain the estimator

$$\widehat{X} = X_\lambda^{(n)} + \frac{n \wedge N_X}{\lambda} = \begin{cases} N_X/\lambda & ; \ N_X < n, \\ X_\lambda^{(n)} + n/\lambda & ; \ N_X \geq n. \end{cases}$$

The following corollary describes the involved variance.

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COROLLARY 4. *In the special case where* $g(x) = x$, *we have*

$$\mathrm{Var}(\widehat{X}) = \mathrm{Var}(X) + \frac{1}{\lambda^2}\mathrm{E}[n \wedge N_X]. \tag{19}$$

The following bound holds:

$$0 \le \mathrm{Var}(\widehat{X}) - \mathrm{Var}(X) \le \frac{n \wedge \mathrm{E}[X]}{\lambda^2}. \tag{20}$$

PROOF: If we let $g(x) = x^2$ then

$$g^*(n, x) = x^2 + \frac{2n}{\lambda}x + \frac{n(n-1)}{\lambda^2} = \left(x + \frac{n}{\lambda}\right)^2 - \frac{n}{\lambda^2}.$$

It follows from (16) and that

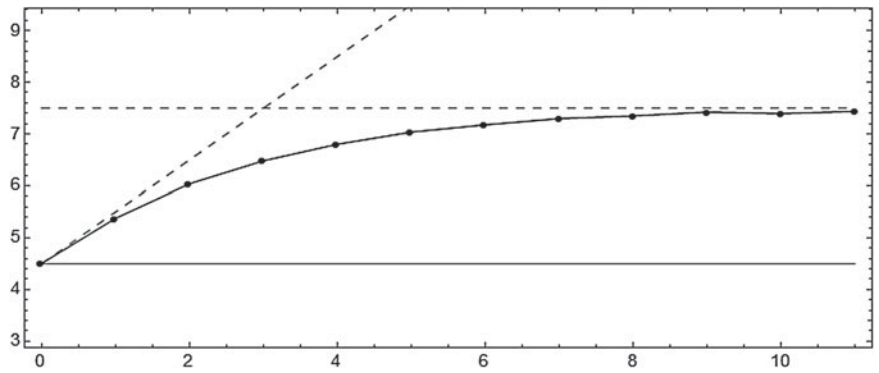
$$\mathrm{E}[X^2] = \mathrm{E}[g^*(n \wedge N_X, X_+^{(n)})] = \widehat{X}^2 - \frac{n \wedge N_X}{\lambda^2}. \tag{21}$$

This and the fact that $\mathrm{E}[X] = \mathrm{E}[\widehat{X}]$ proves (19). The bound is a consequence of the inequality $\mathrm{E}[n \wedge N_X] \le \lambda \mathrm{E}[X]$ which follows from

$$\mathrm{E}[X] = \mathrm{E}[X_\lambda^{(n)}] + \frac{1}{\lambda}\mathrm{E}[n \wedge N_X].$$

■

Example 1. The following figure shows a simulation of the variance of $\widehat{X}$ (one million samples) with n on the abcissa. The random variable X has an inverse Gaussian distribution with $\mathrm{E}[X] = 3$ and $\mathrm{Var}(X) = 9/2$. The dashed lines represent the bounds $\mathrm{Var}(X) + n/\lambda^2$ and $\mathrm{Var}(X) + \mathrm{E}[X]/\lambda^2$, the solid line equals the variance $\mathrm{Var}(X)$.



2.4. Further Properties of the Overshoot Operator

With the aid of Theorem 2 we can prove that the Laplace–Stieltjes transform $\varphi_{\lambda,n}(s)$ of $X_\lambda^{(n)}$ is related to the remainder term in the Taylor series expansion of φ around λ. Compare this with the corresponding result (7) for the stationary excess operator.

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THEOREM 5. *The Laplace–Stieltjes transform of* $X_\lambda^{(n)}$ *is for* $s \geq 0$ *given by*

$$\varphi_{\lambda,n}(s) = \begin{cases} \pi_n \Big(\dfrac{\lambda}{\lambda - s}\Big)^n \Big(\varphi(s) - \displaystyle\sum_{k=0}^{n-1} \frac{(s-\lambda)^k}{k!} \varphi^{(k)}(\lambda)\Big), & ; \ \lambda \neq s, \\ \pi_n \dfrac{(-\lambda)^n}{n!} \varphi^{(n)}(s). & ; \ \lambda = s, \end{cases} \tag{22}$$

PROOF: Letting $g(x) = e^{-sX}$, $s \geq 0$ in (4) we obtain

$$g^*(n,x) = \sum_{k=0}^{n} \binom{n}{k} (-s/\lambda)^k e^{-sx} = \Big(\frac{\lambda - s}{\lambda}\Big)^n e^{-sx}.$$

Consequently, by Theorem 2,

$$\mathrm{E}[e^{-sX}] = \sum_{k=0}^{n-1} \mathrm{P}(N_X = k) \Big(\frac{\lambda - s}{\lambda}\Big)^k + \mathrm{P}(X \geq T_n) \Big(\frac{\lambda - s}{\lambda}\Big)^n \mathrm{E}[e^{-s(X - T_n)} | X \geq T_n].$$

Since $\mathrm{P}(N_X = k) = \frac{(-\lambda)^k}{k!} \varphi^{(k)}(\lambda)$ we obtain

$$\varphi(s) = \sum_{k=0}^{n-1} \frac{(s-\lambda)^k}{k!} \varphi^{(k)}(\lambda) + \mathrm{P}(X \geq T_n) \Big(\frac{\lambda - s}{\lambda}\Big)^n \varphi_{\lambda,n}(s),$$

so that

$$\varphi_{\lambda,n}(s) = \frac{1}{\mathrm{P}(X \geq T_n)} \Big(\frac{\lambda}{\lambda - s}\Big)^n \Big(\varphi(s) - \sum_{k=0}^{n-1} \frac{(s-\lambda)^k}{k!} \varphi^{(k)}(\lambda)\Big).$$

This proves the first equality in (22). The second follows from the limit

$$\lim_{\lambda \to s} \frac{n!}{(s-\lambda)^n} \Big(\varphi(s) - \sum_{k=0}^{n-1} \frac{(s-\lambda)^k}{k!} \varphi^{(k)}(\lambda)\Big) = \varphi^{(n)}(s).$$

■

Given a measurable function $g : [0, \infty) \to \mathbb{R}$, for which $\mathrm{E}|g(X_\lambda^{(n)})|$ is finite, it follows from (14) that

$$\mathrm{E}[g(X_\lambda^{(n)})] = \pi_n \frac{\lambda^n}{(n-1)!} \int_0^\infty g(x) \mathrm{E}[e^{-\lambda(X-x)_+} (X - x)_+^{n-1}]\, dx. \tag{23}$$

Since

$$\int_0^\infty g(x) \mathrm{E}[e^{-\lambda(X-x)_+} (X - x)_+^{n-1}]\, dx = (-1)^{n-1} \frac{d^{n-1}}{d\lambda^{n-1}} \int_0^\infty g(x) \mathrm{E}[e^{-\lambda(X-x)_+}]\, dx$$

we obtain the formula

$$\begin{aligned} \mathrm{E}[g(X_\lambda^{(n)})] &= -\pi_n \frac{(-\lambda)^n}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} \int_0^\infty g(x) \mathrm{E}[e^{-\lambda(X-x)_+}]\, dx \\ &= -\pi_n \frac{(-\lambda)^n}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} \left(\frac{1}{\lambda \pi_1} \mathrm{E}[g(X_\lambda^{(1)})]\right). \end{aligned} \tag{24}$$

We can use this to prove the following theorem, which demonstrates how one can express the moments of $F_\lambda^{(n)}$ in terms of the derivatives of the Laplace–Stieltjes transform φ_m of $F_0^{(m)}$.

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THEOREM 6. *Suppose that* $\mathrm{E}[X^{m+1}] < \infty$. *Then*

$$\mathrm{E}[(X_\lambda^{(n)})^m] = -\pi_n \frac{(-\lambda)^n}{(n-1)!} \frac{\mathrm{E}[X^{m+1}]}{(m+1)} \varphi_{m+1}^{(n-1)}(\lambda), \quad n \in \mathbb{N}. \tag{25}$$

PROOF: We first show (25) for $n = 1$. Using (23) we obtain

$$\begin{aligned}
\frac{1}{\lambda\pi_1}\mathrm{E}[(X_\lambda^{(1)})^m] &= \int_0^\infty x^m \mathrm{E}[e^{-\lambda(X-x)}\mathbb{1}_{\{X>x\}}]\,dx = \lambda \mathrm{E}[e^{-\lambda X}\int_0^X x^m e^{\lambda x}\,dx] \\
&= \mathrm{E}\Big[\sum_{k=0}^m (-1)^{m-k}\frac{m!X^k}{k!\lambda^{m-k+1}} + (-1)^{m+1}\frac{m!}{\lambda^{m+1}}e^{-\lambda X}\Big] \\
&= \frac{m!(-1)^m}{\lambda^{m+1}}\Big(-\sum_{k=0}^m \frac{(-1)^k\lambda^k \mathrm{E}[X^k]}{k!} + \varphi(\lambda)\Big) \\
&= \frac{\mathrm{E}[X^{m+1}]}{m+1}\frac{(m+1)!}{(-\lambda)^{m+1}\mathrm{E}[X^{m+1}]}\Big(\varphi(\lambda) - \sum_{k=0}^m \frac{(-\lambda)^k}{k!}\mathrm{E}[X^k]\Big) \\
&= \frac{\mathrm{E}[X^{m+1}]}{m+1}\varphi_{m+1}(\lambda).
\end{aligned}$$

Then (25) follows from (24):

$$\begin{aligned}
\mathrm{E}[(X_\lambda^{(n)})^m] &= -\pi_n \frac{(-\lambda)^n}{(n-1)!}\frac{d^{n-1}}{d\lambda^{n-1}}\left(\frac{1}{\lambda\pi_1}\mathrm{E}[(X_\lambda^{(1)})^m]\right) \\
&= -\pi_n \frac{(-\lambda)^n}{(n-1)!}\frac{d^{n-1}}{d\lambda^{n-1}}\left(\frac{\mathrm{E}[X^{m+1}]}{m+1}\varphi_{m+1}(\lambda)\right).
\end{aligned}$$

■

Note that if $\mathrm{E}[X^k] < \infty$, for all $k \geq 0$ then

$$\varphi(\lambda) - \sum_{k=0}^m \frac{(-\lambda)^k}{k!}\mathrm{E}[X^k] = \sum_{k=m+1}^\infty \frac{(-\lambda)^k}{k!}\mathrm{E}[X^k]$$

and after some calculations we obtain the series

$$\mathrm{E}[(X_\lambda^{(n)})^m] = \pi_n \frac{\lambda^n m!}{(n-1)!}\sum_{k=0}^\infty \frac{(k+n-1)!(-\lambda)^k}{(k+n+m)!k!}\mathrm{E}[X^{k+n+m}].$$

Particular cases of (25) are

$$\begin{aligned}
\mathrm{E}[(X_\lambda^{(1)})^m] &= \frac{m!}{(-\lambda)^m}\frac{\varphi(\lambda) - \sum_{k=0}^m \frac{(-\lambda)^k}{k!}\mathrm{E}[X^k]}{\varphi(\lambda) - 1}, \\
\mathrm{E}[X_\lambda^{(1)}] &= \frac{\mathrm{E}[X]}{1-\varphi(\lambda)} - \frac{1}{\lambda}.
\end{aligned}$$

We have already seen that $F_\lambda^{(n)}(x)$ converges to $F_0^{(n)}(x)$ as $\lambda \to 0$. In what follows, we are going to investigate the convergence speed.

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THEOREM 7. *If* $\mathrm{E}[X^{n+1}] < \infty$ *then as* $\lambda \to 0$

$$F_\lambda^{(n)}(x) = F_0^{(n)}(x) + \lambda \frac{n\mathrm{E}[X^{n+1}]}{(n+1)\mathrm{E}[X^n]} \Big(F_0^{(n)}(x) - F_0^{(n+1)}(x) \Big) + O(\lambda^2). \tag{26}$$

PROOF: Let

$$\beta_{\lambda,n}(x) = \int_0^\infty (1 - F(x+w)) w^{n-1} e^{-\lambda w}\, dw.$$

and define an absolutely continuous probability distribution function H_x by its density function

$$h_x(u) = \frac{1}{\beta_{0,n}(x)} (1 - F(x+u)) u^{n-1}.$$

Then H_x has the Laplace–Stieltjes transform

$$\int_0^\infty e^{-\lambda u} dH_x(u) = \frac{\beta_{\lambda,n}(x)}{\beta_{0,n}(x)}$$

and jth moment $\nu_j = \beta_{0,n+j}(x)/\beta_{0,n}(x)$. Since $\mathrm{E}[X^{n+m}] < \infty$ it follows that the mth moment of H_x is finite and the Laplace–Stieltjes transform can be expanded into a power series around $\lambda = 0$, namely

$$\beta_{\lambda,n}(x) = \sum_{j=0}^m \frac{\beta_{0,n+j}(x)}{j!} (-\lambda)^j + o(\lambda^m). \tag{27}$$

It follows from (11) that

$$F_\lambda^{(n)}(x) = 1 - \frac{\beta_{\lambda,n}(x)}{\beta_{\lambda,n}(0)} = \frac{\sum_{j=0}^m \frac{\beta_{0,n+j}(0) - \beta_{0,n+j}(x)}{j!} (-\lambda)^j + o(\lambda^m)}{\sum_{j=0}^m \frac{\beta_{0,n+j}(0)}{j!} (-\lambda)^j + o(\lambda^m)}.$$

Suppose that $F_\lambda^{(n)}(x) = \sum_{k=0}^\infty a_k \lambda^k$ for an unknown sequence $(a_k)_{k=0,1,\ldots}$ of coefficients. Then

$$\Big(\sum_{k=0}^\infty a_k \lambda^k \Big) \Big(\sum_{j=0}^m \frac{\beta_{0,n+j}(0)}{j!} (-\lambda)^j + o(\lambda^m) \Big)$$
$$= \sum_{j=0}^m \frac{\beta_{0,n+j}(0) - \beta_{0,n+j}(x)}{j!} (-\lambda)^j + o(\lambda^m).$$

From which we obtain by comparison of the coefficients

$$a_0 \frac{\beta_{0,n}(0)}{n} = \frac{\beta_{0,n}(0) - \beta_{0,n}(x)}{n}$$

and hence $a_0 = 1 - \frac{\beta_{0,n}(x)}{\beta_{0,n}(0)} = F_0^{(n)}(x)$. Similarly

$$-a_0 \beta_{0,n+1}(0) + a_1 \beta_{0,n}(0) = \beta_{0,n+1}(0) - \beta_{0,n+1}(x)$$

and it follows that

$$a_1 = \frac{\beta_{0,n+1}(0)}{\beta_{0,n}(0)} (F_0^{(n)}(x) - F_0^{(n+1)}(x))$$

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and since $\beta_{0,n}(0) = \int_0^\infty (1 - F(w))w^{n-1}\,dw = \frac{1}{n+1}\mathrm{E}[X^{n+1}]$,

$$a_1 = \frac{n\mathrm{E}[X^{n+1}]}{(n+1)\mathrm{E}[X^n]}\Big(F_0^{(n)}(x) - F_0^{(n+1)}(x)\Big),$$

leading to (26). ∎

Next we are going to investigate the behavior of $F_\lambda^{(n)}$ as $n \to \infty$. It turns out that using a simple transformation we can transfer known convergence results from [9] (see also [18]) to our situation. To this end we define for a distribution function F the *min-transform* of F to be the distribution function

$$F_{\{\lambda\}}(x) = 1 - e^{-\lambda x}(1 - F(x)) \tag{28}$$

and let $X_{\{\lambda\}}$ denote a random variable with distribution function $F_{\{\lambda\}}$. Note that $F_{\{\lambda\}}$ is the distribution function of the minimum of X and an independent exponential random variable with mean $1/\lambda$. The following Lemma will be useful in the sequel. It shows that the above transform is to a certain extend compatible with the overshoot operation.

LEMMA 8. *For $n \in \mathbb{N}, \lambda \geq 0, \mu \geq 0$ we have $(F_{\lambda+\mu}^{(n)})_{\{\lambda\}} = (F_{\{\lambda\}})_\mu^{(n)}$.*

PROOF: We use induction. The assertion is clear for $n = 0$. Then, using Proposition 1 twice, plus the induction hypothesis,

$$(F_{\lambda+\mu}^{(n)})_{\{\lambda\}} = ((F_{\lambda+\mu}^{(1)})_{\lambda+\mu}^{(n-1)})_{\{\lambda\}} = ((F_{\lambda+\mu}^{(1)})_{\{\lambda\}})_\mu^{(n-1)} = ((F_{\{\lambda\}})_\mu^{(1)})_\mu^{(n-1)} = (F_{\{\lambda\}})_\mu^{(n)},$$

which proves the assertion. ∎

By letting $\mu \to 0$ it follows in particular that

$$(F_\lambda^{(n)})_{\{\lambda\}} = (F_{\{\lambda\}})_0^{(n)}, \tag{29}$$

that is, the min-transform of the overshoot distribution of F is given by the equilibrium distribution of the min-transform of F.

We will also need the moments of $X_{\{\lambda\}}$, which are given in the following lemma.

LEMMA 9. $\mathrm{E}[(X_{\{\lambda\}})^k] = \frac{k!}{\lambda^k \pi_k}$.

PROOF: By the definition of π_k in (12)

$$\begin{aligned}\frac{1}{\pi_k} &= \frac{\lambda^k}{(k-1)!}\int_0^\infty u^{k-1}e^{-\lambda u}(1 - F(u))\,du \\ &= \frac{\lambda^k}{k!}\int_0^\infty k u^{k-1}e^{-\lambda u}(1 - F(u))\,du = \frac{\lambda^k}{k!}\mathrm{E}[X_{\{\lambda\}}^k].\end{aligned}$$

∎

We want to investigate the convergence of $X_\lambda^{(n)}/c_n$ for suitable sequences c_n to a random variable X^* with distribution function F^*. In [9] possible limits of $X_0^{(n)}/c_n$, with c_n some appropriate scaling sequence, have been studied.

THEOREM 10. *Let $\stackrel{d}{\Rightarrow}$ denote convergence in distribution and let $\varrho(F_{\{\lambda\}}) = \inf\{x \geq 0 : F(x) = 1\}$ denote the right extremity of F.*

1. *If $\varrho(F) < \infty$ then $nX_\lambda^{(n)} \stackrel{d}{\Rightarrow} X^*$ where X^* is exponential with mean $\varrho(F)$.*

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2. *If* $\varrho(F)=\infty$ *and* $\pi_{n+2}\pi_n/(\pi_{n+1})^2\to 1$ *then the limit* $\kappa=\lim_{n\to\infty}\pi_n/\pi_{n+1}$ *exists and is in* $[0,1]$.
   (a) *If* $\kappa=0$ *then*
   $$\frac{\pi_{n+1}}{\pi_n}X_\lambda^{(n)}\stackrel{d}{\Rightarrow}X^*$$
   where X^* *is exponential with mean* $1/\lambda$.
   (b) *If* $\kappa\in(0,1)$ *then*
   $$X_\lambda^{(n)}\stackrel{d}{\Rightarrow}X^*,$$
   where X^* *is exponential with mean* $\kappa/(\lambda(1-\kappa))$.
   (c) *If* $\kappa=1$ *then* $X_\lambda^{(n)}\to\infty$ *in probability.*

PROOF: First note that for any sequence $(c_n)_{n\in\mathbb{N}}$

$$\begin{aligned}
&X_\lambda^{(n)}/c_n\stackrel{d}{\Rightarrow}X^*\\
\Leftrightarrow\quad &F_\lambda^{(n)}(c_nx)-F^*(x)\to 0\\
\Leftrightarrow\quad &(F_\lambda^{(n)})_{\{\lambda\}}(c_nx)-F^*_{\{\lambda c_n\}}(x)\to 0\\
\Leftrightarrow\quad &(F_{\{\lambda\}})_0^{(n)}(c_nx)-F^*_{\{\lambda c_n\}}(x)\to 0,
\end{aligned}\tag{30}$$

for all continuity points x of $F^*(x)$.

1. If $\varrho(F)<\infty$ and we let $c_n=1/n$ then according to Theorem 3.1 in [9] $(F_{\{\lambda\}})_0^{(n)}(c_nx)$ tends to an exponential distribution function with mean $\varrho(F)$ and it follows from (30) that
   $$F^*_{\{\lambda c_n\}}(x)=1-e^{-\lambda c_nx}(1-F^*(x))\to 1-e^{-x/\varrho(F)},$$
   implying that $F^*(x)=1-e^{-x/\varrho(F)}$.
2. For the $\varrho(F)=\infty$ case let $c_n=\frac{\pi_n}{\pi_{n+1}}$ and $\delta_n=\frac{n+2}{n+1}\cdot\frac{\pi_{n+2}\pi_n}{(\pi_{n+1})^2}$. Then it follows from Lemma 9 that
   $$c_n=\lambda\frac{\mathrm{E}[(X_{\{\lambda\}})^{n+1}]}{(n+1)\mathrm{E}[(X_{\{\lambda\}})^n]}\quad\text{and}\quad\delta_n=\frac{\mathrm{E}[(X_{\{\lambda\}})^{n+2}]\mathrm{E}[(X_{\{\lambda\}})^n]}{\mathrm{E}[(X_{\{\lambda\}})^{n+1}]^2}.$$
   Obviously $\delta_n\to 1$ and it follows from Theorem 6.1 in [9] that $(F_{\{\lambda\}})_0^{(n)}(c_nx)$ tends to an exponential distribution function with parameter λ. By (30) it follows that
   $$F^*_{\{\lambda c_n\}}(x)=1-e^{-\lambda c_nx}(1-F^*(x))\to 1-e^{-\lambda x}$$
   as well, showing that c_n converges to some $\kappa\geq 0$ and that $F^*(x)=1-e^{-\lambda(1-\kappa)x}$. That $\kappa\leq 1$ follows from the representation
   $$\frac{\pi_n}{\pi_{n+1}}=\mathrm{P}(X>T_{n+1}|X>T_n),\tag{31}$$
   which is a consequence of (12). We have shown that $X_\lambda^{(n)}/c_n\stackrel{d}{\Rightarrow}X^*$ where X^* has distribution $F^*(x)=1-e^{-\lambda(1-\kappa)x}$. Letting $\kappa=0$ (a) follows. For $\kappa\in(0,1]$ note that $c_n\to\kappa$ and hence $X_\lambda^{(n)}\stackrel{d}{\Rightarrow}\kappa X^*$, yielding the assertions in (b) and (c).

■

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NOTATION

We use the following abbreviations and notations:

N_t, T_k	A Poisson counting process, its kth event time
X, F, φ	A non-negative random variable, the distribution function of X, the Laplace transform of F
π_n	$1/\mathrm{P}(X \geq T_n)$
$F_+^{(n)}, X_+^{(n)}$	The distribution of $(X - T_n)_+$, a random variable with distribution $F_+^{(n)}$
F_0, X_0	The stationary excess distribution, a random variable with distribution F_0
$F_0^{(n)}, X_0^{(n)}, \varphi_n$	The nth stationary excess distribution, a random variable with distribution $F_0^{(n)}$, its Laplace transform
F_λ, X_λ	The overshoot distribution, a random variable with distribution F_λ
$F_\lambda^{(n)}, X_\lambda^{(n)}, \varphi_{\lambda,n}$	The nth overshoot distribution, a random variable with distribution $F_\lambda^{(n)}$, its Laplace transform
$F_{\{\lambda\}}, X_{\{\lambda\}}$	The *min-transform* of F (for the definition see (28)), a random variable with distribution $F_{\{\lambda\}}$

References

1. Abate, J. & Whitt, W. (1996). An operational calculus for probability distributions via Laplace transforms. *Advances in Applied Probability* 28(1): 75–113.
2. Asmussen, S. (2003). *Applied Probability and Queues*, Applications of Mathematics **51**. 2nd edn., Springer.
3. Brown, M. (2006). Exploiting the waiting time paradox. Applications of the size-biasing transformation. *Probability in the Engineering and Informational Sciences* 20(2): 195–230.
4. Cox, D.R. (1962). *Renewal Theory*, John Wiley & Sons.
5. Denuit, M. (2002). *S*-convex extrema, Taylor-type expansions and stochastic approximations. *Scandinavian Actuarial Journal* 2002(1): 45–67.
6. Doob, J.L. (1948). Renewal theory from the point of view of the theory of probability. *Transactions of the American Mathematical Society* 63: 422–438.
7. Ferrari, P.A., Kesten, H., Martinez, S. & Picco, P. (1995). Existence of quasi-stationary distributions: a renewal dynamical approach. *Annals of Probability* 23(2): 501–521.
8. Gupta, R.C. (2007). Role of equilibrium distribution in reliability studies. *Probability in the Engineering Informational Sciences* 21(2): 315–334.
9. Harkness, W.L. & Shantaram, R. (1969). Convergence of a sequence of transformations of distribution functions. *Pacific Mathematical Journal* 31: 403–415.
10. Hesselager, O., Wang, Sh. & Willmot, G.E. (1998). Exponential and scale mixtures and equilibrium distributions. *Scandinavian Actuarial Journal.* 1998(2): 125–142.
11. Huang, J.S. & Lin, G.D. (1995). Characterization of distributions using equilibrium transformations. *Sankhyā, Series A* 57(2): 179–185.
12. Lin, G.D. (1994). On a probabilistic generalization of Taylor's theorem. *Statistics & Probability Letters* 19(3): 237–243.
13. Massey, W.A. & Whitt, W. (1993). A probabilistic generalization of Taylor's theorem. *Statistics & Probability Letters* 16(1): 51–54.
14. Nanda, A.K., Jain, K. & Singh, H. (1996). On closure of some partial orderings under mixtures. *Journal of Applied Probability* 33(3): 698–706.
15. Pakes, A.G. (1996). Length biasing and laws equivalent to the log-normal. *Journal of Mathematical Analysis and Applications* 197(3): 825–854.

https://doi.org/10.1017/S0269964814000308 Published online by Cambridge University Press

16. Pakes, A.G. (2004). Characterization of probability laws via length-biasing and stationary-excess operations. *Communications in Statistics, Theory and Methods* 33(11–12): 2975–2992.
17. Shantaram, R. & Harkness, W. (1972). On a certain class of limit distributions. *Annals of Mathematics and Statistics* 43: 2067–2071.
18. van Beek, P. & Braat, J. (1973). The limits of sequences of iterated overshoot distribution functions. *Stochastic Processes and their Applications* 1: 307–316.
19. Vardi, Y., Shepp, L.A. & Logan, B.F. (1981). Distribution functions invariant under residual-lifetime and length biased sampling. *Zeitschrift fr Wahrscheinlichkeitstheorie und Verwandte Gebiete* 56: 415–426.
20. Whitt, W. (1985). The renewal-process stationary-excess operator. *Journal of Applied Probability* 22: 156–167.

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