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Modulational stability of electron plasma wave spectra

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Analytical models for weakly nonlinear electron plasma waves are considered in order to obtain dynamic equations for the space-time evolution of their local power spectra. The model contains the wave kinetic equation as a limiting case for slow, long wavelength modulations. It is demonstrated that a finite spectral width in wavenumbers has a stabilizing effect on the modulational instability. The results invite a simple heuristic relation between the spectral width and the root-mean-square amplitude of stable stationary turbulent Langmuir wave spectra. A non-local average dispersion relation is derived as a limiting form by using the formalism developed for the spectral dynamics.

1. Introduction

The standard model for weakly nonlinear electron plasma waves (Zakharov 1972) assumes the dominant nonlinearity to be modification of the local plasma density (i.e. the index of refraction). Kinetic effects as Landau damping and nonlinear particle trapping are ignored from the outset, arguing that these effects will become negligible when the characteristic wavelengths λ are sufficiently long. For laboratory conditions it might become difficult to fulfill this condition due to the finite size of the apparatus if we at the same time want to ignore boundary conditions. Nonlinearities associated with, for instance, the adiabatic electron motion will be of the order of $(\lambda_D/\lambda)^2$ and can be ignored (Zakharov 1972; Dysthe and Pécseli 1977; Pécseli 1985), assuming the Debye length λ_D to be much smaller than any relevant wavelength. The origin of the dominating nonlinear term here is the space-time varying density in the flux of the electron continuity equation.

The present study considers the space-time evolution of local high-frequency electrostatic wave spectra. While laboratory investigations often deal with narrow-band processes in frequency and wavenumber, the observations in nature often have a broad-band character (Gurnett et al. 1981; LaBelle et al. 2010). Our analysis addresses these latter phenomena. In a frequency representation, Langmuir waves will usually appear as narrow banded since all wavenumbers give frequencies close to the plasma frequency, ω_{pe}. A useful representation can properly be made only including also the wavenumber variables.

We expect that the ideas presented here are relevant, for instance, for the interpretation of Naturally Enhanced Ion-Acoustic Lines (NEIALs) that are observed in the polar regions (Forme 1999) with radar beams nearly aligned with the magnetic

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Position	$\mathbf{R}$	Spatial separations	$\mathbf{r}$
Absolute time	T	Temporal separations	τ
Spectral local wavevector	$\mathbf{k}$	Real modulation wavevector	$\mathbf{K}$
Real local spectral frequency	Ω	Modulation frequency, generally complex	W

TABLE 1. Definition and explanation of symbols.

field lines. Authoritative numerical studies of this problem (Guio and Forme 2006) use basic equations as those discussed in our Sec. 2 and include a source term accounting for an electron beam that generates an enhanced level of Langmuir waves. Similar numerical simulations can be useful for testing the ideas resulting from the present work. Presumably it is necessary to carry out such a study in minimum two spatial dimensions in order to justify the quasi-normal approximation made in the analysis. On the other hand it might also become instructive to compare simulations in one and two/three spatial dimensions in order to see the effects of details in the approximations.

The present analysis is based on ideas originally formulated for studying wave mechanics (Wigner 1932; Leaf 1968), but they were later applied also to two-dimensional wave-trains on water surfaces (Alber 1978). We use a fluid model for both electrons and ions, thereby ignoring kinetic effects as linear and nonlinear Landau damping (Dysthe and Pécseli 1977; Shatashvili and Tsintsadze 1982; Melrose 1987). Formally, the inclusion of these latter effects is relatively easy in a linearized one-dimensional analysis, but the expressions become rather cumbersome.

There are several symbols used in the paper, and the notion of, for instance, wavenumber and frequency can have different interpretations. A summary Table 1 is included to clarify the meaning of the symbols used. Parts of the analysis are not standard, and therefore presented in some detail where appropriate.

2. Basic equations

As our basic equation for space-time evolution of the electric field $\mathbf{E}(\mathbf{r}, t)$ of weakly nonlinear electron plasma waves we take (Zakharov 1972; Thornhill and ter Haar 1978; Kono and Škorić 2010)

$$\nabla \cdot \left(\frac{\partial^2}{\partial t^2}\mathbf{E} - \upsilon^2 \nabla\nabla \cdot \mathbf{E} + \omega_{pe}^2 \mathbf{E} \right) = -\nabla \cdot \left(\omega_{pe}^2 \frac{\widetilde{n}}{n_0}\mathbf{E} \right). \tag{2.1}$$

The electron plasma frequency is ω_{pe}, the electron Debye length λ_D while υ represents the electron thermal velocity, here defined as $\upsilon \equiv \sqrt{3T_e/m}$. The electron density is $n \equiv n_0 + \widetilde{n}$.

We assume the waves to be electrostatic, i.e. $\nabla \times \mathbf{E} \approx 0$. With $\nabla\nabla \cdot \mathbf{E} = \nabla^2\mathbf{E} + \nabla \times \nabla \times \mathbf{E} \approx \nabla^2\mathbf{E}$ we obtain

$$\frac{\partial^2}{\partial t^2}\mathbf{E} - \upsilon^2 \nabla^2 \mathbf{E} + \omega_{pe}^2 \mathbf{E} = -\left(\omega_{pe}^2 \frac{\widetilde{n}}{n_0}\mathbf{E} \right). \tag{2.2}$$

The physical electric field is related to the amplitude (or envelope) field by $\frac{1}{2}\mathbf{E}(\mathbf{r}, t)e^{-i\omega t}$ + c.c., and it is assumed that the field amplitude $\mathbf{E}(\mathbf{r}, t)$ varies over a timescale much longer than the plasma period $2\pi/\omega_{pe}$. The rapidly varying, high-frequency part of the signal is removed by the standard substitution $\mathbf{E} \rightarrow \mathbf{E}(\mathbf{r}, t)\exp(-i\omega_o t)$, with ω_0 being the carrier wave frequency, here taken to be the

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plasma frequency ω_{pe}. When $\partial\mathbf{E}/\partial t \ll \omega_0\mathbf{E}$ we obtain

$$\left(i\frac{\partial}{\partial t}+\frac{1}{2}\frac{\upsilon^2}{\omega_{pe}}\nabla^2\right)\mathbf{E}=\frac{1}{2}\omega_{pe}\left(\frac{\tilde{n}}{n_0}\mathbf{E}\right). \tag{2.3}$$

Consistency requires that $\nabla\times(\tilde{n}\mathbf{E})=\tilde{n}(\nabla\times\mathbf{E})+(\nabla\tilde{n})\times\mathbf{E}\approx 0$: if not, the term on the right hand side of (2.3) will act as a source of an electromagnetic component. If we have $\nabla\times\mathbf{E}$ small, we require either $\nabla\tilde{n}$ parallel to $\mathbf{E}$, approximately, which means that we deal with essentially one-dimensional processes, or that $\nabla\tilde{n}/\tilde{n}$ is small, which means that the wave envelope varies slowly with the spatial coordinate, as compared to the wavelength of the Langmuir waves.

The expression (2.3) applies for prescribed variations of density $\tilde{n}$ but is most interesting for a self-consistent dynamic problem where the density perturbations are nonlinearly coupled to the density by

$$\left(\frac{\partial^2}{\partial t^2}-C_s^2\nabla^2\right)\frac{\tilde{n}}{n_0}=\frac{\varepsilon_0}{2Mn_0}\nabla^2|E|^2, \tag{2.4}$$

to be completed with initial conditions.

The basic set of (2.3) and (2.4), as they stand, can describe decay of Langmuir waves as well as modulational instabilities and wave collapse in three-dimensional plasmas.

2.1. *Equations for correlation functions*

A general analysis will address the full space-time varying correlation tensor given as $\langle E_\ell(t_1,\mathbf{r}_1)E_j^*(t_2,\mathbf{r}_2)\rangle$. We first derive equations which are suitable for an ensuing statistical average. In the following analysis we reduce the information to contain only the trace of this tensor, i.e. the correlation function $\langle\mathbf{E}(t_1,\mathbf{r}_1)\cdot\mathbf{E}^*(t_2,\mathbf{r}_2)\rangle$ of the electrostatic waves. We rewrite (2.3) for two sets of variables $\mathbf{r}_1, t_1$ and $\mathbf{r}_2, t_2$ (Alber 1978; Marcuvitz 1980; Bhakta and Majumder 1983; Dysthe et al. 1986). After complex conjugating one of the equations we multiply the two equations by $\mathbf{E}^*(t_2,\mathbf{r}_2)$ and $\mathbf{E}(t_1,\mathbf{r}_1)$, respectively. Subtracting the two resulting equations we find

$$\left[i\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)+\frac{1}{2}\frac{\upsilon^2}{\omega_{pe}}\left(\nabla_1^2-\nabla_2^2\right)\right]\rho=\frac{1}{2}\omega_{pe}\left(\eta_1-\eta_2\right)\rho, \tag{2.5}$$

with $\rho=\rho(\mathbf{r}_1,t_1,\mathbf{r}_2,t_2)\equiv\mathbf{E}(\mathbf{r}_1,t_1)\cdot\mathbf{E}^*(\mathbf{r}_2,t_2)$, where ρ is complex in general, while $\eta_j\equiv\tilde{n}_j/n_0$ with $j=1,2$. The ensemble average of ρ gives the space-time varying correlation function of the electrostatic wave field.

We now introduce average and separation coordinates $\mathbf{R}\equiv(\mathbf{r}_1+\mathbf{r}_2)/2$ and $\mathbf{r}\equiv\mathbf{r}_1-\mathbf{r}_2$, respectively, and similarly for the temporal variable $T\equiv(t_1+t_2)/2$ and $\tau\equiv t_1-t_2$, giving $t_1=T+\tau/2$, $\mathbf{r}_1=\mathbf{R}+\mathbf{r}/2$, etc. In terms of these new variables we have $\nabla_1=\frac{1}{2}\nabla_R+\nabla_r$, $\partial/\partial t_1=\frac{1}{2}\partial/\partial T+\partial/\partial\tau$, $\nabla_1^2=\frac{1}{4}\nabla_R^2+\nabla_r^2+\nabla_R\cdot\nabla_r$, etc. when differentiating $\rho=\rho(\mathbf{r}_1,t_1,\mathbf{r}_2,t_2)\to\rho(\mathbf{R},T,\mathbf{r},\tau)$. In a standard symbolic notation we also have

$$\begin{aligned}\eta_1&=\eta\left(\mathbf{R}+\frac{1}{2}\mathbf{r},T+\frac{1}{2}\tau\right)=\eta(\mathbf{R},T)\\&+\left\{\frac{1}{2}\mathbf{r}\cdot\nabla_R\eta(\mathbf{R},T)+\frac{1}{2}\tau\frac{\partial}{\partial T}\eta(\mathbf{R},T)\right\}^{(1)}+\ldots\\&+\frac{1}{n!}\left\{\frac{1}{2}\mathbf{r}\cdot\nabla_R\eta(\mathbf{R},T)+\frac{1}{2}\tau\frac{\partial}{\partial T}\eta(\mathbf{R},T)\right\}^{(n)}+\ldots\end{aligned} \tag{2.6}$$

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and similarly for $\eta_2 = \eta(\mathbf{R} - \frac{1}{2}\mathbf{r}, T - \frac{1}{2}\tau)$. All in all, we find the operator expression

$$\eta_1 - \eta_2 = 2\sum_{n=0}^{\infty} \frac{1}{(1+2n)!} \left\{ \frac{1}{2}\mathbf{r}\cdot\nabla_R \eta(\mathbf{R}, T) + \frac{1}{2}\tau \frac{\partial}{\partial T}\eta(\mathbf{R}, T) \right\}^{(1+2n)}.$$

We will also need

$$\eta_1 + \eta_2 = 2\sum_{n=0}^{\infty} \frac{1}{(2n)!} \left\{ \frac{1}{2}\mathbf{r}\cdot\nabla_R \eta(\mathbf{R}, T) + \frac{1}{2}\tau \frac{\partial}{\partial T}\eta(\mathbf{R}, T) \right\}^{(2n)}.$$

These two forms facilitate an ensuing Fourier transform. From (2.5) we then find after some algebra

$$\left(i\frac{\partial}{\partial T} + \frac{\upsilon^2}{\omega_{pe}}\nabla_R\cdot\nabla_r \right) \rho(\mathbf{R}, T, \mathbf{r}, \tau) =$$
$$\omega_{pe} \sum_n \frac{1}{(1+2n)!} \left\{ \frac{1}{2}\mathbf{r}\cdot\nabla_R \eta(\mathbf{R}, T) + \frac{1}{2}\tau \frac{\partial}{\partial T}\eta(\mathbf{R}, T) \right\}^{(1+2n)} \rho(\mathbf{R}, T, \mathbf{r}, \tau). \quad (2.7)$$

Care should be taken here to interpret the differential operations correctly. The operators in the summations originate from the difference $\eta_1 - \eta_2$, and refer only to $\eta(\mathbf{R}, T)$. A notation introduced in the following removes this ambiguity.

We now Fourier transform $\rho(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) = \rho(\mathbf{R}, T, \mathbf{r}, \tau)$ with respect to both the spatial and temporal separation variables, $\mathbf{r}$ and τ and denote this Fourier transform $F(\mathbf{R}, T, \mathbf{k}, \Omega)$ with $\mathbf{k}$ and Ω both real. For time stationary and spatially homogeneous processes, we hereby obtain the power spectrum for the electrostatic field as function of frequency Ω and wavevector $\mathbf{k}$. We use that $\mathscr{F}t[\mathbf{r}f(\mathbf{r})] = i\nabla_k \widehat{f}(\mathbf{k})$, where $\widehat{f}(\mathbf{k}) \equiv \mathscr{F}t[f(\mathbf{r})]$ denotes the Fourier transform and $\nabla_k \equiv (\partial/\partial k_x)\widehat{\mathbf{x}} + (\partial/\partial k_y)\widehat{\mathbf{y}} + (\partial/\partial k_z)\widehat{\mathbf{z}}$. In the present case we interpret the ensuing result as a set of coupled equations for the space-time variation of the local $(\Omega, \mathbf{k})$-power spectrum $F(\mathbf{R}, T, \Omega, \mathbf{k})$ of the electrostatic field at a position and time $(\mathbf{R}, T)$. We use the series expansions $\cos x = \sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$ and $\sin x = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)!$ when expressing the series expansion of the differential operators.

$$\left(\frac{\partial}{\partial T} + \frac{\upsilon^2}{\omega_{pe}}\mathbf{k}\cdot\nabla_R \right) F(\mathbf{R}, T, \mathbf{k}, \Omega) =$$
$$\omega_{pe}\widehat{S}(\mathbf{R}, \mathbf{k}, T, \Omega|\xi, T')\,\eta(\xi, T')F(\mathbf{R}, T, \mathbf{k}, \Omega). \quad (2.8)$$

In order to make it clear which variables the differential operators refer to, we introduced the operators (Wigner 1932; Leaf 1968; Tappert 1971)

$$\widehat{S}(\mathbf{R}, \mathbf{k}, T, \Omega|\xi, T')\eta(\xi, T')F(\mathbf{R}, T, \mathbf{k}, \Omega) \equiv$$
$$\int\int_{-\infty}^{\infty} d^3\xi\, dT'\ \delta(\mathbf{R} - \xi)\delta(T - T')$$
$$\times \sin\left(\frac{1}{2}\nabla_k\cdot\nabla_\xi - \frac{1}{2}\frac{\partial}{\partial\Omega}\frac{\partial}{\partial T'} \right) \eta(\xi, T')F(\mathbf{R}, T, \mathbf{k}, \Omega),$$

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and for later use also

$$\widehat{C}(\mathbf{R},\mathbf{k},T,\Omega|\xi,T')\eta(\xi,T')F(\mathbf{R},T,\mathbf{k},\Omega)\equiv$$
$$\int\int_{-\infty}^{\infty} d^3\xi\, dT'\ \delta(\mathbf{R}-\xi)\delta(T-T')$$
$$\times\cos\left(\frac{1}{2}\nabla_k\cdot\nabla_\xi-\frac{1}{2}\frac{\partial}{\partial\Omega}\frac{\partial}{\partial T'}\right)\eta(\xi,T')F(\mathbf{R},T,\mathbf{k},\Omega),$$

where the integrals run over all ξ-space, and also all times T'.

We obtained two equations by writing (2.3) as functions of two sets of variables labeled by 1 and 2. By the difference of these two equations we obtained (2.8) after some manipulations as outlined before. We can also take the *sum* of the two equations and after the very similar operations as before we obtain another equation in the form

$$\left(i\left(\frac{\partial}{\partial t_1}-\frac{\partial}{\partial t_2}\right)+\frac{1}{2}\frac{\upsilon^2}{\omega_{pe}}\left(\nabla_1^2+\nabla_2^2\right)\right)\rho=\frac{1}{2}\omega_{pe}\left(\eta_1+\eta_2\right)\rho. \tag{2.9}$$

After some algebra, following the previous procedure, we find

$$\left(2\Omega+\frac{\upsilon^2}{\omega_{pe}}\left(\frac{1}{4}\nabla_R^2-k^2\right)\right)F(\mathbf{R},T,\mathbf{k},\Omega)=$$
$$\omega_{pe}\widehat{C}(\mathbf{R},\mathbf{k},T,\Omega|\xi,T')\,\eta(\xi,T')F(\mathbf{R},T,\mathbf{k},\Omega). \tag{2.10}$$

The relations (2.8) and (2.10) presume that $\eta(\mathbf{k},t)$ is given: it could in principle be representing some a priori given density variations, a dc-density gradient for instance (Kontar 2001; Kontar and Pécseli 2002) where η is time-stationary so the expressions become somewhat simpler. For the present study we are mostly interested in a self-consistent nonlinear model relating $F(\mathbf{R},T,\mathbf{k},\Omega)$ and $\eta(\mathbf{R},T)$.

Most related studies restrict the analysis to the a difference equation as (2.5), but it should be emphasized that the starting point is two complex equations, in the present case (2.3) for two sets of variables $(t_1,\mathbf{r}_1)$ and $(t_2,\mathbf{r}_2)$. Reducing this set to only one implies loss of information, which should be accounted for by some means. Here we retain both the sum and difference equations (2.5) and (2.9), which contain the same information as the two original equations, albeit in a different form which is at first sight not as accessible. A detailed analysis of the relations (2.8) and (2.10) will, however, make the physical content and the usefulness of these equations more evident.

2.2. *Equations for the low-frequency density variations*

We now relate the variations of the high-frequency electric field to the slow variations in plasma density. The simplest relation relevant here assumes that ponderomotive forces associated with the spatial variation of the high-frequency wave field set the electrons into slow bulk motions, and the resulting charge imbalance gives rise to slowly varying electric fields that sets also the ion component into motion. If the variations are very slow, as measured by the ion plasma period $2\pi/\Omega_{pi}$, we can use a relation for normalized quasi-neutral plasma density variations (2.4) and have with the present variables

$$\left(\frac{\partial^2}{\partial T^2}-C_s^2\nabla_R^2\right)\eta(\mathbf{R},T)=\frac{\varepsilon_0}{4Mn_0}\nabla_R^2|E(\mathbf{R},T)|^2, \tag{2.11}$$

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where $C_s \equiv \sqrt{T_e/M}$ is the ion sound speed, assuming cold ions and where we have $|E(\mathbf{R}, T)|^2 = \int \int_{-\infty}^{\infty} F(\mathbf{R}, T, \mathbf{k}, \Omega) dk^3 d\Omega$.

The relation (2.11) has to be completed with initial conditions. It is common practice to assume the simple limit for slow time variations

$$-C_s^2 \eta(\mathbf{R}, T) \approx \frac{\varepsilon_0}{4Mn_0} |E(\mathbf{R}, T)|^2, \tag{2.12}$$

but this approximation implicitly imposes some restrictions on the initial conditions, and these conditions need not always apply.

The equations (2.8) and (2.10) together with (2.11) are so far merely another representation of the original set (2.3) and (2.4), and are capable of describing the same physics. We have two functions and four variables, where two of the variables have been Fourier transformed. Symbolically we can write for the functions η and $\rho \rightarrow F$, with variables $\mathbf{R}, T, \mathbf{r} \rightarrow \mathbf{k}, \tau \rightarrow \Omega$. In the following we focus on the modulational instability of a Langmuir wave spectrum and demonstrate the advantage of the present representation. Differences between the two sets of model equations will result from a statistical averaging and in particular by an assumption of statistical homogeneity which is restrictive. The problem is not fully defined before the ensemble is specified: if all realizations were identical, the averaging would have no effect. The most common assumption concerning the ensemble is the random-phase approximation (Nicholson 1983), where the phases of the individual Fourier components are assumed to be uniformly distributed over $\{0; 2\pi\}$, while the Fourier amplitudes can in principle be the same in all realizations, or have a statistical distribution as well.

2.3. *Plane-wave solutions*

Inserting $F = F_0(\mathbf{k}, \Omega)$ into (2.8) gives a trivial null-result, while we by (2.10) obtain the condition $(2\Omega - (v^2/\omega_{pe})k^2 - \omega_{pe}\eta_0)F_0(\mathbf{k}, \Omega) = 0$. The first term in the expansion of the $\widehat{C}$-operator gives unity. Recalling that $x\delta(x) = 0$, we have that an equation of the form $\xi F(\xi) = 0$ has the solution $F = A\delta(\xi)$, with A being a constant to be determined by an independent normalization condition (van Kampen and Felderhof 1967). Consequently, we obtain the steady state solution

$$F_0(\mathbf{k}, \Omega) = G_0(\mathbf{k}) \delta \left(\Omega - \frac{1}{2} \frac{v^2}{\omega_{pe}} k^2 - \frac{1}{2} \omega_{pe} \eta_0 \right). \tag{2.13}$$

Since the model does not include internal frequency-wavenumber couplings in the spectrum, we have $G_0(\mathbf{k})$ to be an arbitrary function, determined by the initial conditions. In the argument of the δ-function we recognize the approximation to the linear dispersion relation

$$\Omega + \omega_0 = \sqrt{v^2 k^2 + \omega_{pe}^2} \approx \omega_{pe} \left(1 + \frac{3}{2} (k\lambda_D)^2 \right),$$

as obtained from (2.2), and recalling our approximation $\omega_0 \approx \omega_{pe}$ so that the local plasma frequency is $\omega_{pe}\sqrt{1+\eta} \approx \omega_{pe}(1 + \frac{1}{2}\eta)$.

2.4. *Limit of long scale-lengths and slow variations*

The expressions given by (2.8) and (2.10) have a simpler limit when the variations with R and T are slow. Retaining everywhere the lowest order terms in gradients of

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R and derivatives of T, we find from (2.8) the approximation

$$\left(\frac{\partial}{\partial T}+\frac{\upsilon^2}{\omega_{pe}}\mathbf{k}\cdot\nabla_R\right)F(\mathbf{R},T,\mathbf{k},\Omega)=$$
$$\frac{\omega_{pe}}{2}\nabla_R\eta(\mathbf{r},T)\cdot\nabla_k F(\mathbf{R},T,\mathbf{k},\Omega)-\frac{\omega_{pe}}{2}\frac{\partial}{\partial T}\eta(\mathbf{r},T)\frac{\partial}{\partial\Omega}F(\mathbf{R},T,\mathbf{k},\Omega). \tag{2.14}$$

By ignoring also the ∇_R^2 operator on the left side of (2.10) we find

$$\left(2\Omega-k^2\frac{\upsilon^2}{\omega_{pe}}\right)F(\mathbf{R},T,\mathbf{k},\Omega)=\omega_{pe}\eta(\mathbf{R},T)F(\mathbf{R},T,\mathbf{k},\Omega). \tag{2.15}$$

The solution of (2.15) is obtained as in Sec. 2.3 to be

$$F(\mathbf{R},T,\mathbf{k},\Omega)=G(\mathbf{k})\delta\left(\Omega-\frac{1}{2}k^2\frac{\upsilon^2}{\omega_{pe}}-\frac{1}{2}\omega_{pe}\eta(\mathbf{r},T)\right), \tag{2.16}$$

where the argument of the δ-function can be interpreted as a local dispersion relation with $\omega_{pe}(1+\frac{1}{2}\eta(\mathbf{r},T))$ being the local plasma frequency.

2.5. *The wave kinetic equation*

Taking the original equations (2.8) and (2.10) with vanishing time separation $t_1=t_2$ so that $T=t$ and $\tau=0$, we find an equation describing the time evolution of the local wavenumber spectrum $F(\mathbf{R},t,\mathbf{k})$ (Kadomtsev 1965; Marcuvitz 1980).

$$\left(\frac{\partial}{\partial t}+\frac{\upsilon^2}{\omega_{pe}}\mathbf{k}\cdot\nabla_R\right)F(\mathbf{R},t,\mathbf{k})=\frac{\omega_{pe}}{2}\nabla_R\eta(\mathbf{R},t)\cdot\nabla_k F(\mathbf{R},t,\mathbf{k}), \tag{2.17}$$

which is now decoupled from (2.15). The result (2.17) could as well have been obtained from (2.14) after integration with respect to Ω.

The form (2.17) has an interesting similarity to the Vlasov equation, here with a force given by the local density gradient $-\nabla_R\eta(\mathbf{r},t)$. For example, a linear density variation, which can be realized at least locally with $\eta=AX/\mathscr{L}$, gives a constant force $-\nabla_R\eta=-(A/\mathscr{L})\hat{\mathbf{x}}$ on a wavepacket represented by a wavevector $\mathbf{k}$.

A self consistent space-time evolution of the plasma density is obtained from (2.11) as

$$\left(\frac{\partial^2}{\partial t^2}-C_s^2\nabla_R^2\right)\eta(\mathbf{R},t)=\frac{\varepsilon_0}{4Mn_0}\nabla_R^2\int_{-\infty}^{\infty}F(\mathbf{R},t,\mathbf{k})dk^3, \tag{2.18}$$

which relates η to the wavenumber distribution F. This is a *dynamic* equation in contrast to the Poisson equation that relates electric fields and charge density in the classical studies of electrostatic waves in plasmas.

The form (2.17) was obtained by semi-heuristic arguments (Vedenov et al. 1967; Hasegawa 1975; Breizman and Malkin 1980), with some earlier related studies by Camac et al. (1961). Other closely related studies can be found as well (Bloomberg 1972; Kono and Yajima 1976; Kono 1978; Tracy and Boozer 1989; Mendonça and Bingham 2002; Mendonça et al. 2014). A formalism that includes also parts of the present results was presented by Tappert (1971). As shown here, (2.17) is merely a limiting case of a more general result.

The wave kinetic equation exhibits many phenomena related to collisionless plasmas, such as 'echo' (Nycander et al. 1986). The similarity between (2.17) and the Vlasov equation allows also BGK-like (Bernstein et al. 1957) steady state solutions to be found and also this question has been analyzed (Hasegawa 1975, 1977; Pécseli and Trulsen 1990). Solutions in the form of a localized 'plasmon blob' were found

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in numerical simulation by Pécseli and Trulsen (1990, 1992). These solutions should not be confused with the exact nonlinear soliton-solutions of (2.3) together with (2.4). These stationary soliton-solutions deal with spatially modulated plane nonlinear waves with coherent phase relations.

The similarity between (2.17) and the Vlasov equation is interesting also by inviting a numerical method for simulating the space-time evolution of nonlinear Langmuir waves by use of Particle-in-Cell (PIC) methods (here properly denoted as 'wavenumber-in-cell' method) or by more standard Vlasov equation solvers. The sketchy ideas of Tappert and Cole (1971) were later implemented in practice and results from numerical simulations in one spatial dimension were presented by Pécseli and Trulsen (1990, 1992). One numerical problem should be emphasized in relation to use of (2.17) together with (2.11): the numerical noise originating from the discreteness of the PIC particles introduces through (2.11) a proportional noise in η, and this noise is then further emphasized by the differentiation implied in the force term $-\nabla_R \eta(\mathbf{r}, t)$. In the standard problem for analyzing electrostatic plasma perturbations by PIC methods, this problem is of minor significance in comparison since Poisson's equation $\nabla \cdot \mathbf{E} = e(n_i - n_e)/\varepsilon_0$ gives a reduction in the electric field noise originating from the discrete particles. For practical application of the wavenumber in cell method the noise has to be reduced for instance by filtering (Pécseli and Trulsen 1992).

3. Fluid closure

The simplest closure of the equations can be obtained by a fluid-like hierarchy. We can introduce a 'plasmon density' as

$$\mathcal{N}(\mathbf{R}, T) \equiv \int\int_{-\infty}^{\infty} F(\mathbf{R}, T, \mathbf{k}, \Omega) dk^3 d\Omega, \tag{3.1}$$

and an effective phase velocity

$$\Upsilon(\mathbf{R}, T) \equiv \frac{\upsilon^2}{\omega_{pe}\mathcal{N}} \int\int_{-\infty}^{\infty} \mathbf{k} F(\mathbf{R}, T, \mathbf{k}, \Omega) dk^3 d\Omega \,.$$

Integrating (2.8) with respect to Ω and $\mathbf{k}$ we obtain a continuity equation of the form

$$\frac{\partial}{\partial T}\mathcal{N} + \nabla_R \cdot (\Upsilon \mathcal{N}) = 0, \tag{3.2}$$

justifying the interpretation of Υ in terms of a velocity. We introduce a general notation by an 'overbar' as $\overline{Y} \equiv \int\int_{-\infty}^{\infty} Y F(\mathbf{R}, T, \mathbf{k}, \Omega) dk^3 d\Omega / \mathcal{N}$ and find from (2.10)

$$\overline{\Omega} + \frac{\upsilon^2}{\omega_{pe}} \frac{1}{8\mathcal{N}} \nabla_R^2 \mathcal{N} - \frac{1}{2} \frac{\upsilon^2}{\omega_{pe}} \overline{k^2} = \frac{1}{2} \omega_{pe} \eta, \tag{3.3}$$

in terms of $\mathcal{N}$ defined by (3.1) and η. This relation has the form of a nonlinear and non-local dispersion relation for $\overline{\Omega}$ and $\overline{k^2}$. The right hand side enters as a correction to the local plasma frequency, while $\frac{1}{8}(\upsilon^2/\omega_{pe})\nabla_R^2 \mathcal{N}/\mathcal{N}$ is the non-local contribution. The relation contains no time differentiation so it is local in time. The result (3.3) is exact, given the premises in terms of (2.3), while the local dispersion relation obtained by (2.16) is an approximation for slowly varying η and $\mathcal{N}$. On the other hand (2.16) applies for k as a variable, giving a variable ω, while (3.3) refers to averages over the spectra, i.e. $\overline{\Omega}$ and $\overline{k^2}$.

From (2.8) we obtain an equivalent of a momentum equation for $\Upsilon(\mathbf{R}, T)$ by multiplication with $\mathbf{k}$ and then integrating. A hierarchy of fluid-like equations is in

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principle recovered also for the present problem, although the sin and cos operators offer non-trivial differences and we also have *two* basic equations, (2.8) and (2.10). In addition to these equations we have (2.18) where $\nabla_R^2 \mathcal{N}$ enters the right hand side. A closure will require $\overline{\mathbf{kk}}$ to be expressed in terms of Υ, etc. This problem will not be pursued here. The important new result of the present discussion is the non-local average dispersion relation (3.3).

4. Ensemble averages

As they stand, the set of basic equations (2.8), (2.10) and (2.11) are deterministic, and contain basically the same information as the original two equations obtained from (2.3) in terms of the two sets of coordinates $(\mathbf{r}_1, t_1)$ and $(\mathbf{r}_2, t_2)$. For proceeding with the analysis we assume that a statistical analysis is meaningful (Zaslavskiĭ and Sagdeev 1967). Ensemble averaging of the original set of equations (2.3) and (2.11) does not give useful results for the present analysis. The ensemble average $\langle \rho(\mathbf{R}, T, \mathbf{r}, \tau) \rangle$ is on the other hand interpreted as the space-time varying local auto-correlation function and the corresponding average $\langle F(\mathbf{R}, T, \mathbf{k}, \Omega) \rangle$ is the space-time varying local power spectrum of the electric field fluctuations. In case the system constitutes a homogeneous time stationary random process these expressions will be independent of $\mathbf{R}$ and T, and we find the usual expression for the correlation function $\langle \rho(\mathbf{r}, \tau) \rangle$ and power-spectrum $\langle F(\mathbf{k}, \Omega) \rangle$. This ensemble averaging is straight forward for most of the terms, since they are linear in ρ and F, except from the product $\langle \eta(\mathbf{r}, T) F(\mathbf{R}, T, \mathbf{k}, \Omega) \rangle$ appearing in (2.8) and (2.10). To discuss the averaging of these terms we note that the one-dimensional version (2.11) can be solved analytically with the result (Scott 1982; Dysthe et al. 1986)

$$\eta(Z, T) = -\frac{\varepsilon_0}{4Mn_0} \int_0^T dT' \int_{Z-C_s(T-T')}^{Z+C_s(T-T')} dZ' \frac{\partial^2}{\partial Z'^2} |E(Z', T')|^2. \tag{4.1}$$

The assumed initial condition is $\eta(Z, T = 0) = 0$, which is natural if the high-frequency wave field is to be generated initially in an otherwise uniform plasma background density. This initial condition would not be consistent with the approximation (2.12). More general conditions at $T = 0$ can be introduced by adding the appropriate condition to (4.1). Partial integration is not making (4.1) any simpler because the integral limits have to be retained.

To get some insight into the structure of the solution (4.1) we assume merely as an illustration, that the integrand has an analytical structure propagating with some characteristic velocity U, which could for instance represent a group velocity of a wavepacket, i.e. $\partial^2 |E(Z', T')|^2 / \partial Z'^2 \sim H(Z' - UT')$, where H is derived from the shape of the wavepacket. For this simple one-dimensional model case we find

$$\eta(Z, T) = -\frac{\varepsilon_0}{8Mn_0} \left(2\frac{H(Z - UT)}{C_s^2 - U^2} - \frac{H(Z - C_s T)}{C_s(C_s - U)} - \frac{H(Z + C_s T)}{C_s(C_s + U)} \right). \tag{4.2}$$

The first term in the parenthesis is the response in η to the forcing by H, while the two other terms represent left and right traveling sound pulses ('free sound') that are needed in order to fulfill the assumed initial condition. From this simple model we can conclude that for an infinitely extended initially modulated wave field we might for late times at any spatial position expect sound pulses to arrive from widely separated spatial regions. If the separation of these regions is sufficiently large, we expect that the corresponding contributions can be taken to be statistically independent (Benney

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and Saffman 1966). When η at any position simultaneously receives many statistically independent contributions, we might by the central limit theorem argue that η is a Gaussian random process.

If the initial condition on $|\nabla\phi|^2$ is spatially compact, we can for late times assume that the sound pulses corresponding to the two last terms in (4.2) have disappeared, so that the precise initial condition is of minor importance. If the spatial support of the initial condition is $\mathscr{L}$, the time for this assumption to be applicable is $t > \mathscr{L}/C_s$. In the present study we will be mostly interested in spatially widely distributed systems, and will consider the limit where $\mathscr{L}/C_s \to \infty$. For the case studied here we can assume that η has a local Gaussian distribution to a good approximation (Benney and Saffman 1966).

In (2.8) and (2.10) we encounter ensemble averages as $\langle\eta(\xi, T')F(\mathbf{R}, T, \mathbf{k}, \Omega)\rangle$. In the present work we will advocate the approximation

$$\langle\eta(\xi, T')F(\mathbf{R}, T, \mathbf{k}, \Omega)\rangle \approx \langle\eta(\xi, T')\rangle\langle F(\mathbf{R}, T, \mathbf{k}, \Omega)\rangle. \tag{4.3}$$

If the electric field fluctuations associated with the electron waves constituted a Gaussian random process, the relation (4.3) would be exact. We have good arguments for suggesting that $\eta(\mathbf{R}, T)$ is a Gaussian random process, but this does not necessarily imply that this will be true also for $F(\mathbf{R}, T, \mathbf{k}, \Omega)$. For a one-dimensional limit, the integral in (4.1) is also randomly varying, and we can safely assume that for large times it follows a Gaussian statistic and this is after all what is relevant here. We can assume that making the assumption (4.3) will not imply any significant differences in the statistical properties of the integral term in (4.1).

Retaining the assumption of Gaussian random processes, we could use that $\eta(\mathbf{R}, T)$ is functionally dependent on $F(\mathbf{R}, T, \mathbf{k}, \Omega)$ through (4.1). Rather than (4.3) we could use the Novikov–Furutsu theorem (Furutsu 1963; Novikov 1964) to obtain $\langle\eta(\xi, T')F(\mathbf{R}, T, \mathbf{k}, \Omega)\rangle$. It is, however, doubtful that any significant improvement will be obtained by this, and by (4.1) it assumes an approximation for one spatial dimension only.

5. Perturbation analysis

In the following we consider ensemble averaged quantities and carry out a stability analysis of $\langle F_0(\mathbf{k}, \Omega)\rangle$ by a perturbation analysis of $\langle F(\mathbf{R}, T, \mathbf{k}, \Omega)\rangle = F_0(\mathbf{k}, \Omega) + \langle F_1(\mathbf{R}, T, \mathbf{k}, \Omega)\rangle$ with $\langle F_1\rangle \ll F_0$ and $\langle\eta\rangle = \eta_0 + \langle\eta_1\rangle$ with $\langle\eta_1\rangle \ll \eta_0$. Ignoring second-order terms we find that several terms disappear in the right hand sides of (2.8) and (2.10). We obtain

$$\left(\frac{\partial}{\partial T} + \frac{\upsilon^2}{\omega_{pe}}\mathbf{k}\cdot\nabla_R\right)\langle F_1(\mathbf{R}, T, \mathbf{k}, \Omega)\rangle$$
$$= \omega_{pe}\widehat{S}(\mathbf{R}, \mathbf{k}, T, \Omega|\xi, T')\langle\eta_1(\xi, T')\rangle F_0(\mathbf{k}, \Omega)\,, \tag{5.1}$$

and similarly

$$\left(2\Omega + \frac{\upsilon^2}{\omega_{pe}}\left(\frac{1}{4}\nabla_R^2 - k^2\right)\right)\langle F_1(\mathbf{R}, T, \mathbf{k}, \Omega)\rangle$$
$$= \omega_{pe}\widehat{C}(\mathbf{R}, \mathbf{k}, T, \Omega|\xi, T')\langle\eta_1(\xi, T')\rangle F_0(\mathbf{k}, \Omega), \tag{5.2}$$

with time differentiation entering on the right hand side only through $\widehat{C}$.

We now Fourier transform with respect to T, so that $\partial/\partial T \to -iW$, and spatial positions so that $\nabla_R \to i\mathbf{K}$. The operators $\widehat{C}$ and $\widehat{S}$ can be expressed in terms of series

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containing W and $\mathbf{K}$, best seen by (2.7). The Fourier transform of a function will be identified by $\check{}$, together with its arguments. We use the series expansions of cos$h$ and sin$h$ together with the appropriate Taylor expansions of $F_0(\mathbf{k}\pm\frac{1}{2}\mathbf{K}, \Omega\pm\frac{1}{2}W)$, see also (2.6), and find

$$\left(W-\frac{\upsilon^2}{\omega_{pe}}\mathbf{k}\cdot\mathbf{K}\right)\left\langle\check{F}_1(\mathbf{K},W,\mathbf{k},\Omega)\right\rangle = -\frac{1}{2}\omega_{pe}\langle\check{\eta}_1(\mathbf{K},W)\rangle\left(F_0\left(\mathbf{k}+\frac{1}{2}\mathbf{K},\Omega+\frac{1}{2}W\right)-F_0\left(\mathbf{k}-\frac{1}{2}\mathbf{K},\Omega-\frac{1}{2}W\right)\right). \quad (5.3)$$

Similarly we find

$$\left(2\Omega-\frac{\upsilon^2}{\omega_{pe}}\left(\frac{1}{4}K^2+k^2\right)\right)\left\langle\check{F}_1(\mathbf{K},W,\mathbf{k},\Omega)\right\rangle = \frac{1}{2}\omega_{pe}\langle\check{\eta}_1(\mathbf{K},W)\rangle\left(F_0\left(\mathbf{k}+\frac{1}{2}\mathbf{K},\Omega+\frac{1}{2}W\right)+F_0\left(\mathbf{k}-\frac{1}{2}\mathbf{K},\Omega-\frac{1}{2}W\right)\right) + \omega_{pe}\eta_0\left\langle\check{F}_1(\mathbf{K},W,\mathbf{k},\Omega)\right\rangle. \quad (5.4)$$

Both (5.3) and (5.4) have

$$\langle\check{\eta}_1(\mathbf{K},W)\rangle = -\frac{\varepsilon_0}{4Mn_0}K^2\frac{\int\int\left\langle\check{F}_1(\mathbf{K},W,\mathbf{k}',\Omega')\right\rangle dk'^3d\Omega'}{C_s^2K^2-W^2}. \quad (5.5)$$

Generally we have complex W. Using the background steady state solution (2.13) with the general relation $g(x)\delta(x-a)=g(a)\delta(x-a)$, we obtain, for instance, from (5.3) the expression

$$\langle\check{F}_1(\mathbf{K},W,\mathbf{k},\Omega)\rangle = \frac{\varepsilon_0\omega_{pe}}{8Mn_0}K^2\frac{\int\int\langle\check{F}_1(\mathbf{K},W,\mathbf{k}',\Omega')\rangle dk'^3d\Omega'}{C_s^2K^2-W^2} \times\frac{F_0\left(\mathbf{k}+\frac{1}{2}\mathbf{K},\Omega+\frac{1}{2}W\right)-F_0\left(\mathbf{k}-\frac{1}{2}\mathbf{K},\Omega-\frac{1}{2}W\right)}{W-\frac{\upsilon^2}{\omega_{pe}}\mathbf{k}\cdot\mathbf{K}}. \quad (5.6)$$

We could use the same procedure on (5.4) and obtain the same result: seemingly one of the equations is redundant. This is, however, an observation based on the restricted linearized set of equations, and will not be true for the ensemble averaged full nonlinear set (2.8) and (2.10).

With complex W there are no problems in integrating (5.6) and we find the implicit complex dispersion relation

$$W^2-C_s^2K^2 = -\frac{\varepsilon_0\omega_{pe}}{8Mn_0}K^2\int_{-\infty}^{\infty}\frac{G_0\left(\mathbf{k}+\frac{1}{2}\mathbf{K}\right)-G_0\left(\mathbf{k}-\frac{1}{2}\mathbf{K}\right)}{W-\frac{\upsilon^2}{\omega_{pe}}\mathbf{k}\cdot\mathbf{K}}dk^3 = \frac{\varepsilon_0\upsilon^2}{8Mn_0}K^4\int_{-\infty}^{\infty}\frac{G_0(\mathbf{k}')}{(W-\frac{\upsilon^2}{\omega_{pe}}\mathbf{k}'\cdot\mathbf{K})^2-(\frac{1}{2}\frac{\upsilon^2}{\omega_{pe}}K^2)^2}dk'^3, \quad (5.7)$$

where now the Ω-integration has been carried out explicitly by use of (2.13) and we write e.g.

$$G_0\left(\mathbf{k}+\frac{1}{2}\mathbf{K}\right)\equiv\int_{-\infty}^{\infty}F_0\left(\mathbf{k}+\frac{1}{2}\mathbf{K},\Omega+\frac{1}{2}W\right)d\Omega=\int_{-\infty}^{\infty}F_0\left(\mathbf{k}+\frac{1}{2}\mathbf{K},\Omega'\right)d\Omega'.$$

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As $\mathrm{Im}\{W\} \to 0$ the integration paths for (5.6) or in (5.7) should be understood as Landau contours when F_0 or G_0 are continuous functions.

The expression (5.7) can be simplified by integrating over $\mathbf{k}'$-components perpendicular to $\mathbf{K}$. By $k_\parallel$ we denote the $\mathbf{K}$-parallel component of $\mathbf{k}'$ and have

$$W^2 - C_s^2 K^2 = \frac{\varepsilon_0 \upsilon^2}{8 M n_0} K^4 \int_{-\infty}^{\infty} \frac{G_* \left(k_\parallel\right)}{(W - \frac{\upsilon^2}{\omega_{pe}} k_\parallel K)^2 - (\frac{1}{2}\frac{\upsilon^2}{\omega_{pe}} K^2)^2} dk_\parallel, \tag{5.8}$$

where $G_*(k_\parallel) \equiv \int_{-\infty}^{\infty} G_0(\mathbf{k}') d\mathbf{k}'_\perp$ results from integration with respect to components of $\mathbf{k}'$ perpendicular to $\mathbf{K}$.

For $K \to \pm\infty$ we find

$$\lim_{K \to \pm\infty} (W^2 - C_s^2 K^2) = -\frac{\varepsilon_0 \omega_{pe}^2}{2 M n_0 \upsilon^2} \int G_0 \left(\mathbf{k}'\right) dk'^3 ,$$

i.e. $W^2 \approx C_s^2 K^2$ for $K \to \pm\infty$ and the wave spectrum is stable in this limit.

For $K \to 0$ we find $W^2 \to 0$ and the spectrum is stable also in this limit. These conclusions hold for any $G_0(\mathbf{k})$. Consequently we can argue that in case the system is unstable, there will most likely be at least one finite value for K that separates the stable and unstable range of frequencies W, a possible exception being that the system is stable only asymptotically as $K \to \pm\infty$. In particular, for certain $G_0(\mathbf{k})$ we might find that the system is stable for all K.

5.1. *Narrow wavebands*

The simplest case found is one allowing the approximation $G_0 = A^2 \delta(\mathbf{k} - \mathbf{k}_0)$ in (5.7). The resulting dispersion relation can have pairs of complex-conjugate solutions where one corresponds to instability. Also the angle θ between $\mathbf{k}_0$ and $\mathbf{K}$ enters, the result becoming particularly simple when $\mathbf{k}_0 \perp \mathbf{K}$, corresponding to a filamentation instability. This limiting case is particularly interesting by representing an absolute (rather than convective) instability. By (5.8) we have

$$(W^2 - C_s^2 K^2) \left(\left(W - \frac{\upsilon^2}{\omega_{pe}} \mathbf{k}_0 \cdot \mathbf{K} \right)^2 - \frac{\upsilon^4}{4\omega_{pe}^2} K^4 \right) = \frac{\varepsilon_0 \upsilon^2}{8 M n_0} K^4 A^2. \tag{5.9}$$

We thus have two natural modes of wave propagation with dispersion relations $W = \pm C_s K$ (i.e. ion acoustic sound waves) and $W = (\upsilon^2/\omega_{pe})(\mathbf{k}_0 \cdot \mathbf{K} \pm \frac{1}{2} K^2)$, corresponding to propagating and dispersing wavepackets. These two waveforms are coupled by the finite amplitude, nonlinear effects accounted for by the right hand side of (5.9). If we take the weak turbulence, long wavelength limit of the wave kinetic equation as discussed in Sec. 2.5, the K^4-term on the left side of (5.9) will be missing, and the wavepackets are propagating with the group velocity $\upsilon^2 k_0/\omega_{pe}$, but not dispersing.

For presenting a normalized figure, we introduce normalizations for frequency $W' \equiv (W/\Omega_{pi})\sqrt{M/m}$ and for wavenumbers $\mathbf{K}' \equiv \mathbf{K}\lambda_{De}\sqrt{M/m}$ and also $\mathbf{k}'_0 \equiv 3\mathbf{k}_0\lambda_{De}\sqrt{M/m}$, while the coupling coefficient becomes $C' \equiv (\frac{1}{2}\varepsilon_0 A^2/n_0 T_e)\frac{3}{4}(M/m)$. Relevant Langmuir wavenumbers will be in the range $0 < |k_0| \leqslant \frac{1}{3}\sqrt{m/M}/\lambda_{De}$, while low-frequency modulations will have frequencies $W \leqslant \Omega_{pi}\sqrt{m/M}$ and wavenumbers $|K| \leqslant \sqrt{m/M}/\lambda_{De}$, giving $|k'_0| \leqslant 1$, $W' \leqslant 1$ and $|K'| \leqslant 1$. We also have $|W'/K'| = |W/K|/\lambda_{De}\Omega_{pi} = |W/K|/C_s$. Taking a wave with energy density $\frac{1}{2}\varepsilon_0 E^2 \approx (m/M) n_0 T_e$ will imply $C' \approx 3/4$.

The expression (5.9) with real coefficients gives a fourth-order equation for W. A pair of complex roots will imply instability. Plotting $G(W', K') \equiv (W'^2 - K'^2)((W' -$

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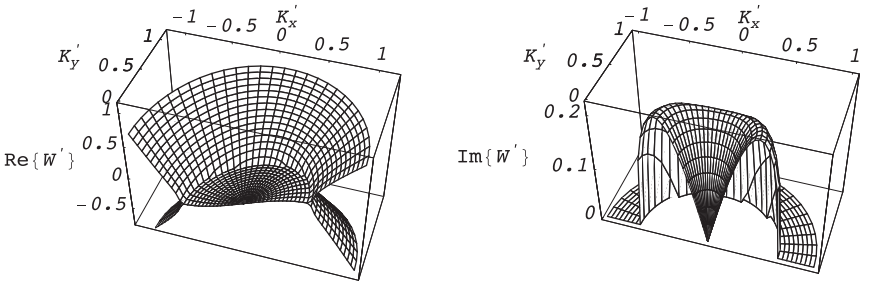


FIGURE 1. Numerical solution of the normalized equation (5.9) in the form $(W'^2 - K'^2)((W' - \mathbf{k}_0' \cdot \mathbf{K}')^2 - \frac{9}{4}K'^4) = C'K'^4$. We have taken $C' = 0.75$ and $k_0' = 0.75$ for the sake of presentation. Real and imaginary parts of the normalized frequency are shown. The largest growth rates for the modulational instability are found for $\mathbf{K} \parallel \mathbf{k}_0 \parallel \hat{\mathbf{x}}$.

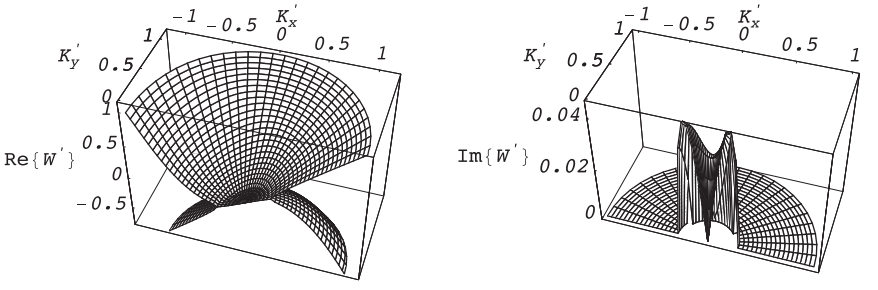


FIGURE 2. Numerical solution of $(W'^2 - K'^2)((W' - \mathbf{k}_0' \cdot \mathbf{K}')^2 - \frac{9}{4}K'^4) = C'K'^4$, here with $C' = 0.1$ and all other parameters as in Fig. 1. Note the large anisotropy (i.e. stronger θ-dependence) of the growth rate for the present small amplitude case, as compared to the large amplitude case in Fig. 1.

$\mathbf{k}_0' \cdot \mathbf{K}')^2 - \frac{9}{4}K'^4)/K'^4$ we can determine values for $\mathbf{K}'$ that makes $G(W', K') = C'$. Depending on parameters we can find cases with four real roots, two real and a set of two complex conjugate roots, and finally there is for extreme conditions also a possibility for two sets of complex conjugate roots. Real and imaginary parts of the frequency for unstable solutions of the normalized version of (5.9) are shown in Figs. 1 and 2 for two amplitude levels. For large amplitudes the growth rates are almost isotropic in wavenumber space, for small amplitudes almost one-dimensional.

An approximate solution of (5.9) can be found as

$$W \approx \frac{\upsilon^2}{\omega_{pe}} \mathbf{k}_0 \cdot \mathbf{K} \pm \frac{1}{2} \sqrt{\frac{\upsilon^4}{\omega_{pe}^2} K^4 + \frac{\varepsilon_0 \upsilon^2}{Mn_0} \frac{K^2 A^2}{\left(\frac{\upsilon^2}{\omega_{pe}} \mathbf{k}_0 \cdot \hat{\mathbf{K}}\right)^2 - C_s^2}}, \tag{5.10}$$

where $\hat{\mathbf{K}}$ is a unit vector along $\mathbf{K}$. In this limit we have the possibility for instability for small K^2 when $|\upsilon^2 \mathbf{k}_0 \cdot \hat{\mathbf{K}}/\omega_{pe}| < C_s$. By comparison with the numerical solution in Fig. 1 we find that (5.10) offers a satisfactory approximation for a wide range of parameters. The instability growth rate can become large as $\upsilon^2 \mathbf{k}_0 \cdot \hat{\mathbf{K}}/\omega_{pe} \to C_s$, corresponding to the case where wavepackets propagating with their group velocity become resonant with sound waves. The singularity at exact resonance is avoided by solving the full equation (5.9) where all growth rates are finite.

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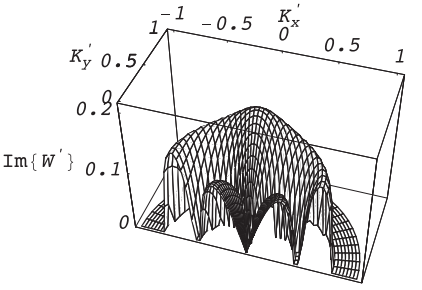


FIGURE 3. Growth rates for the modulational instability of two narrow-band counter-propagating wave-trains centered at wavenumbers $\pm\mathbf{k}_0$.

A characteristic critical modulational wavenumber K_c is found for the filamentation case (where $\mathbf{k}_0 \cdot \widehat{\mathbf{K}} = 0$) as

$$|K_c| = \frac{\omega_{pe}}{\upsilon}\sqrt{\frac{\varepsilon_0 A^2}{n_0 T_e}}\,.$$

This wavenumber is found to be an important parameter for the present study, and it will be used as a later reference. In normalized units we have $K_c' = K_c\lambda_D\sqrt{M/m} = \frac{2}{3}\sqrt{2C'}$. If we take $\frac{1}{2}\varepsilon_0 A^2 \approx (m/M)n_0 T_e$ we find $|K_c|\lambda_{De} \approx \sqrt{m/M}$. Modulationally unstable wavenumbers are smaller or comparable to the maximum unstable wavenumber $K_m = K_c/(1 - (\upsilon^2/\omega_{pe})\mathbf{k}_0 \cdot \widehat{\mathbf{K}}/C_s^2)$ within the approximations implied in (5.10).

5.2. *Two wave modes*

Langmuir waves can be reflected by solid boundaries or density enhancements and we can easily envisage a situation where we have two counter-propagating wave-trains. If we have an ideal 'mirror-reflection' the two wave-components will be phase-locked, but even small random fluctuations at the reflection point will imply a slight broadening of the reflected wave-train (Dysthe et al. 1983) and the phase relations will be lost, corresponding to the random phase approximation (Nicholson 1983). If a phase relation between the in-going and reflected waves remains, the standing wave-component gives rise to a density modulation, as demonstrated experimentally (Michelsen et al. 1977). This density modulation disappears in the random phase model of two counter-propagating waves. As an approximation we can retain a narrow-band approximation of the two waves, and provided that there are no losses at the reflection, we can take their energy densities to be the same. The analytical expression (5.9) is easily generalized to this case. The resulting unstable wavenumber range is strongly modified, as illustrated in Fig. 3. Also the real parts of the dispersion relation are calculated, but the results are too complicated for simple presentation. The most conspicuous feature is a banded structure in $\mathrm{Im}\{W'\}$, noting also that the peak values of the growth rate are diminished as compared to the single mode case with the same power. The angular distributions of the growth rates are particularly affected, as seen by comparing Figs. 1 and 3. With the present choice of propagation direction, the filamentary instability is, however, the same when the sum of amplitude squares is the same.

We solved also the problem with three wave modes and found more bands appearing, and the peak growth rates in the direction along $\mathbf{k}_0$ were further reduced.

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5.3. *Approximate solutions for wide-band spectra*

Considering now the case where G_0 represents a wide band of electron waves, we want to determine the stability of such a spectrum. We have no a priori universal form for such a spectrum, and in this sense the problem is not well-defined. We have the option of postulating some spectrum and then solve the Hilbert integrals entering (5.7), but the result will apply only for that particular choice of $G_0(\mathbf{k})$. The problem is even more restrictive than outlined here, since in reality only few functions are known where the integrals in (5.7) can be obtained analytically, the plasma dispersion function being one of them. The problem has the normalized independent variables $\mathbf{K}'$ and $W' = W'(\mathbf{K}')$, and parameters C', $\mathbf{k}_0'$, hereby also the angle θ between $\mathbf{K}'$ and $\mathbf{k}_0'$, and finally the spectral width Δ' which could in principle depend on θ as well. The most important parameters are C' and Δ' and these will be discussed most.

5.3.1. *Box approximation*

As the simplest model that suffices to give an estimate for the stability conditions and ranges we can approximate G_* in (5.8) by a box function so that $G_*(k_\parallel) = A^2/\Delta$ for $\mathbf{k}_0\cdot\hat{\mathbf{K}}-\Delta/2 \leqslant k_\parallel \leqslant \mathbf{k}_0\cdot\hat{\mathbf{K}}+\Delta/2$ while $G_*(k_\parallel) = 0$ elsewhere so that $\int_{-\infty}^{\infty} G_*(k_\parallel)dk_\parallel = A^2$. In that case we can evaluate the integrals entering (5.7) to obtain the form

$$W^2 - C_s^2K^2 = A^2\frac{\varepsilon_0\upsilon^2}{2Mn_0}K^4 \times \frac{\mathrm{ArcTanH}\left(\frac{\left(2\mathbf{k}_0\cdot\hat{\mathbf{K}}-\Delta\right)\frac{\upsilon^2}{\omega_{pe}}K-2W}{\upsilon^2K^2/\omega_{pe}}\right) - \mathrm{ArcTanH}\left(\frac{\left(2\mathbf{k}_0\cdot\hat{\mathbf{K}}+\Delta\right)\frac{\upsilon^2}{\omega_{pe}}K-2W}{\upsilon^2K^2/\omega_{pe}}\right)}{\upsilon^4K^3\Delta/\omega_{pe}^2}, \tag{5.11}$$

which in the limit $\Delta \to 0$ reproduces (5.9). The dispersion relation have either real roots or in complex conjugate pairs, see also Fig. 4.

With the normalized units from before, we can write (5.11) in the form

$$W'^2 - K'^2 = \frac{2C'K'}{9\Delta'} \times \left[\mathrm{ArcTanH}\left(\frac{(2\mathbf{k}_0'\cdot\hat{\mathbf{K}} - 3\Delta)K' - 2W'}{3K'^2}\right) - \mathrm{ArcTanH}\left(\frac{(2\mathbf{k}_0'\cdot\hat{\mathbf{K}} + 3\Delta')K' - 2W'}{3K'^2}\right)\right]. \tag{5.12}$$

Taking the reciprocal of the term containing ArcTanH and expanding (5.11) in powers of Δ we have

$$\frac{1}{2}\frac{\upsilon^4K^3\Delta/\omega_{pe}^2}{\mathrm{ArcTanH}\left(\frac{\left(2\mathbf{k}_0\cdot\hat{\mathbf{K}}-\Delta\right)\frac{\upsilon^2}{\omega_{pe}}K-2W}{\upsilon^2K^2/\omega_{pe}}\right) - \mathrm{ArcTanH}\left(\frac{\left(2\mathbf{k}_0\cdot\hat{\mathbf{K}}+\Delta\right)\frac{\upsilon^2}{\omega_{pe}}K-2W}{\upsilon^2K^2/\omega_{pe}}\right)} \approx \left(W - \frac{\upsilon^2}{\omega_{pe}}\mathbf{k}_0\cdot\mathbf{K}\right)^2 - \frac{\upsilon^4}{4\omega_{pe}^2}K^2(K^2+\Delta^2), \tag{5.13}$$

to lowest order in Δ^2 and K^2, with the next higher order correction being of the form $-(\upsilon^8/12\omega_{pe}^4W^2)K^6\Delta^2$. For $\Delta \to 0$ we recover the second parenthesis on the left side of (5.9). Since relevant values of the modulational wavenumber are $K \leqslant \sqrt{m/M}/\lambda_{De}$ we can to a reasonable approximation here insert (5.13) instead of the second parenthesis

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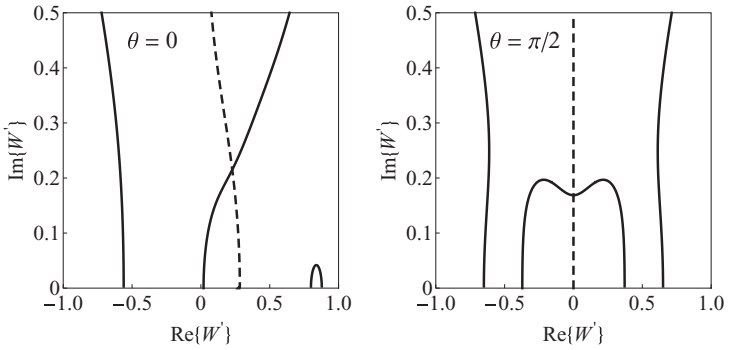


FIGURE 4. Numerical solution of the normalized equation (5.12) for the complex frequency W'. The figure is showing the zero-lines for the real part (full line) and similarly for the imaginary part (dashed lines) in the complex frequency plane. Parameters are $K' = 0.5, k_0' = 0.75, \Delta' = 0.2K_c'$ and $C' = 0.75$. The full figure is symmetric with respect to the real W' axis. Solutions are shown for two angles, $\theta = 0$ and $\theta = \pi/2$. The latter figure is symmetric with respect to $\mathrm{Re}\{W'\} = 0$ as well.

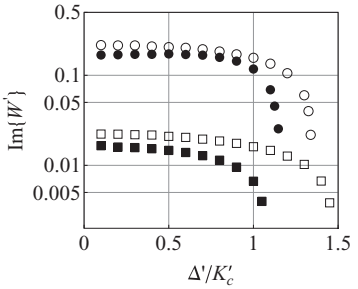


FIGURE 5. Variation of $\mathrm{Im}\{W'\}$ for fixed parameters $C' = 0.75, K' = 0.5$ and $k_0' = 0.75$ shown as a function of Δ'/K_c'. Open circles are for $\theta = 0$, filled circles for filamentation, $\theta = \pi/2$. Results for other angles θ (other parameters unchanged) fall between the two datasets shown here. Filled and open squares show similar results for $C' = 0.05$, $K' = 0.1$, with other parameters unchanged. Note that C' enters the normalizing quantity K_c' on the abscissa.

on the left side of (5.9) and find with reference to (5.10) that the correction due to $\Delta > 0$ enhances the magnitude of the positive first term with K^4. A finite bandwidth thus has a stabilizing effect of the modulational instability of a wave-train. In particular we note that the filamentation instability is stabilized if $\Delta \approx K_c$ within the present approximation. This result has an interesting physical interpretation: if we collect all the wave energy in a single plane wave with an infinitely long correlation length, we have the maximum unstable filamentational modulation to be at K_c. As the wave spectrum broadens, the correlation length reduces approximately as $1/\Delta$. When the correlation length is approximately K_c, the wave spectrum is stabilized, see Fig. 5, which is obtained by solutions like those illustrated in Fig. 4. Note that the stabilization effect is faster than exponential for large Δ'. With increasing Δ', we find that filamentation is stabilized before perpendicular modulation, $\theta = 0$. The growth rate for filamentation ($\theta = \pi/2$) is slightly smaller than for $\theta = 0$ also for small Δ'. For the present parameters, the modulational instability is completely stabilized for any θ when $\Delta > 1.7K_c$. The criterion $\Delta \approx K_c$ is thus not an exact result, rather an order of magnitude estimate. Also, we might find that the growth rate may not vanish, but become so small that we might consider the wave-train as effectively stabilized,

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i.e. 'stable for all practical purposes'. This question will be discussed in more detail later.

The simple box approximation provides some basic insight into features of spectra with finite bandwidths and allows some integrals to be found analytically. In particular the model serves to illustrate cases where the spectrum is strictly confined, i.e. vanishes completely outside a certain wavenumber range. The draw-back is that no reasonable three-dimensional power-spectrum can be projected on a direction given by **K** to give a box function. To substantiate the conclusions from the box approximation we consider another case to demonstrate that the essence of the conclusions of the present section is robust.

5.3.2. *Wide-band Gaussian spectra*

We can write (5.7) in a general form as

$$W^2 - C_s^2 K^2 = K^2 \frac{\varepsilon_0 \omega_{pe}}{8 M n_0} (A(W, \mathbf{K}) + i B(W, \mathbf{K})).$$

For real W we have by definition both A and B to be real functions of the real variables. Taking the integration path in the Landau sense when $\mathrm{Re}\{W\} \to 0$, we have

$$B(W, \mathbf{K}) = \pi \left[G_* \left(\frac{W}{K} \frac{\omega_{pe}}{\upsilon^2} - \frac{1}{2} K \right) - G_* \left(\frac{W}{K} \frac{\omega_{pe}}{\upsilon^2} + \frac{1}{2} K \right) \right]$$

for real W.

We have found that an arbitrary wave spectrum is modulationally stable for $K \to \infty$ and $K = 0$. In case the spectrum is unstable for some intermediate K we would expect that for some real $(W, \mathbf{K})$ we have $B(W, \mathbf{K}) = 0$ implying $G_*((W/K)(\omega_{pe}/\upsilon^2) - \frac{1}{2}K) = G_*((W/K)(\omega_{pe}/\upsilon^2) + \frac{1}{2}K)$ there. For small K this condition implies $G_*'((W/K)(\omega_{pe}/\upsilon^2)) \approx 0$ with $'$ here denoting differentiation with respect to the argument of G_*. This reasoning cannot directly be applied for a narrow spectrum as in (5.10), but even in that case we find $\mathrm{Im}\{W\} = 0$ at an intermediate K-value where we have the transition from unstable to stable modulations.

From the arguments summarized here we conclude that an exact stabilization of a wave spectrum can be achieved only when

$$G_* \left(\frac{W}{K} \frac{\omega_{pe}}{\upsilon^2} - \frac{1}{2} K \right) = G_* \left(\frac{W}{K} \frac{\omega_{pe}}{\upsilon^2} + \frac{1}{2} K \right),$$

for some finite value of K. For practical purposes we can argue that an acceptable criterion for spectral stability can be taken as $\mathrm{Im}\{W\}/\mathrm{Re}\{W\} \ll 1$ for some K value where $\mathrm{Re}\{W\}$ and $\mathrm{Im}\{W\}$ denote real and imaginary parts of W respectively.

Unfortunately, the integrals in (5.7) or (5.8) are not easily solved, and tabulated solutions are found only for a few examples for the integrand. Some relatively accurate general approximations can, however, be found. Considering the first one of the integrals in (5.7) we have

$$I^{\pm}(W, \mathbf{K}) \equiv \int_{-\infty}^{\infty} \frac{G_0 \left(\mathbf{k} \pm \frac{1}{2} \mathbf{K} \right)}{W - \frac{\upsilon^2}{\omega_{pe}} \mathbf{k} \cdot \mathbf{K}} dk^3$$

$$= - \int_{-\infty}^{\infty} \frac{G_* \left(k_\parallel + \mathbf{k}_0 \cdot \hat{\mathbf{K}} \right)}{\frac{\upsilon^2}{\omega_{pe}} (k_\parallel K + \mathbf{k}_0 \cdot \mathbf{K}) \pm \frac{1}{2} \frac{\upsilon^2}{\omega_{pe}} K^2 - W} dk_\parallel,$$

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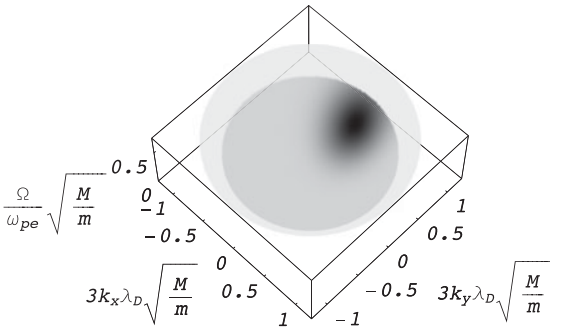


FIGURE 6. Illustration of a wide-band Gaussian wavenumber distribution function, shown with grey color-coding superposed the reduced Langmuir wave dispersion relation $\Omega = \frac{1}{2}k^2\upsilon^2/\omega_{pe}$. White regions for $k > (3\lambda_D)^{-1}\sqrt{m/M}$ decay and should ideally not be included. We have here $k_0' = 0.75$ and $\Delta' = 0.35$.

for complex W, where the integrand has no singularities for unstable condition. In general, the integration path is assumed to be in the Landau sense. We assumed that the maximum value of G_0 is at some $\mathbf{k}_0$ and that the wavenumber spectrum is symmetric with respect to this maximum, i.e. $G_*(\mathbf{k}_0 \cdot \widehat{\mathbf{K}} - k_\parallel) = G_*(\mathbf{k}_0 \cdot \widehat{\mathbf{K}} + k_\parallel)$ for all $k_\parallel$. We can write the complex dispersion relation $D(W, \mathbf{K})$ for real $\mathbf{K}$ and complex W as

$$D(W, \mathbf{K}) \equiv W^2 - C_s^2 K^2 + K^2 \frac{\varepsilon_0 \omega_{pe}}{8 M n_0} \left(I^+(W, \mathbf{K}) - I^-(W, \mathbf{K}) \right).$$

For small $\mathrm{Im}\{W\} \equiv W_2$ we can expand $D(W, \mathbf{K}) = D_1(W, \mathbf{K}) + i D_2(W, \mathbf{K})$ to give $D_1(W_1, \mathbf{K}) \approx 0$, which determines the real part of the dispersion relation

$$\mathrm{Re}\{W\} \equiv W_1(\mathbf{K}), \quad \text{and} \quad W_2 \approx \frac{D_2(W_1, \mathbf{K})}{\partial D_1(W_1, \mathbf{K})/\partial W_1},$$

which gives the growth or decay rate for modulations of the wave spectrum.

We choose to consider a simplified model spectrum where

$$G_0(\mathbf{k}) = A^2 \left(\frac{1}{\pi \Delta^2} \right)^{3/2} e^{-\left((k_x - k_{0x})^2 + (k_y - k_{0y})^2 + (k_z - k_{0z})^2 \right)/\Delta^2},$$

see also Fig. 6. This model gives $G_*(k_\parallel) = (A^2/\sqrt{\pi \Delta^2}) e^{-(k_\parallel - \mathbf{k}_0 \cdot \widehat{\mathbf{K}})^2/\Delta^2}$ after integration over all wavenumber components perpendicular to the $k_\parallel$-direction. We can then introduce the complex plasma dispersion function as $Z(z)$ and write

$$I^\pm(W, \mathbf{K}) = -A^2 \frac{\omega_{pe}}{\upsilon^2 K \Delta} Z \left(\frac{\omega_{pe}}{\upsilon^2} \frac{W}{K\Delta} \mp \frac{K}{2\Delta} - \mathbf{k}_0 \cdot \widehat{\mathbf{K}}/\Delta \right),$$

giving

$$\frac{W^2}{K^2} - C_s^2 = A^2 \frac{\varepsilon_0 \omega_{pe}^2}{8 M n_0 \upsilon^2 K \Delta} \left[Z \left(\frac{\omega_{pe}}{\upsilon^2} \frac{W}{K\Delta} + \frac{K}{2\Delta} - \mathbf{k}_0 \cdot \widehat{\mathbf{K}}/\Delta \right) - Z \left(\frac{\omega_{pe}}{\upsilon^2} \frac{W}{K\Delta} - \frac{K}{2\Delta} - \mathbf{k}_0 \cdot \widehat{\mathbf{K}}/\Delta \right) \right]. \tag{5.14}$$

Note that we have no singularity at $K \to 0$ or $\Delta \to 0$, since the term in the angular brackets vanishes in both these limits. We have the coefficient of the angular brackets

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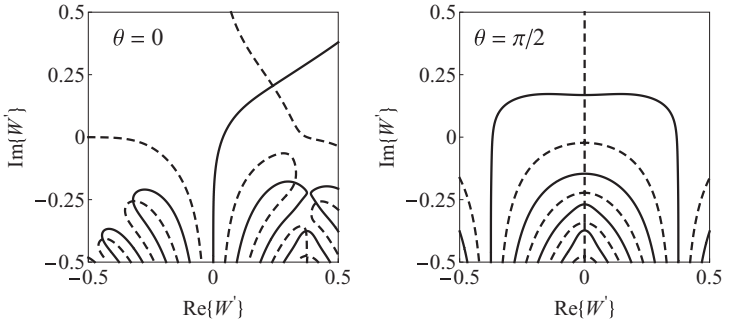


FIGURE 7. Numerical solution of the normalized equation (5.15), showing the zero-lines for the real part (full line) and similarly for the imaginary part (dashed lines) in the complex frequency plane. The two cases shown are for $\theta = 0$ and $\theta = \pi/2$, the latter case being symmetric since the assumed spectrum is symmetric. A solution for $D(W', K') = 0$ is found where dashed and full lines are crossing. We have on each figure one unstable solution and several damped ones.

to be

$$A^2 \frac{\varepsilon_0 \omega_{pe}^2}{8 M n_0 \upsilon^2 K \Delta} = \frac{1}{2} \frac{A^2 \varepsilon_0}{n_0 T_e} \frac{\Omega_{pi}^2}{12 K \Delta} = C' \frac{m}{9M} \frac{\Omega_{pi}^2}{K \Delta} > C' \frac{\Omega_{pi}^2 \lambda_{De}}{3K} \sqrt{\frac{m}{M}},$$

where we used C' introduced in Sec. 5.1. We also have

$$\frac{\omega_{pe}}{\upsilon^2} \frac{W}{K \Delta} = \frac{1}{3 \Delta \lambda_{De}} \frac{W}{K C_s} \sqrt{\frac{m}{M}} > \frac{W}{K C_s},$$

where we arrived at the inequalities by noting that we cannot have spectra wider than a critical wavenumber approximately given as $\frac{1}{3} \sqrt{m/M} / \lambda_D$, since wavenumbers larger than this will decay to another Langmuir wave and an ion sound wave, see also Fig. 6.

With the normalization used before we have an implicit expression for the normalized frequency W' in the form

$$D(W', K') \equiv W'^2 - K'^2 - \frac{C'}{9} \frac{K'}{\Delta'} \left[Z \left(\frac{W'}{3K'\Delta'} + \frac{K'}{2\Delta'} - \frac{\mathbf{k}_0' \cdot \hat{\mathbf{K}}}{3\Delta'} \right) - Z \left(\frac{W'}{3K'\Delta'} - \frac{K'}{2\Delta'} - \frac{\mathbf{k}_0' \cdot \hat{\mathbf{K}}}{3\Delta'} \right) \right] = 0. \tag{5.15}$$

The asymptotic limits of the plasma dispersion function are: $Z_1(z_1 \to \infty) \to -1/z_1$ and $Z_2(z_1 \to \infty) \to 0$ where $Z \equiv Z_1 + i Z_2$. Using these limiting forms that are exact for a δ-function distribution, we recover the narrow-band result (5.9). With improved accuracy we can use $Z_1(z_1 \to \infty) \to -(1 + 1/(2z_1^2))/z_1$ and $Z_2(z_1) = \sqrt{\pi} \exp(-z_1^2)$. The latter approximations contain effects of the finite spread in wavenumbers.

Illustrative results for solutions of $D(W', K') = 0$ are shown in Fig. 7 for the parameters $k_0' = 0.75$, $K' = 0.5$, $C' = 0.75$ and a narrow-band condition with $\Delta' = 0.2 K_c'$. Note the left–right symmetry of the figure for $\theta = \pi/2$. In addition to the unstable modes we have also a number of damped solutions with $\mathrm{Im}\{W'\} < 0$. These have no counterpart in a box approximation, see Fig. 4, but for $\mathrm{Im}\{W'\} > 0$ the two dispersion relations agree qualitatively in spite of the differences in the underlying spectral models. The variation of the normalized growth rates with the spectral widths of Gaussian spectral distributions is illustrated in Fig. 8.

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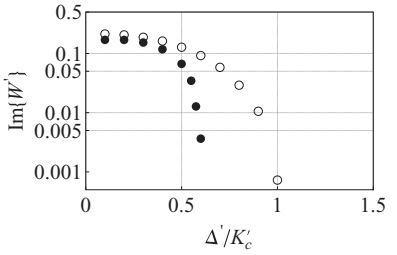


FIGURE 8. Imaginary parts of the normalized frequency $\mathrm{Im}\{W'\}$ for fixed parameters $C' = 0.75$, $K' = 0.5$ and $k_0' = 0.75$, shown as a function of Δ'/K_c'. Filled circles refer to filamentation, $\theta = \pi/2$, open circles for $\theta = 0$, with results for other θ values falling between these two datasets for the given parameters.

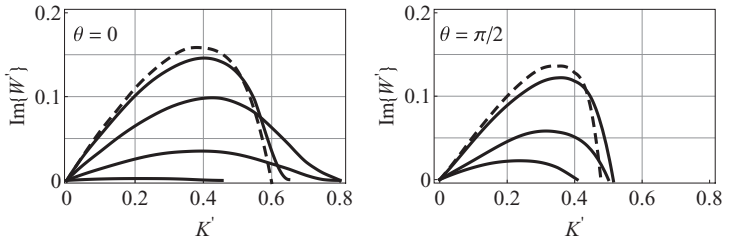


FIGURE 9. Imaginary parts of the normalized frequency $\mathrm{Im}\{W'\}$ for fixed parameters $C' = 0.5, \theta = 0$ and $\theta = \pi/2$ with $k_0' = 0.75$ shown as a function of K' for different values of Δ'/K_c'. Curves are shown for $\Delta'/K_c' = 0.1, 0.25, 0.5, 0.75$ and 1, for $\theta = 0$ and $\Delta'/K_c' = 0.1, 0.25, 0.5$ and 0.6, for $\theta = \pi/2$, the curves being recognized by a steadily decreasing peak value of $\mathrm{Im}\{W'\}$ in both cases.

The variation of $\mathrm{Im}\{W'\}$ with modulation wavenumber K' as found by numerical solutions of (5.15) is shown in Fig. 9 for selected values of Δ'/K_c' and a fixed intermediate wave energy density $C' = 0.5$, and $k_0' = 0.75$ with $\theta = 0$ and $\theta = \pi/2$. Results for $\Delta'/K_c' = 0.1$ are given with dashed line, these are close to the growth rates found for a $\delta(k - k_0)$ spectrum. For $\Delta'/K_c' \approx 1$ we find that the instability is nearly stabilized. We find a systematic decrease of the peak growth rate for increasing Δ'. In a transient interval there is an increase in the bandwidth of the unstable modulation wavenumbers. Similar results were found for a spectral box distribution. The results in Fig. 9 are representative for all angles. For $\theta \neq \pi/2$ we find that one solution retains a value of $\mathrm{Im}\{W'\}$ close to zero as Δ' increases.

We find generally that $(\mathrm{Re}\{W\} + i\,\mathrm{Im}\{W\})/K \approx$ const. for small K, with the phase velocity $\mathrm{Re}\{W\}/K < C_s$. This is true for narrow as well as wide-band unstable spectra, see for instance the narrow-band results in Figs. 1 and 2. While the unstable cases are somewhat similar for the box and Gaussian distributions (compare Figs. 4 and 7 for positive $\mathrm{Im}\{W'\}$), the stable conditions are fundamentally different in the sense that for the stable box distribution an initial perturbation will continue to oscillate without damping, while it will be damped out for the Gaussian case. The latter features are universal for normalizable, continuous and differentiable wide-band stable power spectra.

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6. Conclusions

In the present study we investigated the stability of random Langmuir wave-trains. Explicit results for narrow-band spectra were presented, together with numerical solutions of the pertinent parts of the dispersion relations. The stability of broad spectra was analyzed by a quasi-normal approximation in the sense discussed in the text. The results gave the following hypothesis: based on the general observation that an initially narrow nonlinear wave spectrum tends to become broader with time, we can postulate that given a wave-train that is initially modulationally unstable, the instability gives rise to a broadening of the local wave spectrum until the modulational instability is stabilized. At steady state conditions, this hypothesis implies a simple scaling between the width of the wavenumber spectrum and root-mean-square (RMS) amplitude as $\Delta \sim A$ which is amenable to experimental and numerical tests. The hypothesis can be formulated physically as the assumption that if the correlation length of the random wave field is comparable to the minimum unstable modulational wavelength obtained for a *narrow wave-train with same RMS energy density*, then the modulational instability is arrested. For a modulational wavelength λ to be unstable, it is required that the correlation length of the turbulence is longer than this λ. If the wave-field is initiated with a spectrum having a bandwidth larger than required for marginal stability, this spectral width will remain for a long time. It is interesting that lack of correlation over a modulational wavelength seems to be a sufficient condition for stabilization of the modulational instability. Intuitively, we would expect that statistical independence would be required, and this is a stronger criterion.

Details in the transition process from a narrow spectrum to a broader one is not discussed here. Quasi-linear models for the saturation of the turbulence was discussed by Mendonça and Bingham (2002), but only for the limiting form of a wave kinetic equation, see Sec. 2.5, which is restrictive. The lowest order corrections from what corresponds to our equations (2.8) and (2.10) integrated over Ω were included by Mendonça et al. (2014).

In case there is a non-vanishing average wavenumber $\mathbf{k}_0$ for the spectrum, we expect the saturated stable wave spectrum to be slightly anisotropic, wider in the direction parallel to $\mathbf{k}_0$ as compared to the perpendicular direction, the anisotropy being most pronounced for small wave energy densities, see for instance Fig. 2. We arrived at these hypotheses by considering spectra that strictly speaking are not physically realistic. The box spectrum is thus clearly not physical. The Gaussian spectrum has a shortcoming in including short wavelengths in that part of the spectral range where we can have decay of Langmuir waves (Zakharov 1972; Thornhill and ter Haar 1978; Kono and Škorić 2010) which is not included in the present study. Since, however, the results concerning the spectral stability were robust within numerical factors of order unity, we expect that our basic results have general applicability for Langmuir wave spectra.

In nature as well as laboratory it is easy to imagine situations where an antenna or a source region continuously emits a wave field. Provided the source is spatially localized, some interesting generalizations of the foregoing results might be suggested, provided the wave field contains a dominant directional component given by for instance a unit vector $\widehat{\mathbf{k}}_0$. This will be the case for a large plane surface emitting a spectral band of Langmuir waves propagating normally to the emitter. The distance measured from the antenna can be taken to be a time-like coordinate and the foregoing procedure will apply qualitatively here as well (Dysthe et al. 1978; Pécseli 1984).

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The analysis of the present study is based on the standard model for interactions of Langmuir waves and ion sound (Zakharov 1972), ignoring nonlinear internal interactions in the Langmuir wave spectrum. For small wave amplitudes $\varepsilon_0 E^2/n\kappa T_e \leqslant m/M$ this approximation is generally assumed to be justified, although it is also known that even very weak electron plasma instabilities with slow growth rates can excite Langmuir waves with clear anharmonic features. Numerical studies of beam instabilities with cold electron beams of relative density as small as 1% or less demonstrated slow exponential growth in wave amplitude (Daldorff et al. 2011) with harmonic structures developing within a time less than an ion plasma period. A possibly universal form of saturated turbulent Langmuir spectra is determined by effects not included in the basic equations used here. The present analysis thus does not give results for universal Langmuir wave spectra, only derive conditions for their stability.

The analysis assumed cold ions. Inclusion of a warm fluid ion model is relatively simple, a kinetic ion model is more challenging. It is feasible but leads to long and complicated results. The basic physical features are to some extent covered by related studies of weakly nonlinear plane Langmuir waves (Dysthe and Pécseli 1977; Pécseli 1985; Daldorff et al. 2011). For the filamentary instability, which is non-propagating and aperiodically growing, there are no particles resonating with a propagating wave, and kinetic effects have little consequence.

The present analysis assumes that the electron velocity distribution is close to a Maxwellian, so that a plasma temperature in the thermodynamic sense (and thereby a thermal velocity) can be determined uniquely. If the electron velocity distribution is strongly distorted by, for instance, having distinct warm and cold components a new channel of electron decay opens due to weakly damped electron acoustic waves (Khirseli and Tsintsadze 1980; Hanssen et al. 1994). Such cases are not considered here.

A limiting case is where $C' \approx 1$: energy densities larger than this are outside the range of the expansion assumed in the basic equations. To stabilize the maximum RMS amplitude we need $\Delta' \approx 1$, which in reality implies that the wave spectrum fills out all the available wavenumber range, recalling the requirement $|\mathbf{k}_0' - \mathbf{K}'| < 1$ for all K' in the spectrum to avoid Langmuir wave decay, see also Fig. 6.

The problem of plasma-wave collapse (Zakharov 1972; Rasmussen and Rypdal 1986) has not been addressed here. It is important to recall the precise form of the criterion addressed in that problem: it can be demonstrated that under suitable conditions, with $\bar{\mathbf{r}} \equiv \int_{\mathrm{Vol}} \mathbf{r}|E|^2 d^3r/\mathcal{N}$ and $\mathcal{N} \equiv \int_{\mathrm{Vol}} |E|^2 d^3r$, the average $\Delta\mathbf{r}^2(t) \equiv \int_{\mathrm{Vol}} (\mathbf{r} - \bar{\mathbf{r}})^2 |E|^2 d^3r/\mathcal{N} \rightarrow 0$ in a finite time. This result requires, however, that the integrals listed exist, and this is not the case for a spatially homogeneous wide-band turbulent spectrum for $|E(\mathbf{r}, t)|^2$. If the spectrum is modulationally unstable, it can break up into localized large amplitude wavepackets in a saturated stage of this instability, and these might eventually collapse individually. The modulationally stable homogeneous spectrum will not generate 'seeds' for Langmuir wave collapse in the form of large amplitude localized bursts. These spectra will not give collapse, irrespective of the RMS amplitude, provided the stability criterion for the spectral width is fulfilled. When the wave energy densities are close to the maximum level, $C' \approx 1$, it seems unlikely that a broadening of the spectrum can arrest the modulational instability completely, and in this limit we expect that nucleation of wave energy can be observed. Here it is interesting to note that the problem of Langmuir wave collapse has a close counterpart in lower hybrid wave collapse (Shapiro et al. 1993). For conditions in a space environment where this phenomenon could be observed

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by instrumented satellites, it was concluded that evidence for nonlinear lower hybrid wave phenomena were observed, but these had nothing to do with bursts of collapsing lower hybrid waves (Pécseli et al. 1996).

The numerical and experimental studies of Langmuir wave collapse (see for instance the review by Rubenchik et al. 1985) have without exception (to the authors knowledge) started out with a localized burst of large amplitude wave activity, and the present stability analysis of a homogeneous wave-train does not apply, although the basic equations (2.8) and (2.10) together with (2.11) do.

One principal problem remains, however. This is common for all studies of the present type: the results address ensemble averages without guarantee that they apply directly to an individual experiment, which is after all what is observed. In classical statistical mechanics we have an ergodic theorem (Yaglom 1962) stating that averages over a long time record give the same result as ensemble averages, implying that given time, any individual system will eventually cover all of the available states. It is very easy to construct models without an ergodic property, and in particular no such theorem is given for the system described by the basic equations (2.3) and (2.4). These comments invite the following numerical or experimental study: initialize the system with a moderate amplitude, $C' < 1$, electron plasma wave with a narrow wavenumber bandwidth around a wavenumber $|\mathbf{k}_0'| < 1$. It will then be possible to follow the competition between the modulational instability and the broadening of the spectrum. It is essential for the problem that the simulation domain is large enough to confine the free sound contribution (see discussion in Sec. 4). The standard model for Langmuir collapse (Zakharov 1972; Rasmussen and Rypdal 1986) assumes that these sound contributions escape, and do not interfere with the cavitation process.

Decay of a Langmuir wave to a sound wave and another Langmuir wave, $\ell_0 \rightarrow \ell_1 + s$, with wavenumbers $k_0' > 1$, see e.g. Fig. 6, was not considered here. The basic hypothesis emerging from the present study concerning stability of a wide-band spectrum can be generalized to the decay by anticipating that for the decay to become effective, the wavelength of the sound wave participating in the process has to be shorter than the correlation length of the spectrum from where the primary wave k_0 is taken.

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