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Axisymmetric transmission through an interface layer

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Abstract

It is shown that if we know the Love's strain function due to the existence of any arbitrary axisymmetric singularity in the elastic whole space, then the corresponding Love's function due to the existence of the same singularity in the interior of one of two dissimilar semi-infinite solids separated by a dissimilar thick layer can be obtained by differentiation and integration of that for the elastic whole space. This discovery rests on the principle that to every singularity in the layered space there will correspond an infinite system of images. Remarkably, the number of images is finite when the phases have equal rigidities but different compressibilities. The analysis thus brings under one general theorem the solutions due to such influencing singularities as a normal point force, a normal point force doublet, an infinitesimal prismatic dislocation loop, a centre of dilatation and a dilatation doublet. From the theorem, we readily infer that the elastic field at large distances from any arbitrary axisymmetric influencing singularity becomes nearly that for two perfectly bonded semi-infinite solids without an interface layer.

1. *Introduction*

When an elastic free layer is under the influence of a known distribution of internal singularities the induced elastic field can be determined by the method of Hankel integrals. For this purpose, we require to express the elastic field of the same system of singularities in an unbounded body in terms of infinite integrals involving Bessel functions and we then find a second set of infinite integrals which expresses the perturbation due to the influence of the free parallel plane boundaries of the elastic layer.

By the use of this powerful method of analysis, Dougall [**4**] determined the elastic field of a free layer under the influence of a given system of internal point singularities and he also solved the companion problem when the layer is under the influence of given distribution of surface forces. These problems have been pursued at great length by modern investigators who have confirmed the accuracy of Dougall.

It does not appear to have been generally observed that the second set of infinite integrals in Dougall's analysis expresses the effect due to an imaginary infinite system of elastic singularities which have no physical existence but which may be called image singularities. The infinite system of images reduces to a finite set if the layer becomes a semi-infinite solid as shown by Mindlin and Cheng [**10**] in their analysis of a half-space with an expanding or contracting inclusion and Aderogba [**1**] who

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connects the Papkovitch potentials due to any arbitrary influencing singularity in the interior of a half-space with the corresponding potentials for the homogeneous infinite solid.

It is the primary purpose of the present paper to show that for the generalized case of an arbitrary axisymmetric singularity in the interior of one of two semi-infinite solids separated by a thick layer, the successive images of the singularity also form an infinite series which can be properly weighted to fulfil the condition of the applicability of the principle of images. The resulting solution is general for the layered solid within the limits of the linear axisymmetric theory of elasticity. Certainly, other more general types of influencing singularities, inside or outside the interface layer, permit the use of the image method.

Maxwell [**7**] seems to have been the first who saw fully the utility of applying the method of images to the analogous problem of electrical conduction through a plate separating two media. Generally, these images lead to an infinite series representation of the solution, which is useful when only the direct singularity and perhaps its first few reflected images are the most important. This will occur whenever the observation point is very close to the influencing singularity, as in the inspiring problem of Bacon [**3**] of the force acting on a centre of dilatation in and near a surface layer.

2. *Formulation*

Let (r, θ, z) denote circular cylindrical coordinates. A linearly elastic isotropic material of shear modulus μ_1 and Poisson's ratio ν_1 occupies the semi-infinite region $z > 0$, $r > 0$, while a different linearly elastic isotropic material of corresponding constants μ_2 and ν_2 occupies the region $-H < z < 0$, $r > 0$. A third linearly elastic isotropic material with corresponding shear modulus μ_3 and Poisson's ratio ν_3 occupies the remaining region of space defined by $z < -H$, $r > 0$. It is assumed that the three materials are perfectly bonded together at the parallel interfaces $z = 0$ and $z = -H$.

The layered system is assumed to be under an axisymmetric deformation, so that its cylindrical displacements u_r, u_z and stresses σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, σ_{rz} are representable in terms of the Love's strain function $\Psi(r, z)$ according to the relations

$$\left.\begin{aligned} 2\mu u_r &= -\frac{\partial^2 \Psi}{\partial r \partial z}, \\ 2\mu u_z &= 2(1-\nu)\nabla^2 \Psi - \frac{\partial^2 \Psi}{\partial z^2}, \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} \sigma_{rr} &= \frac{\partial}{\partial z}\left[\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right]\Psi, \\ \sigma_{\theta\theta} &= \frac{\partial}{\partial z}\left[\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right]\Psi, \\ \sigma_{zz} &= \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\Psi, \\ \sigma_{rz} &= \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\Psi, \end{aligned}\right\} \tag{2}$$

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provided that

$$\nabla^4\Psi \equiv \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right]^2 \Psi = 0. \tag{3}$$

To these equations we adjoin the continuity conditions

$$\left.\begin{aligned} u_r^{(1)} &= u_r^{(2)}, \quad u_z^{(1)} = u_z^{(2)}, \\ \sigma_{rz}^{(1)} &= \sigma_{rz}^{(2)}, \quad \sigma_{zz}^{(1)} = \sigma_{zz}^{(2)}, \end{aligned}\right\} \tag{4}$$

when $z = 0$, $r > 0$, as well as the additional continuity conditions

$$\left.\begin{aligned} u_r^{(2)} &= u_r^{(3)}, \quad u_z^{(2)} = u_z^{(3)}, \\ \sigma_{rz}^{(2)} &= \sigma_{rz}^{(3)}, \quad \sigma_{zz}^{(2)} = \sigma_{zz}^{(3)}, \end{aligned}\right\} \tag{5}$$

when $z = -H$, $r > 0$, the superscripts 1, 2 and 3 in parentheses being used to distinguish between the elastic fields in the three regions $z > 0$, $-H < z < 0$ and $z < -H$, respectively.

Equations (1)–(3), together with the continuity conditions (4) and (5), are necessary and sufficient for the complete determination of the elastic fields of the three media, supposing the mechanism of loading is specified.

3. *Analysis*

Suppose then that in the unbounded homogeneous isotropic elastic solid of moduli μ_1 and ν_1 the loading is confined to a bounded subregion of $z > 0$ and is characterizable by the Love's function $\Psi(r, z)$ which is free from singularities in the region $z < h$, where h is a positive constant. If now the same loading is applied to our layered solid, the elastic field is modified.

We assume that $\Psi(r, z)$ admits the integral representation

$$\Psi(r,\ z) = \int_0^\infty (f + ztg)e^{tz} J_0(rt)\, dt \tag{6}$$

which holds good for all points of space such that $z < h$, where f and g are suitably defined weighting functions of t, while J_0 denotes the Bessel function of the first kind of zero order.

For a compact representation of the desired results, it is convenient in the remaining analysis to introduce for $i = 1, 2$ the auxiliary parameters

$$\left.\begin{aligned} \alpha_i &= \frac{\mu_{i+1} - \mu_i}{\mu_{i+1}(3 - 4\nu_i) + \mu_i}, \\ \beta_i &= \frac{\mu_{i+1}(3 - 4\nu_i) - \mu_i(3 - 4\nu_{i+1})}{\mu_{i+1} + \mu_i(3 - 4\nu_{i+1})}, \\ \overline{\alpha}_i &= \frac{\mu_i - \mu_{i+1}}{\mu_i(3 - 4\nu_{i+1}) + \mu_{i+1}}, \\ \overline{\beta}_i &= \frac{\mu_i(3 - 4\nu_{i+1}) - \mu_{i+1}(3 - 4\nu_i)}{\mu_i + \mu_{i+1}(3 - 4\nu_i)}, \end{aligned}\right\} \tag{7}$$

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which are clearly related through

$$\left.\begin{aligned} \alpha_i\beta_i &= \overline{\alpha}_i\overline{\beta}_i, \\ (1+\alpha_i)(1+\overline{\alpha}_i) &= (1+\beta_i)(1+\overline{\beta}_i) = 1-\alpha_i\beta_i. \end{aligned}\right\} \tag{8}$$

It may be observed that $\overline{\alpha}_i$ and $\overline{\beta}_i$ are obtainable from α_i and β_i, respectively, by merely interchanging the roles of the subscripts i and $i+1$.

As we have remarked, the value of Love's function at any point inside the layered solid may be expressed as the sum of that for the homogeneous infinite solid and a perturbational part representing the influence of the parallel interfaces. To obtain this perturbational part in a well-structured form, it is necessary to take advantage of some results of Aderogba [**2**] in the case of two elastic media separated by a plane surface.

We therefore construct the desired Love's functions in the three phases as follows: for the first material in $z > 0$,

$$\left.\begin{aligned} \Psi_1 = \Psi(r,z) &+ \alpha_1\left[1-4\nu_1+2z\frac{\partial}{\partial z}-z^2\nabla^2\right]\Psi(r,-z) \\ &+\frac{1}{4}[\beta_1-\alpha_1(1-4\nu_1)^2]\int\!\!\int\nabla^2\Psi(r,-z)\,dz\,dz \\ &-2z\alpha_1(1-2\nu_1)\int\nabla^2\Psi(r,-z)\,dz \\ &+(1+\overline{\alpha}_1)\,\Phi_1(r,z)+\frac{1}{2}z(\overline{\beta}_1-\overline{\alpha}_1)\int\nabla^2\Phi_1(r,z)\,dz \\ &+\frac{1}{4}[(1+\overline{\beta}_1)(1-4\nu_1)-(1+\overline{\alpha}_1)(1-4\nu_2)]\int\!\!\int\nabla^2\Phi_1(r,z)\,dz\,dz; \end{aligned}\right\} \tag{9}$$

for the second material in $-H < z < 0$,

$$\left.\begin{aligned} \Psi_2 = (1+\alpha_1)\,\Psi(r,z) &+ \frac{1}{2}z(\beta_1-\alpha_1)\int\nabla^2\Psi(r,z)\,dz \\ &+\frac{1}{4}[(1+\beta_1)(1-4\nu_2)-(1+\alpha_1)(1-4\nu_1)]\int\!\!\int\nabla^2\Psi(r,z)\,dz\,dz \\ &+\Phi_1(r,z)+\overline{\alpha}_1\left[1-4\nu_2+2z\frac{\partial}{\partial z}-z^2\nabla^2\right]\Phi_1(r,-z) \\ &+\frac{1}{4}[\overline{\beta}_1-\overline{\alpha}_1(1-4\nu_2)^2]\int\!\!\int\nabla^2\Phi_1(r,-z)\,dz\,dz \\ &-2z\overline{\alpha}_1(1-2\nu_2)\int\!\!\int\nabla^2\Phi_1(r,-z)\,dz; \end{aligned}\right\} \tag{10}$$

and for the third material in $z < -H$,

$$\Psi_3 = \Phi_2(r,z), \tag{11}$$

where $\Psi(r,z)$ is once again the undisturbed Love's function for the homogeneous infinite solid, while, guided by (6), we judiciously select

$$\left.\begin{aligned} \Phi_1(r,z) &= \int_0^\infty [A_1+(z+H)tB_1]\,e^{-t(z+2H)}J_0(tr)\,dt, \\ \Phi_2(r,z) &= \int_0^\infty [A_2+(z+H)tB_2]\,e^{tz}J_0(tr)\,dt. \end{aligned}\right\} \tag{12}$$

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The four as yet arbitrary functions A_i, B_i, $i = 1, 2$, of the integration parameter t need to be determined in accordance with the continuity conditions (5) at the second interface $z = -H$, since it may be verified that the representations (9) and (10) ensure the automatic satisfaction of the continuity conditions (4) at the first interface $z = 0$. The tedious task of satisfying these conditions ultimately yields the explicit expressions

$$\left.\begin{aligned} 2A_1 + (1 - 4\nu_2)B_1 &= -\beta_2\Delta_1\Delta, \\ B_1 &= \alpha_2\Delta_2\Delta, \\ 2A_2 - (1 - 4\nu_3)B_2 &= -(1 + \alpha_2)\Delta_2\Delta, \\ B_2 &= -(1 + \beta_2)\Delta_1\Delta, \end{aligned}\right\} \tag{13}$$

where

$$\left.\begin{aligned} \Delta_1 &= 2tH\overline{\alpha}_1\alpha_2E(1 + \alpha_1)[2f - (1 - 4\nu_1)\,g] + g(1 + \beta_1)(\alpha_2\overline{\beta}E - 1), \\ \Delta_2 &= (1 + \alpha_1)(\overline{\alpha}_1\beta_2E - 1)[2f - (1 - 4\nu_1)\,g] + 2tHg(1 + \beta_1), \\ \Delta &= [1 - (\overline{\alpha}_1\beta_2 + \alpha_2\overline{\beta}_1)\,E + \overline{\alpha}_1\overline{\beta}_1\alpha_2\beta_2E^2 - (2tH)^2\overline{\alpha}_1\alpha_2E]^{-1}, \\ E &= e^{-2tH}. \end{aligned}\right\} \tag{14}$$

Equations (9)–(12), in conjunction with (13)–(14), completely determine the desired Love's strain functions. Consequently, in view of (1) and (2), the displacements and stresses induced by any axisymmetric imperfection situated in the first medium are fully known. The forces acting on such imperfections are then computable by means of the formulae of Eshelby [**5**].

Finally, it is seen from (6) that

$$\left.\begin{aligned} \int_0^\infty fe^{tz}J_0(tr)\,dt &= \Psi(r, z) - \frac{1}{2}z\int\nabla^2\Psi(r, z)\,dz, \\ \int_0^\infty tge^{tz}J_0(tr)\,dt &= \frac{1}{2}\int\nabla^2\Psi(r, z)\,dz. \end{aligned}\right\} \tag{15}$$

These relations will prove useful in the remaining sections. Furthermore, since equation (13) reveals that the weighting functions A_1 and B_1 are linearly related to the weighting functions A_2 and B_2 by means of

$$\left.\begin{aligned} A_1 &= \frac{\alpha_2(1 - 4\nu_2)}{1 + \alpha_2}A_2 + \frac{1}{2}\left[\frac{\beta_2}{1 + \beta_2} - \frac{\alpha_2(1 - 4\nu_2)(1 - 4\nu_3)}{1 + \alpha_2}\right]B_2, \\ B_1 &= -\frac{\alpha_2}{1 + \alpha_2}[2A_2 - (1 - 4\nu_3)\,B_2], \end{aligned}\right\} \tag{16}$$

it follows that if we know the Love's function $\Phi_2(= \Psi_3)$ for the third material on the other side of the interface layer, the calculation of the perturbation function Φ_1 for the other two materials reduces to the operations of differentiation and integration

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only. Indeed, we find that

$$\left.\begin{aligned}\Phi_1(r,z) = {}& \frac{\alpha_2(1-4\nu_2)}{1+\alpha_2}\left[\Phi_2(r,-z-2H) - \frac{1}{2}(z+H)\int \nabla^2\Phi_2(r,-z-2H)\,dz\right]\\ &+\frac{\alpha_2(z+H)}{1+\alpha_2}\left[\left\{-(z+H)\nabla^2 + 2\frac{\partial}{\partial z}\right\}\Phi_2(r,-z-2H)\right.\\ &\left.-\left\{1+\frac{1}{2}(1-4\nu_3)\right\}\int \nabla^2\Phi_2(r,-z-2H)\,dz\right]\\ &+\frac{1}{4}\left[\frac{\beta_2}{1+\beta_2} - \frac{\alpha_2(1-4\nu_2)(1-4\nu_3)}{1+\alpha_2}\right]\iint \nabla^2\Phi_2(r,-z-2H)\,dz\,dz.\end{aligned}\right\} \tag{17}$$

This result is again true for any distribution of axisymmetric singularity in the first material occupying the half-space $z > 0$. It can thus be used to deal with the case when an ellipsoidal region in $z > 0$ undergoes an axisymmetric change of shape.

4. *Discussion*

It would be nice if one could take the general integral solution in the last section and deduce from it the types of images that occur. In order to get a clear picture of such a physical interpretation, it is perhaps interesting to consider first a special composite model of Hill [**6**] when the phases have equal rigidities but different compressibilities.

We therefore suppose now that $\mu_1 = \mu_2 = \mu_3$, which implies that

$$\beta_i = \frac{\nu_{i+1} - \nu_i}{1 - \nu_{i+1}} = \lambda_i \quad (i = 1, 2).$$

Then, it may be shown that the required Love's functions Ψ_i for the composite solid are explicitly dependent on the undisturbed function Ψ by means of the relations

$$\left.\begin{aligned}\Psi_1 = {}& \left\{\begin{array}{l}\Psi(r,z) + \frac{1}{4}\lambda_1\iint\nabla^2\Psi(r,-z)\,dz\,dz\\ +\frac{1}{4}\lambda_2(1+\lambda_1)\iint\nabla^2\Psi(r,-z-2H)\,dz\,dz\end{array}\right\},\\ \Psi_2 = {}& \left\{\begin{array}{l}\Psi(r,z) + \frac{1}{2}z\lambda_1\int\nabla^2\Psi(r,z)\,dz\\ +\frac{1}{4}[(1+\lambda_1)(1-4\nu_2) - (1-4\nu_1)]\iint\nabla^2\Psi(r,z)\,dz\,dz\\ +\frac{1}{4}\lambda_2(1+\lambda_1)\iint\nabla^2\Psi(r,-z-2H)\,dz\,dz\end{array}\right\},\\ \Psi_3 = {}& \left\{\begin{array}{l}\Psi(r,z) + \frac{1}{2}[(1+\lambda_1)(1+\lambda_2)(z+H) - H(1+\lambda_1) - z]\int\nabla^2\Psi(r,z)\,dz\\ +\frac{1}{4}[(1+\lambda_1)(1+\lambda_2)(1-4\nu_3) - (1-4\nu_1)]\iint\nabla^2\Psi(r,z)\,dz\,dz\end{array}\right\}.\end{aligned}\right\} \tag{18}$$

We therefore deduce that if the influencing singularity is a concentrated point force in the z direction, located at the point $(0,0,h)$ and characterized by $\Psi(r,z) = c[r^2 + (z-h)^2]^{\frac{1}{2}}$, where c is a constant, the complete elastic field induced in the third medium is produced by a combination of a concentrated point force acting along the z-axis, a centre of dilatation, both located at $(0,0,h)$, together with a semi-

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infinite line of centres of dilatation from $(0,0,h)$ to $(0,0,\infty)$. On the other hand, the complete elastic field induced in the interface layer is produced by a combination of a concentrated point force acting along the z-axis; a centre of dilatation, both located at the point $(0,0,h)$, and a semi-infinite line of centres of dilatation from $(0,0,h)$ to $(0,0,\infty)$, together with another semi-infinite line of centres of dilatation from $(0,0,-h-2H)$ to $(0,0,-\infty)$. The elastic field induced in the first medium is produced by a combination of the initial influencing concentrated force at $(0,0,h)$, a semi-infinite line of centres of dilatation from $(0,0,-h)$ to $(0,0,-\infty)$ and another semi-infinite line of centres of dilatation from $(0,0,-h-2H)$ to $(0,0,-\infty)$. Therefore, the distant effect in this first medium of the influencing singularity is produced by the superposition of the singularity solutions of a normal point force at the origin and a semi-infinite line of centres of dilatation along the negative z-axis.

If the influencing singularity is an infinitesimal prismatic dislocation loop of Burgers vector $(0,0,b_3)$ and normal $(0,0,1)$ located in the first medium at the point $(0,0,h)$, then the undisturbed function Ψ is given by

$$\Psi(r,z) = c_1\left[2\nu_1\log(R_1 - z + h) + \frac{\partial}{\partial z}R_1\right], \tag{19}$$

where c_1 is another constant, while $R_1^2 = r^2 + (z-h)^2$. Substitution into (18) therefore shows that the complete elastic field induced in the first medium is produced by the superposition of special singularity solutions for the homogeneous infinite solid, namely, a centre of dilatation at $(0,0,h)$, a force doublet in the z direction at $(0,0,h)$, a centre of dilatation at $(0,0,-h)$, as well as another centre of dilatation at $(0,0,-h-2H)$. The Galerkin functions characterizing all these singularity solutions have been comprehensively discussed by Mindlin [**9**].

Let us next consider the case when only the second and third materials have equal rigidities but different compressibilities, so that $\mu_2 = \mu_3$ and $\alpha_2 = 0$, $\beta_2 = \lambda_2$. Then it may be shown that equations (9)–(11) once again give the desired Love's functions for the layered system provided that the perturbation functions Φ_1 and Φ_2 are dependent on the undisturbed function Ψ for the homogeneous infinite solid by means of the relations

$$\left.\begin{aligned}
\Phi_1(r,z) &= \frac{1}{4}\lambda_2(1+\beta_1)\sum_{n=0}^{\infty}(\overline{\alpha}_1\lambda_2)^n\int\!\!\int\nabla^2\Psi(r,-z-2H-2Hn)\,dz\,dz,\\
\Phi_2(r,z) &= (1+\alpha_1)\left[\Psi(r,z) - \frac{1}{2}z\int\nabla^2\Psi(r,z)\,dz\right]\\
&\quad -\frac{1}{4}(1-4\nu_1)(1+\alpha_1)\int\!\!\int\nabla^2\Psi(r,z)\,dz\,dz\\
&\quad -\frac{1}{2}H(1+\beta_1)\sum_{n=0}^{\infty}(\overline{\alpha}_1\lambda_2)^n\int\nabla^2\Psi(r,z-2Hn)\,dz\\
&\quad +\frac{1}{2}(z+H)(1+\lambda_2)(1+\beta_1)\sum_{n=0}^{\infty}(\overline{\alpha}_1\lambda_2)^n\int\nabla^2\Psi(r,z-2Hn)\,dz\\
&\quad +\frac{1}{4}(1-4\nu_3)(1+\lambda_2)(1+\beta_1)\sum_{n=0}^{\infty}(\overline{\alpha}_1\lambda_2)^n\int\!\!\int\nabla^2\Psi(r,z-2Hn)\,dz\,dz.
\end{aligned}\right\} \tag{20}$$

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It is thus seen that the desired Love's function in each phase is equal to that due to the superposition of an infinite system of special singularity solutions for the homogeneous infinite solid, placed at certain regular points along the normal to the interface layer, through the influencing singularity. For example, if the influencing singularity is a concentrated force in the z direction, located at the point $(0, 0, h)$, the elastic field induced in the third medium is produced by the superposition of the singularity solutions of a family of concentrated forces and centres of dilatation, located at the points $(0, 0, h + 2Hn)$, together with another family of lines of centres of dilatation from the points $(0, 0, h + 2Hn)$ to $(0, 0, \infty)$, where n is a non-negative integer.

From equations (9) and (20), we also deduce that in the present particular case the far field in the first medium due to the presence of any arbitrary axisymmetric singularity is explicitly determinable by the Love's function

$$\left.\begin{aligned}\Psi_1 = \Psi_0(r,z)\,\alpha_1 &\left[1 - 4\nu_1 + 2z\frac{\partial}{\partial z} - z^2\nabla^2\right]\Psi_0(r,-z)\\ &-2z\alpha_1(1-2\nu_1)\int \nabla^2\Psi_0(r,-z)\,dz\\ &+\frac{1}{4}[\beta_1 - \alpha_1(1-4\nu_1)^2 + \lambda_2(1+\overline{\alpha}_1)(1+\beta_1)(1-\overline{\alpha}_1\lambda_2)^{-1}]\\ &\times\int\int \nabla^2\Psi_0(r,-z)\,dz\,dz,\end{aligned}\right\} \tag{21}$$

where $\Psi_0(r, z)$ is the corresponding Love's function when the singularity is located at the origin. Therefore, the far field of a centre of dilatation will be produced by a combination at the origin of a centre of dilatation and a force doublet in the z direction, while that of a normal point force will be produced by a combination of a normal point force at the origin and a semi-infinite line of centres of dilatation along the negative z-axis. Thus, there is a fundamental difference in the far-field effects due to the presence of these two basic singularities. However, it may be confirmed that the far-field effect of an infinitesimal prismatic dislocation loop is also produced by the combined influence of a centre of dilatation and a force doublet at the origin.

From (20), the far field in the third medium due to any influencing axisymmetric singularity is also given explicitly by the Love's function

$$\left.\begin{aligned}\Psi_3 = (1+\alpha_1)&\left[\Psi_0(r,z) - \frac{1}{2}z\int\nabla^2\Psi_0(r,z)\,dz - \frac{1}{4}(1-4\nu_1)\int\int\nabla^2\Psi_0(r,z)\,dz\,dz\right]\\ &+\frac{1}{2}(1+\beta_1)(1+\lambda_2)(1-\overline{\alpha}_1\lambda_2)^{-1}[z\int\nabla^2\Psi_0(r,z)\,dz\\ &+\frac{1}{2}(1-4\nu_3)\int\int\nabla^2\Psi_0(r,z)\,dz\,dz],\end{aligned}\right\} \tag{22}$$

where $\Psi_0(r, z)$ is once again the corresponding Love's function when the singularity is located at the origin. Therefore, the far-field effects in the third medium due to an influencing normal point force and an infinitesimal prismatic dislocation loop are the same as in the first medium, while the far-field effects in the two media due to an influencing centre of dilatation are physically different.

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Let us now consider the interesting particular case of the effect of a thick layer of elastic constants μ_2 and ν_2 that is completely embedded in a medium of elastic constants μ_1 and ν_1 which is under the influence of an arbitrary axisymmetric singularity. The solution to this particular problem can be recovered from the general solution in the third section by separating it to the case when the first medium is the same as the third, so that $\mu_3 = \mu_1$, and $\nu_3 = \nu_1$ and $\alpha_2 = \overline{\alpha}_1$, and $\beta_2 = \overline{\beta}_1$. Then, it may be verified that the significant perturbation functions Φ_1 and Φ_2 occurring in Love's functions (9)–(11) reduce to the lengthy expressions

$$\left.\begin{aligned}
\Phi_1(r,z) = {} & \frac{1}{2}\overline{\alpha}_1(1+\alpha_1)[1-4\nu_2+2(z+H)L] \\
& \times \sum\sum \Delta_{mn}(-1)^n L^n \left[\Psi(r,-z_2) - \frac{1}{2}z_2\int \nabla^2\Psi(r,-z_2)\,dz\right] \\
& -\frac{1}{8}\{\overline{\alpha}_1(1+\alpha_1)(1-4\nu_1)[1-4\nu_2+2(z+H)L] - \overline{\beta}_1(1+\beta_1)\} \\
& \times \sum\sum \Delta_{mn}(-1)^n L^n \iint \nabla^2\Psi(r,-z_2)\,dz\,dz \\
& -\frac{1}{8}(1+\beta_1)[1-4\nu_2+2(z+H)L] \\
& \times \sum\sum \Delta^*_{mn}(-1)^n L^n \iint \nabla^2\Psi(r,-z_1)\,dz\,dz \\
& -\frac{1}{2}\alpha_1\beta_1(1+\alpha_1)\sum\sum \Delta^*_{mn}(-1)^n L^n \left[\Psi(r,-z_3) - \frac{1}{2}z_3\int \nabla^2\Psi(r,-z_3)\,dz\right] \\
& +\frac{1}{8}\alpha_1\beta_1(1+\alpha_1)(1-4\nu_1)\sum\sum \Delta^*_{mn}(-1)^n L^n \iint \nabla^2\Psi(r,-z_3)\,dz\,dz,
\end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned}
\Phi_2(r,z) = {} & \frac{1}{4}(1+\beta_1)(1+\overline{\beta}_1)[2(\nu_1-\nu_3)+(z+H)L] \\
& \times \sum\sum \Delta_{mn} L^n \iint \nabla^2\Psi(r,w_1)\,dz\,dz \\
& +\frac{1}{2}(1+\beta_1)(1+\overline{\beta}_1)\sum\sum \Delta_{mn} L^n \left[\Psi(r,w_1) - \frac{1}{2}w_1\int \nabla^2\Psi(r,w_1)\,dz\right] \\
& -\frac{1}{2}\overline{\alpha}_1(1+\alpha_1)(1+\overline{\beta}_1)[1-4\nu_3+2(z+H)L] \\
& \times \sum\sum \Delta^*_{mn} L^n \left[\Psi(r,w_2) - \frac{1}{2}w_2\int \nabla^2\Psi(r,w_2)\,dz\right] \\
& +\frac{1}{8}\overline{\alpha}_1(1+\alpha_1)(1+\overline{\beta}_1)(1-4\nu_1)[1-4\nu_3+2(z+H)L] \\
& \times \sum\sum \Delta^*_{mn} L^n \iint \nabla^2\Psi(r,w_2)\,dz\,dz \\
& +\frac{1}{8}(1+\overline{\alpha}_1)(1+\beta_1)(\overline{\alpha}_1)^{-1}\sum\sum \Delta^*_{mn} L^n \iint \nabla^2\Psi(r,w_0)\,dz\,dz,
\end{aligned}\right\} \tag{24}$$

where $\Sigma\Sigma$ represents double summation from $m = 0$ and $n = 0$ to infinity, while

$$\left.\begin{aligned}
z_k &= z + H(2m+n+k), \quad w_k = z - H(2m+n+k-1), \quad L = \frac{\partial}{\partial z}, \\
\Delta_{mn} &= \frac{(m+n)!}{m!n!}(\alpha_1\beta_1)^m(2H\overline{\alpha}_1)^n[1+(-1)^n], \\
\Delta^*_{mn} &= \frac{(m+n)!}{m!n!}(\alpha_1\beta_1)^m(2H\overline{\alpha}_1)^n[1-(-1)^n].
\end{aligned}\right\} \tag{25}$$

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Thus if the influencing singularity is located at the point $(0,0,h)$ in the first medium, the elastic field induced in the third medium is obtained by placing appropriate singularities and their derivatives at the points $(0,0,h+H\{2m+n+k-1\})$, where m, n and k are non-negative integers. On the other hand, the image singularities for generating the elastic field in the first medium are located at the points $(0,0,-h-H\{2m+n+k\})$.

The general case

Let us finally consider the general case when the six elastic parameters of the layered system are different from one another. It makes for conciseness to introduce the quantities

$$\left.\begin{aligned} \Delta_{mnk} &= \frac{(m+n+k)!}{m!n!k!}(-\alpha_1\alpha_2\beta_1\beta_2)^m(\overline{\alpha}_1\beta_2+\alpha_2\overline{\beta}_1)^2(4H^2\overline{\alpha}_1\alpha_2)^k, \\ x_p &= z+2H(2m+n+k+p), \quad y_p = z-2H(2m+n+k+p). \end{aligned}\right\} \tag{26}$$

Then, it may be shown that the following general theorem holds.

An axisymmetric interface layer theorem

Suppose $\Psi(r,z)$ is the Love's function for an axisymmetric deformation of an unbounded isotropic homogeneous solid of elastic moduli (μ_1,ν_1), all its singularities being in the region $z>0$. Then when two other different elastic isotropic solids of moduli (μ_2,ν_2) and (μ_3,ν_3) are introduced into the regions $-H<z<0$ and $-\infty<z<-H$, respectively, so that perfect bonds are maintained at the parallel interfaces $z=0$ and $z=-H$, the new Love's functions for the layered system are given by equations (9)–(11), provided that the auxiliary functions Φ_1 and Φ_2 are defined by

$$\left.\begin{aligned} \Phi_1(r,z) = {} & \alpha_2(1+\alpha_1)[1-4\nu_2+2(z+H)L]\sum\sum\sum \Delta_{mnk}L^{2k}\Big[\Psi(r,-x_1) \\ & -\frac{1}{2}x_1\int\nabla^2\Psi(r,-x_1)\,dz-\frac{1}{4}(1-4\nu_1)\iint\nabla^2\Psi(r,-x_1)\,dz\,dz \\ & -\overline{\alpha}_1\beta_2\Big\{\Psi(r,-x_2)-\frac{1}{2}x_2\int\nabla^2\Psi(r,-x_2)\,dz-\frac{1}{4}(1-4\nu_1) \\ & \times\iint\nabla^2\Psi(r,-x_2)\,dz\,dz\Big\}\Big]+\frac{1}{2}H\alpha_2(1+\beta_1)[1-4\nu_2 \\ & +2(z+H)\,L]\sum\sum\sum\Delta_{mnk}L^{2k+1}\iint\nabla^2\Psi(r,-x_1)\,dz\,dz \\ & +2H\overline{\alpha}_1\alpha_2\beta_2(1+\alpha_1)\sum\sum\sum\Delta_{mnk}L^{2k+1}\Big[\Psi(r,-x_2) \\ & -\frac{1}{2}x_2\int\nabla^2\Psi(r,-x_2)\,dz-\frac{1}{4}(1-4\nu_1)\iint\nabla^2\Psi(r,-x_2)\,dz\,dz\Big] \\ & +\frac{1}{4}\beta_2(1+\beta_1)\sum\sum\sum\Delta_{mnk}L^{2k} \\ & \times\iint[\nabla^2\Psi(r,-x_1)-\alpha_2\overline{\beta}_1\nabla^2\Psi(r,-x_2)]\,dz\,dz, \end{aligned}\right\} \tag{27}$$

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$$\left.\begin{aligned}\Phi_2(r,z) = {}& \frac{1}{4}(1+\beta_1)(1+\beta_2)[1-4\nu_2+2(z+H)L] \\ & \times \sum\sum\sum \Delta_{mnk} L^{2k} \iint \left[\nabla^2\Psi(r,y_0) - \frac{1}{2}\alpha_2\overline{\beta}_1 \nabla^2\Psi(r,y_1)\,dz\,dz\right] \\ & -2H\overline{\alpha}_1\alpha_2(1+\alpha_1)(1+\beta_2)[1-4\nu_3+2(z+H)\,L] \\ & \times \sum\sum\sum \Delta_{mnk} L^{2k+1} \left[\Psi(r,y_1) - \frac{1}{2}y_1 \int \nabla^2\Psi(r,y_1)\,dz \right. \\ & \left. -\frac{1}{4}(1-4\nu_1)\iint \nabla^2\Psi(r,y_1)\,dz\,dz\right] \\ & +(1+\alpha_1)(1+\alpha_2)\sum\sum\sum \Delta_{mnk} L^{2k} \\ & \times \left[\Psi(r,y_0) - \frac{1}{2}y_0 \int \nabla^2\Psi(r,y_0)\,dz \right. \\ & -\frac{1}{4}(1-4\nu_1)\iint \nabla^2\Psi(r,y_0)\,dz\,dz \\ & -\overline{\alpha}_1\beta_2\left\{\Psi(r,y_1) - \frac{1}{2}y_1\int \nabla^2\Psi(r,y_1)\,dz \right. \\ & \left.\left. -\frac{1}{4}(1-4\nu_1)\iint \nabla^2\Psi(r,y_1)\,dz\,dz\right\}\right] \\ & -\frac{1}{2}H(1+\alpha_2)(1+\beta_1)\sum\sum\sum \Delta_{mnk} L^{2k+1} \iint \nabla^2\Psi(r,y_0)\,dz\,dz,\end{aligned}\right\} \tag{28}$$

where $\sum\sum\sum$ serves to indicate triple summation from $m = 0$, $n = 0$ and $k = 0$ to infinity. Therefore, the dominant distant effect in the third medium due to any arbitrary axisymmetric singularity in the first medium is given by

$$\left.\begin{aligned}\Psi_3 = {}& \frac{1}{4}\frac{(1+\beta_1)(1+\beta_2)}{1-\overline{\alpha}_1\beta_2}\left[1-4\nu_3+2z\frac{\partial}{\partial z}\right]\iint \nabla^2\Psi_0(r,z)\,dz\,dz \\ & +\frac{(1+\alpha_1)(1+\alpha_2)}{1-\alpha_2\overline{\beta}_1}\left[\Psi_0(r,z) - \frac{1}{2}z\int \nabla^2\Psi_0(r,z)\,dz \right. \\ & \left. -\frac{1}{4}(1-4\nu_1)\iint \nabla^2\Psi_0(r,z)\,dz\,dz\right],\end{aligned}\right\} \tag{29}$$

where $\Psi_0(r,z)$ is the corresponding Love's function when the singularity is located at the origin. Hence, if the influencing singularity is a concentrated normal force, the dominant far field in the third medium is obtained by superposition of the effect of a concentrated normal force in an unbounded elastic medium, applied at the origin, and the effect of a semi-infinite line of centres of dilatation along the positive z-axis. On the other hand, if the influencing singularity is representable in terms of a harmonic Love's function, then the dominant far field in the third medium is also representable in terms of a harmonic Love's function.

Because

$$\left.\begin{aligned}\frac{(1+\alpha_1)(1+\alpha_2)}{1-\alpha_2\overline{\beta}_1} &= \frac{4\mu_3(1-\nu_1)}{\mu_1+\mu_3(3-4\nu_1)}, \\ \frac{(1+\beta_1)(1+\beta_2)}{1-\overline{\alpha}_1\beta_2} &= \frac{4\mu_3(1-4\nu_1)}{\mu_3+\mu_1(3-4\nu_3)},\end{aligned}\right\} \tag{30}$$

one could also say that the dominant far field in the third medium is independent of the physical properties of the inhomogeneous interface layer, that is it does not feel the presence of the layer.

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The simplest possible application of the foregoing theorem is to the case when the layered system is under the influence of a centre of dilatation of strength T applied at the point $(0, 0, h)$ in the material of moduli (μ_1, ν_1). In the absence of material discontinuities, Love's function for the unbounded homogeneous solid is

$$\Omega(r, z) = -2\mu_1 T \log(R_1 - z + h), \tag{31}$$

where $R_1^2 = r^2 + (z-h)^2$. When the components of the composite solid are different, the theorem shows that the new Love's functions are, for the first material in $z > 0$,

$$\left.\begin{aligned}\Psi_1 = \Omega(r,z) &+ \alpha_1\left[1 - 4\nu_1 + 2z\frac{\partial}{\partial z}\right]\Omega(r,-z)\\ &+(1+\overline{\alpha}_1)\Phi(r,z) + \frac{1}{2}z(\overline{\beta}_1 - \overline{\alpha}_1)\int \nabla^2\Phi(r,z)\,dz\\ &+\frac{1}{4}[(1+\overline{\beta}_1)(1-4\nu_1) - (1+\overline{\alpha}_1)(1-4\nu_2)]\int\int \nabla^2\Phi(r,z)\,dz\,dz,\end{aligned}\right\} \tag{32}$$

for the second material in $-H < z < 0$,

$$\left.\begin{aligned}\Psi_2 = (1+\alpha_1)\,\Omega(r,z) &- 2z\overline{\alpha}_1(1-2\nu_2)\int \nabla^2\Phi(r,-z)\,dz\\ &+\Phi(r,z) + \overline{\alpha}_1\left[1 - 4\nu_2 + 2z\frac{\partial}{\partial z} - z^2\nabla^2\right]\Phi(r,-z)\\ &+\frac{1}{4}\left[\overline{\beta}_1 - \overline{\alpha}_1(1-4\nu_2)^2\right]\int\int \nabla^2\Phi(r,-z)\,dz\,dz\end{aligned}\right\} \tag{33}$$

and for the third material in $-\infty < z < -H$,

$$\left.\begin{aligned}\Psi_3 = (1+\alpha_1)(1+\alpha_2)\sum\sum\sum \Delta_{mnk}L^{2k}[\Omega(r,y_0) - \overline{\alpha}_1\beta_2\Omega(r,y_1)]\\ -2H\overline{\alpha}_1\alpha_2(1+\alpha_1)(1+\beta_2)[1-4\nu_2+2(z+H)L]\\ \times\sum\sum\sum \Delta_{mnk}L^{2k+1}\Omega(r,y_1),\end{aligned}\right\} \tag{34}$$

where

$$\left.\begin{aligned}\Phi(r,z) = \alpha_2(1+\alpha_1)[1-4\nu_2+2(z+H)L]\\ \times\sum\sum\sum \Delta_{mnk}L^{2k}[\Omega(r,-x_1) - \overline{\alpha}_1\beta_2\Omega(r,-x_2)]\\ +2H\overline{\alpha}_1\alpha_2\beta_2(1+\alpha_1)\sum\sum\sum \Delta_{mnk}L^{2k+1}\Omega(r,-x_2).\end{aligned}\right\} \tag{35}$$

The force F acting on the centre of dilatation may now be computed by means of Eshelby's [**5**] formula

$$F = 4\pi(1-\nu_1)\,TL\nabla\nabla^2\Psi_1 \tag{36}$$

which gives the non-vanishing component

$$\left.\begin{aligned}F_3 = 16\pi T(1-\nu_1)\,L^4[\alpha_1\Omega(r,-z) + \alpha_2(1+\alpha_1)(1+\overline{\beta}_1)\\ \times\sum\sum\sum \Delta_{mnk}L^{2k}\{\Omega(r,-x_1) - \overline{\alpha}_1\beta_2\Omega(r,-x_2)\}]\end{aligned}\right\} \tag{37}$$

evaluated at the point $(0, 0, h)$.

Far away from the influencing singularity, this component is dominated by the term

$$F_3 = 16\pi T(1-\nu_1)\frac{\mu_3 - \mu_1}{\mu_3(3-4\nu_1) + \mu_1}L^4\Omega(r,-z). \tag{38}$$

It is thus seen, as anticipated, that far away the effect of the interface inhomogeneity

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is unnoticeable and the singularity feels only the other side of the matrix with the shear modulus μ_3. Since the dilatation is clearly attracted far away if $\mu_3 < \mu_1$ and repelled by the layer nearly if $\mu_1 < \mu_2$ it must have a position of equilibrium in the material of moduli (μ_1, ν_1) if $\mu_3 < \mu_1 < \mu_2$.

Far from the interface layer, equations (32) and (35) also combine to yield the dominant term

$$\Psi_1 = \Psi_0(r, z) + \frac{\mu_3 - \mu_1}{\mu_3(3 - 4\nu_1) + \mu_1} \left[1 - 4\nu_1 + 2z\frac{\partial}{\partial z}\right] \Psi_0(r, -z), \tag{39}$$

where $\Psi_0(r, z)$ is Love's function when the dilatation is applied at the origin. This dominant term clearly shows that the far field in the first medium primarily consists of the superposition at the origin of the effects of a centre of dilatation, a centre of compression and a normal force doublet. One could also say that the far field becomes more and more nearly that for two perfectly bonded semi-infinite solids of elastic moduli (μ_1, ν_1) and (μ_3, ν_3) without an interface layer.

5. *Closure*

We have been inspired by the hydrodynamic circle theorem of Milne-Thomson [**8**] connecting the complex potential in any two-dimensional irrotational flow past a rigid circular cylinder with the corresponding complex potential when the cylinder is absent. In the present paper, we also connect the Love's strain function due to any arbitrary axisymmetric singularity in an elastic half-space separated from a dissimilar elastic half-space by a thick elastic layer with the corresponding Love's function for the unbounded homogeneous solid. The principal advantage is that the displacements and stresses induced by any axisymmetric singularity operative in the matrix can be determined directly by routine differentiation only, without the decidedly formidable task of searching for suitable stress functions and matching continuity conditions at the interfaces for each loading condition.

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