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MORE ON LOGARITHMIC SUMS OF CONVEX BODIES

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Abstract. We prove that the log-Brunn–Minkowski inequality (log-BMI) for the Lebesgue measure in dimension n would imply the log-BMI and, therefore, the B-conjecture for any even log-concave measure in dimension n. As a consequence, we prove the log-BMI and the B-conjecture for any even log-concave measure in the plane. Moreover, we prove that the log-BMI reduces to the following: for each dimension n, there is a density f_n, which satisfies an integrability assumption, so that the log-BMI holds for parallelepipeds with parallel facets, for the density f_n. As a byproduct of our methods, we study possible log-concavity of the function $t \mapsto |(K +_p \cdot \mathrm{e}^t L)^\circ|$, where $p \geqslant 1$ and K, L are symmetric convex bodies, which we are able to prove in some instances and, as a further application, we confirm the variance conjecture in a special class of convex bodies. Finally, we establish a non-trivial dual form of the log-BMI.

§1. *Introduction.* Let K, L be convex bodies in $\mathbb{R}^n$ (i.e. compact, convex sets, with non-empty interiors), that contains the origin in their interiors. For $p \geqslant 1$, the L^p-Minkowski–Firey sum $a \cdot K +_p b \cdot L$ of K and L with respect to some positive numbers a, b is defined by its support function

$$h_{a\cdot K +_p b\cdot L} = (a h_K^p + b h_L^p)^{1/p}. \tag{1}$$

The case $p = 1$ corresponds to the classical Minkowski sum $aK + bL = \{ax + by \mid x \in K, y \in L\}$. In the pioneer work of Böröczky *et al* [5], the L^p-convex combination of K and L, with respect to some $\lambda \in (0, 1)$, for all $p \geqslant 0$ is defined:

$$\lambda \cdot K +_p (1-\lambda) \cdot L = \left\{ x \in \mathbb{R}^n \mid x \cdot u \leqslant [\lambda h_K^p(u) + (1-\lambda) h_L^p(u)]^{1/p}, \ \forall u \in S^{n-1} \right\}$$

and

$$\lambda \cdot K +_0 (1-\lambda) \cdot L = \{x \in \mathbb{R}^n \mid x \cdot u \leqslant h_K^\lambda(u) h_L^{1-\lambda}(u), \forall u \in S^{n-1}\}.$$

Note that if $0 \leqslant p < 1$, $\lambda \cdot K +_p (1-\lambda) \cdot L$ cannot be defined by (1), simply because the resulting function is not always convex. Nevertheless, the two definitions coincide for $p \geqslant 1$. Let us state the fundamental L^p-Brunn–Minkowski inequality (see e.g. [18, 19, 42]), for $p \geqslant 1$ in its dimension-free form:

$$|\lambda \cdot K +_p (1-\lambda) \cdot L| \geqslant |K|^\lambda |L|^{1-\lambda},$$

where $|\cdot| = |\cdot|_n$ is the n-dimensional Lebesgue measure.

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Although for $p \geqslant 1$, the L^p-Brunn–Minkowski theory has been considerably developed in the previous years (see e.g. [32–36]), much less seems to be known for $0 \leqslant p < 1$. The following is conjectured in [5] (without the equality cases).

CONJECTURE 1.1 (The logarithmic-Brunn–Minkowski inequality). *Let K, L be symmetric convex bodies in $\mathbb{R}^n$ and $\lambda \in (0, 1)$. Then,*

$$|\lambda \cdot K +_0 (1-\lambda) \cdot L| \geqslant |K|^{\lambda} |L|^{1-\lambda},$$

with equality in the following case: whenever $K = K_1 \times \cdots \times K_m$, for some convex sets $K_1, \ldots, K_m$, that cannot be written as Cartesian products of lower-dimensional sets, then there exist positive numbers $c_1, \ldots, c_m$, such that $L = c_1 K_1 \times \cdots \times c_m K_m$.

The conjecture can easily be seen to be wrong for general convex bodies, even for $n = 1$. Note, also, that for $0 \leqslant p \leqslant q$, $\lambda \cdot K +_p (1-\lambda) L \subseteq \lambda \cdot K +_q (1-\lambda) L$, thus the log-Brunn–Minkowski inequality (if true) implies the L^p-Brunn–Minkowski inequality, for symmetric convex bodies, for all $p > 0$.

The cone-volume measure of K is defined as $S_0(K, \cdot) = h_K S(K, \cdot)$, where $S(K, \cdot)$ is the surface area measure of K, viewed as a measure on the sphere (see e.g. [42]). In [6], a necessary and sufficient condition was discovered (the planar case was treated by Stancu [43, 44]; see also [7] for other applications of the cone-volume measure and [9] for a possible functional generalization of the classical Minkowski problem). A confirmation of Conjecture 1.1 would answer the following open problem: when do two symmetric convex bodies K, L have proportional cone-volume measures? If Conjecture 1.1 was proven to be true, the pairs (K, L) would be exactly those for which equality holds in the log-Brunn–Minkowski inequality. The planar case was settled in [5].

THEOREM A [5]. *Conjecture 1.1 is true in dimension two.*

It was shown in [41] that Conjecture 1.1 holds true for pairs of unconditional bodies with respect to the same orthonormal basis. Actually, the proof (based on a result from [10]) shows that this result (as for the inequality) remains true if we replace the Lebesgue measure with any unconditional log-concave measure in $\mathbb{R}^n$. Recall that a measure μ is called log-concave if for all convex bodies K, L, it satisfies the Brunn–Minkowski inequality:

$$\mu(\lambda K + (1-\lambda) L) \geqslant \mu(K)^{\lambda} \mu(L)^{1-\lambda}, \quad \lambda \in (0, 1).$$

By a result of Borell [4], the absolutely continuous log-concave measures in $\mathbb{R}^n$ are exactly those having log-concave densities, i.e. their logarithms are concave functions. It is reasonable to conjecture the following.

CONJECTURE 1.2 (Log-Brunn–Minkowski conjecture for even log-concave measures). *Let μ be an even log-concave measure in $\mathbb{R}^n$, K, L be symmetric convex bodies and $\lambda \in (0, 1)$. Then, $\mu(\lambda \cdot K +_0 (1-\lambda) \cdot L) \geqslant \mu(K)^{\lambda} \mu(L)^{1-\lambda}$.*

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Conjecture 1.2 is closely connected with (actually implies; see Corollary 3.2) the so-called B-conjecture.

CONJECTURE 1.3 (B-conjecture). *Let μ be an even log-concave measure in $\mathbb{R}^n$ and K be a symmetric convex body. Then, the function $\mathbb{R} \ni t \mapsto \mu(e^t K)$ is log-concave.*

This was conjectured in [**10**, **28**]. This was previously conjectured by Banaszczyk in [**30**] for the standard Gaussian measure γ_n (i.e. the measure that has density $e^{-\|x\|_2^2/2}$) and it was confirmed by Cordero-Erasquin *et al* in [**10**], a fact known as the B-theorem (see also [**29**] for an application). More generally, the following result was shown to hold.

THEOREM B (The B-theorem for the Gaussian measure [**10**]). *Let A be a diagonal $n \times n$ matrix and K be a symmetric convex body. Then, the function $\mathbb{R} \ni t \mapsto \gamma_n(e^{tA} K)$ is log-concave. In particular, the standard Gaussian measure satisfies the B-conjecture.*

A connection of the B-theorem for the Gaussian measure with another conjectured version of the Brunn–Minkowski inequality for the Gaussian measure was established by Gardner and Zvavitch [**20**] (see also [**38**]).

The following fact, also from [**10**], will be used.

THEOREM C [**10**]. *Let A be a diagonal $n \times n$ matrix, μ be an unconditional log-concave measure and K be an unconditional convex body. Then, the function $\mathbb{R} \ni t \mapsto \mu(e^{tA} K)$ is log-concave.*

We will say that a measure μ in $\mathbb{R}^n$ satisfies the log-Brunn–Minkowski inequality (or that the log-Brunn–Minkowksi inequality holds for the measure μ) if for any two symmetric convex bodies K, L in $\mathbb{R}^n$, the function $(0, 1) \ni \lambda \mapsto \mu(\lambda \cdot K +_0 (1-\lambda) \cdot L)$ is log-concave. Similarly, we will say that μ satisfies the B-conjecture (or satisfies the B-theorem) if for any symmetric convex body K, the function $\mathbb{R} \ni t \mapsto \mu(e^t K)$ is log-concave.

A connection between the log-Brunn–Minkowski inequality for the Lebesgue measure (Conjecture 1.1 and the B-conjecture for uniform measures of symmetric convex bodies (i.e. measures of the form $|K \cap \cdot|$, where K is a symmetric convex body) was established in [**41**]. Namely, it was proven that (i) the log-Brunn–Minkowski inequality for the Lebesgue measure in dimension n implies the B-conjecture for uniform measures in dimension n. Thus, by Theorem A, the B-conjecture for uniform measures in the plane follows (this fact was proven independently in [**31**]). (ii) The log-Brunn–Minkowski inequality for the Lebesgue measure holds in any dimension if and only if, in any dimension, the function $|(e^{tA} C_n) \cap K|$ is log-concave in t for any symmetric convex body K and for any diagonal matrix A. Here C_n denotes the cube $[-1, 1]^n$.

Our first goal is to continue the ideas from [**41**] and extend the aforementioned results even further. Let us briefly describe our main results towards this

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direction. In §3 we prove (see Theorem 3.1) that actually the log-Brunn–Minkowski inequality for the Lebesgue density implies the log-Brunn–Minkowski inequality for any even log-concave density and, therefore, the B-conjecture in full generality. Thus, again by Theorem A, we establish (see Corollary 3.3) Conjectures 1.2 and 1.3 in the plane.

On the other hand, in §5 we modify the proof of fact (ii) mentioned earlier to prove that actually in order to confirm Conjecture 1.2, one needs for any dimension n to find a density f_n which satisfies a mild integrability assumption and the function

$$t \mapsto \int_{e^{tA}C_n} f_n(Tx)\,dx$$

is log-concave for any choice of the diagonal matrix A and for any invertible linear map T. The reader should focus in the case of the Gaussian density.

Our second goal is to study log-concavity and log-convexity properties for dual bodies. In §4, as a byproduct of our method from §3, we show (see Proposition 4.3) that the B-conjecture for uniform measures or for measures with densities of the form $e^{-\|x\|_K^p}$, $p \geqslant 1$ would imply the log-concavity of the function

$$t \mapsto |(K +_p \cdot e^t L)^\circ|,$$

where M° stands for the dual body of M. Using the cases where the B-conjecture is known to hold, we establish this log-concavity property in some special cases (see Corollary 4.6). As a further application, in Theorem 4.8 we confirm the variance conjecture (see §4 for more information) in a special class of convex bodies.

Finally, in §6, we establish the L^0-analogue (where 1-sum is replaced by logarithmic sum) of Firey's dual Brunn–Minkowski inequality [**14**] (and its extension to other quermassintegrals):

$$|(\lambda K + (1-\lambda)L)^\circ| \leqslant |K^\circ|^\lambda |L^\circ|^{1-\lambda}.$$

Note that the L^0-version is clearly a stronger inequality. Also, since no explicit formula is valid for the support function of the logarithmic sum, no classical arithmetic inequalities (such as Hölder) can be used directly towards the proof. Therefore, our inequality is a non-trivial extension of Firey's result.

§2. *Preliminaries.* Let us state some results that will be needed subsequently. We refer to [**18**, **42**] for more information.

Let K be a convex body that contains zero in its interior. The polar body of K is defined as

$$K^\circ = \{x \in \mathbb{R}^n \mid x \cdot y \leqslant 1, \forall y \in K\}.$$

Then, K° is also a convex body that contains zero in its interior and $(K^\circ)^\circ = K$.

The ith quermassintegral $W_i(K)$ of K is defined by the Steiner formula

$$|K + tB_2^n| = \sum_{i=0}^{n} \binom{n}{i} W_i(K) t^i, \quad t > 0,\ i = 0, \ldots, n,$$

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where B_2^n is the Euclidean unit ball. Note that $W_0(K)$ is the volume of K and $W_1(K)$, $W_{n-1}(K)$, $W_n(K)$ are proportional to the surface area, the mean width and the Euler characteristic, respectively. Moreover, the functional W_i is $(n-i)$-homogeneous, that is $W_i(tK) = t^{n-i}W_i(K)$. A useful formula for the quermassintegrals of K is the Kubota recursion formula:

$$W_i(K) = \int_{\mathcal{G}_{n,n-i}} |K|H|_{n-i}\, dH, \quad i = 1, \dots, n-1.$$

Here, $\int_{\mathcal{G}_{n,n-i}} \cdot\, dH$ denotes the integral of a function defined on the Grassmannian $\mathcal{G}_{n,n-i}$, with respect to the Haar measure on $\mathcal{G}_{n,n-i}$ and $K|H$ is the orthogonal projection of K onto the subspace H.

For the rest of this section, K will be denoting a symmetric convex body (i.e. $K = -K$). The norm $\|\cdot\|_K$ of K is the unique norm in $\mathbb{R}^n$, such that $K = \{x \mid \|x\|_K \leqslant 1\}$. Recall that every norm in $\mathbb{R}^n$ is the norm of a unique symmetric convex body.

The support function of K is defined by $h_K(x) = \max_{y\in K} x \cdot y$, $x \in \mathbb{R}^n$. There is a duality relation between the norm and the support function of K (recall that K is symmetric): $h_K = \|\cdot\|_{K^\circ}$.

The inradius and the outradius of K are defined as

$$\text{inradius}(K) = \min_{x\in\partial K} \|x\|_2, \qquad \text{outradius}(K) = \max_{x\in K} \|x\|_2.$$

Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a homogeneous of degree p function. Then, by integration in polar coordinates, we have

$$\int_K f(x)\, dx = \frac{1}{n+p} \int_{S^{n-1}} f(x)\|x\|_K^{-n-p}\, dx,$$

where $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$, the unit sphere in $\mathbb{R}^n$.

The Prékopa–Leindler inequality is probably the most famous functional generalization of the Brunn–Minkowski inequality. We will need the one-dimensional Prékopa–Leindler inequality in the following form, proven in [4] (see also [2, 45]).

THEOREM D (Multiplicative version of the Prékopa–Leindler inequality). *Let $\lambda \in (0,1)$, $f, g, h : \mathbb{R}_+ \to \mathbb{R}_+$ be non-negative measurable functions, such that $h(x^\lambda y^{1-\lambda}) \geqslant f^\lambda(x) g^{1-\lambda}(y)$, for all $x, y > 0$. Then, the following holds:*

$$\int_{\mathbb{R}_+} h(x)\, dx \geqslant \left[\int_{\mathbb{R}_+} f(x)\, dx\right]^\lambda \left[\int_{\mathbb{R}_+} g(x)\, dx\right]^{1-\lambda}.$$

It is well known that there exists a unique up to isometry volume-preserving linear map T such that the quantity

$$L_{TK}^2 := \frac{1}{|K|^{(n+2)/n}} \int_{TK} (x \cdot y)^2\, dx$$

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is constant as a function of $y \in S^{n-1}$. Then, TK is said to be isotropic and the number L_{TK} is called the isotropic constant of K (see [39] for basic results on this concept). It is true that

$$L_K^2 = \frac{1}{n|K|^{(n+2)/n}} \min_{T \in SL_n} \int_{TK} \|x\|_2^2 \, dx.$$

It has been conjectured that the isotropic constants of symmetric convex bodies are bounded from above by an absolute constant; this problem is known as the slicing problem. The isotropic constant is known to be bounded from below by an absolute constant (see again [39]). The best estimate to date for the upper bound is of the order $n^{1/4}$ due to Klartag [26], after improving the previous estimate $Cn^{1/4} \log n$ by Bourgain [8].

Let H be a k-dimensional subspace of $\mathbb{R}^n$. Define the Schwarz symmetrization $S_H(K)$ of K with respect to H as the set that is constructed by replacing every cross-section, orthogonal to H, of K with a Euclidean ball of the same $(n-k)$-dimensional volume. It is an easy application of the Brunn–Minkowski inequality that $S_H(K)$ is also a convex body. If $H = \mathbb{R}u$, for some unit vector u, we abbreviate $S_u(K) = S_H(K)$. Note, furthermore, that $h_K(u) = h_{S_u(K)}(u)$.

§3. *On the log-Brunn–Minkowski inequality for even log-concave measures.* The main result of this section is the following.

THEOREM 3.1. *Assume that the log-Brunn–Minkowksi inequality holds in dimension n for the Lebesgue measure. Then, the log-Brunn–Minkowksi inequality holds in dimension n for any even log-concave measure.*

Before proving Theorem 3.1, let us first state some of its consequences.

COROLLARY 3.2. *Assume that the log-Brunn–Minkowksi inequality holds in dimension n for the Lebesgue measure. Then, the B-conjecture holds in dimension n, for any even log-concave measure.*

Proof. Let μ be an even log-concave measure, K be a symmetric convex body, $\lambda \in (0, 1)$ and $s, t \in \mathbb{R}$. Then, by Theorem 3.1, the log-Brunn–Minkowski inequality holds for the measure μ, therefore

$$\mu(\mathrm{e}^{\lambda s+(1-\lambda)t} K) = \mu(\lambda \cdot (\mathrm{e}^s K) +_0 (1-\lambda) \cdot (\mathrm{e}^t K)) \geqslant \mu(\mathrm{e}^s K)^{\lambda} \mu(\mathrm{e}^t K)^{1-\lambda}$$

and the assertion follows. □

Combining Theorem 3.1, Corollary 3.2 and Theorem A, we immediately obtain the following corollary.

COROLLARY 3.3. *The log-Brunn–Minkowski inequality and the B-conjecture hold in the plane, for any even log-concave measure.*

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Corollary 3.3 (in particular, the B-theorem in the plane) together with [38, Proposition 3.1] (see also [20]) immediately imply the following.

COROLLARY 3.4. *Let μ be an even log-concave measure in the plane, M be a symmetric convex body in the plane and $\lambda \in (0,1)$. Then for every K, $L \in \{\alpha M; \alpha \geqslant 0\}$, one has*

$$\mu(\lambda K + (1-\lambda)L)^{1/2} \geqslant \lambda \mu(K)^{1/2} + (1-\lambda)\mu(L)^{1/2}.$$

For the proof of Theorem 3.1, some geometric lemmas are required.

LEMMA 3.5. *Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be an even convex function and $t \in \mathbb{R}$, so that the sets $\{\varphi \leqslant t\}$ and $\{\varphi = \varphi(0)\}$ are convex bodies. Then, there exists $b > 0$, depending only on φ, t, such that for all $u \in S^{n-1}$, $r, s \in \mathbb{R}$ with $\varphi(0) < r < s \leqslant t$, the following inequality holds:*

$$s - r \leqslant b(h_{\{\varphi \leqslant s\}}(u) - h_{\{\varphi \leqslant r\}}(u)).$$

Proof. Note that the restriction of φ into the set $\{\varphi \leqslant t\}$ is Lipschitz with some constant $A > 0$. Let $r < s$, $u \in S^{n-1}$ and $x \in \mathbb{R}^n$, such that $\varphi(x) = r$ and $x \cdot u = h_{\{\varphi \leqslant r\}}(u)$. Then, there exists a $\xi > 1$, such that for $x' := \xi x$, $\varphi(x') = s$. Then, $x' \cdot u \leqslant h_{\{\varphi \leqslant s\}}(u)$. Thus,

$$\begin{aligned} h_{\{\varphi \leqslant s\}}(u) - h_{\{\varphi \leqslant r\}}(u) \geqslant x' \cdot u - x \cdot u &= \|x' - x\|_2 \frac{x' - x}{\|x' - x\|_2} \cdot u \\ &= \|x' - x\|_2 \frac{x}{\|x\|_2} \cdot u \\ &= \|x' - x\|_2 \frac{h_{\{\varphi \leqslant r\}}(u)}{\|x\|_2}. \end{aligned}$$

We have $\varphi(x) = r \leqslant t$, therefore $\|x\|_2 \leqslant \text{outradius}(\{\varphi \leqslant t\})$. Also, $h_{\{\varphi \leqslant r\}}(u) \geqslant h_{\{\varphi = \varphi(0)\}}(u) \geqslant \text{inradius}(\{\varphi = \varphi(0)\})$. Hence,

$$\begin{aligned} h_{\{\varphi \leqslant s\}}(u) - h_{\{\varphi \leqslant r\}}(u) &\geqslant \frac{\varphi(x') - \varphi(x)}{A} \frac{\text{inradius}(\{\varphi = \varphi(0)\})}{\text{outradius}(\{\varphi \leqslant t\})} \\ &= \frac{s - r}{A} \frac{\text{inradius}(\{\varphi = \varphi(0)\})}{\text{outradius}(\{\varphi \leqslant t\})}. \end{aligned}$$

□

LEMMA 3.6. *Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a function and t be a real number, both satisfying the conditions of Lemma 3.5. There exists $c > |\varphi(0)|$, so that if we set*

$$\overline{\varphi}(x) = \begin{cases} \varphi(x) + c, & \varphi(x) \leqslant t, \\ \infty, & \varphi(x) > t, \end{cases} \tag{2}$$

then for any $r, s \geqslant 0$,

$$\{\overline{\varphi} \leqslant r^{\lambda} s^{1-\lambda}\} \supseteq \lambda \cdot \{\overline{\varphi} \leqslant r\} +_0 (1-\lambda) \cdot \{\overline{\varphi} \leqslant s\}.$$

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Proof. Let b be the constant from Lemma 3.5. Set

$$c := b \cdot \mathrm{outradius}(\{\varphi \leqslant t\}) + |\varphi(0)|$$

and define $\overline{\varphi}$ by (2). Clearly, $\overline{\varphi} \geqslant 0$. It suffices to prove that for every $r, s > 0$,

$$h_{\{\overline{\varphi} \leqslant r^\lambda s^{1-\lambda}\}}(u) \geqslant h^\lambda_{\{\overline{\varphi} \leqslant r\}}(u) h^{1-\lambda}_{\{\overline{\varphi} \leqslant s\}}(u). \tag{3}$$

First assume that $\overline{\varphi}(0) \leqslant r, s \leqslant t + c$. Fix $u \in S^{n-1}$. For $\overline{\varphi}(0) \leqslant p \leqslant t + c$, set

$$f(p) := h_{\{\overline{\varphi} \leqslant p\}}(u). \tag{4}$$

Clearly, f is strictly increasing. Let us show that it is also concave. For p_1, $p_2 \in [\overline{\varphi}(0), t + c]$ and for $\lambda \in (0, 1)$, we clearly have

$$\{\overline{\varphi} \leqslant \lambda p_1 + (1 - \lambda) p_2\} \supseteq \lambda\{\overline{\varphi} \leqslant p_1\} + (1 - \lambda)\{\overline{\varphi} \leqslant p_2\},$$

thus

$$\begin{aligned} f(\lambda p_1 + (1 - \lambda) p_2) &= h_{\{\overline{\varphi} \leqslant \lambda p_1 + (1-\lambda) p_2\}}(u) \\ &\geqslant \lambda h_{\{\overline{\varphi} \leqslant p_1\}}(u) + (1 - \lambda) h_{\{\overline{\varphi} \leqslant p_2\}}(u) \\ &= \lambda f(p_1) + (1 - \lambda) f(p_2). \end{aligned}$$

We will show that for $t + c \geqslant s > r \geqslant \overline{\varphi}(0)$,

$$f(r^\lambda s^{1-\lambda}) \geqslant f(r)^\lambda f(s)^{1-\lambda}.$$

Consider the line through the points $(f(r), r)$ and $(f(s), s)$ and suppose that this is defined by the equation $x_2 = c_1 x_1 + d_1$, for points (x_1, x_2) of the plane. Since f is strictly increasing, it is clear that $c_1 > 0$. We claim that $d_1 \geqslant 0$. Indeed, by (4), we have

$$\begin{aligned} d_1 = s - \frac{s - r}{f(s) - f(r)} f(s) &= s - \frac{s - r}{h_{\{\overline{\varphi} \leqslant s\}}(u) - h_{\{\overline{\varphi} \leqslant s\}}(u)} h_{\{\overline{\varphi} \leqslant s\}}(u) \\ &\geqslant s - b h_{\{\overline{\varphi} \leqslant s\}}(u) \\ &\geqslant s - b h_{\{\overline{\varphi} \leqslant t + c\}}(u) \\ &\geqslant s - b \cdot \mathrm{outradius}(\{\varphi \leqslant t\}) \\ &\geqslant \overline{\varphi}(0) - b \cdot \mathrm{outradius}(\{\varphi \leqslant t\}) \\ &= \varphi(0) + |\varphi(0)| \geqslant 0. \end{aligned}$$

Now, the concavity of f implies that

$$r^\lambda s^{1-\lambda} \leqslant c_1 f(r^\lambda s^{1-\lambda}) + d_1.$$

It follows that

$$\begin{aligned} f(r^\lambda s^{1-\lambda}) &\geqslant (r/c_1)^\lambda (s/c_1)^{1-\lambda} - d_1/c_1 \geqslant \left(\frac{r}{c_1} - \frac{d_1}{c_1}\right)^\lambda \left(\frac{s}{c_1} - \frac{d_1}{c_1}\right)^{1-\lambda} \\ &= f(r)^\lambda f(s)^{1-\lambda}. \end{aligned}$$

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This proves (3) in the case where $t+c \geqslant s, r \geqslant \overline{\varphi}(0)$. If $s < \overline{\varphi}(0)$ (or $r < \overline{\varphi}(0)$), then $\{\overline{\varphi} \leqslant s\} = \emptyset$ and (3) holds trivially. On the other hand, if $s > t+c$, then $\{\overline{\varphi} \leqslant s\} = \{\overline{\varphi} \leqslant t+c\}$ and

$$\{\overline{\varphi} \leqslant r^{\lambda} s^{1-\lambda}\} \supseteq \{\overline{\varphi} \leqslant r^{\lambda}(t+c)^{1-\lambda}\}. \tag{5}$$

If, in addition, $r \leqslant t+c$, then by (5) and by the case $t+c \geqslant s, r \geqslant \overline{\varphi}(0)$, it follows that

$$\begin{aligned}\{\overline{\varphi} \leqslant r^{\lambda} s^{1-\lambda}\} &\supseteq \lambda \cdot \{\overline{\varphi} \leqslant r\} +_0 (1-\lambda) \cdot \{\overline{\varphi} \leqslant t+c\} \\ &= \lambda \cdot \{\overline{\varphi} \leqslant r\} +_0 (1-\lambda) \cdot \{\overline{\varphi} \leqslant s\}.\end{aligned}$$

In the remaining case we have $r, s > t+c$. Then, $\{\overline{\varphi} \leqslant r\} = \{\overline{\varphi} \leqslant s\} = \{\overline{\varphi} \leqslant r^{\lambda} s^{1-\lambda}\} = \{\overline{\varphi} \leqslant t+c\}$ and the assertion is again trivial. □

LEMMA 3.7. *Let $\lambda \in (0,1)$, $a_1, a_2 > 0$, μ be a measure, $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a non-negative even convex function and K, L be symmetric convex bodies. Assume that for all $r_1, r_2 > 0$,*

$$\mu([\lambda \cdot K +_0 (1-\lambda) \cdot L] \cap \{\varphi \leqslant r_1^{\lambda} r_2^{1-\lambda}\}) \geqslant \mu(K \cap \{\varphi \leqslant r_1\})^{\lambda} \mu(L \cap \{\varphi \leqslant r_2\})^{1-\lambda}.$$

Then,

$$\begin{aligned}&\int_{\lambda \cdot K +_0 (1-\lambda) \cdot L} \mathrm{e}^{-a_1^{\lambda} a_2^{1-\lambda} \varphi(x)}\, d\mu(x) \\ &\quad \geqslant \left[\int_K \mathrm{e}^{-a_1 \varphi(x)}\, d\mu(x)\right]^{\lambda} \left[\int_L \mathrm{e}^{-a_2 \varphi(x)} d\mu(x)\right]^{1-\lambda}.\end{aligned}$$

Proof. By the Fubini theorem we have

$$\begin{aligned}&\int_{\lambda \cdot K +_0 (1-\lambda) \cdot L} \mathrm{e}^{-a_1^{\lambda} a_2^{1-\lambda} \varphi(x)} d\mu(x) \\ &= \int_0^{\infty} \mu([\lambda \cdot K +_0 (1-\lambda) \cdot L] \cap \{\mathrm{e}^{-a_1^{\lambda} a_2^{1-\lambda} \varphi} \geqslant s\})\, ds \\ &= \int_0^{1} \mu([\lambda \cdot K +_0 (1-\lambda) \cdot L] \cap \{a_1^{\lambda} a_2^{1-\lambda} \varphi \leqslant -\log s\})\, ds \\ &= \int_0^{\infty} \mu([\lambda \cdot K +_0 (1-\lambda) \cdot L] \cap \{a_1^{\lambda} a_2^{1-\lambda} \varphi \leqslant r\}) \mathrm{e}^{-r}\, dr.\end{aligned}$$

Set $h(r) = \mu([\lambda \cdot K +_0 (1-\lambda) \cdot L] \cap \{a_1^{\lambda} a_2^{1-\lambda} \varphi \leqslant r\}) \mathrm{e}^{-r}$, $f(r) = \mu(K \cap \{a_1 \varphi \leqslant r\}) \mathrm{e}^{-r}$ and $g(r) = \mu(L \cap \{a_2 \varphi \leqslant r\}) \mathrm{e}^{-r}$. We will make use of the multiplicative form of the Prékopa–Leindler inequality. If $r_1, r_2 > 0$, using our assumption, we have

$$\begin{aligned}h(r_1^{\lambda} r_2^{1-\lambda}) &= \mu([\lambda \cdot K +_0 (1-\lambda) \cdot L] \cap \{\varphi \leqslant (a_1^{-1} r_1)^{\lambda} (a_2^{-1} r_2)^{1-\lambda}\}) \mathrm{e}^{-r_1^{\lambda} r_2^{1-\lambda}} \\ &\geqslant \mu(K \cap \{\varphi \leqslant a_1^{-1} r_1\})^{\lambda} \mu(L \cap \{\varphi \leqslant a_2^{-1} r_2\})^{1-\lambda} \mathrm{e}^{-r_1^{\lambda} r_2^{1-\lambda}}\end{aligned}$$

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$$\geqslant \mu(K \cap \{\varphi \leqslant a_1^{-1} r_1\})^{\lambda} \mu(L \cap \{\varphi \leqslant a_2^{-1} r_2\})^{1-\lambda} \mathrm{e}^{-[\lambda r_1 + (1-\lambda) r_2]}$$
$$= [\mu(K \cap \{a_1 \varphi \leqslant r_1\}) \mathrm{e}^{-r_1}]^{\lambda} [\mu(L \cap \{a_2 \varphi \leqslant r_2\}) \mathrm{e}^{-r_2}]^{1-\lambda}$$
$$= f(r_1)^{\lambda} g(r_2)^{1-\lambda}.$$

Thus, by Theorem D, we have

$$\int_{\lambda \cdot K + 0 (1-\lambda) \cdot L} \mathrm{e}^{-a_1^{\lambda} a_2^{1-\lambda} \varphi(x)} \, d\mu(x) = \int_0^{\infty} h(r) \, dr$$
$$\geqslant \left[\int_0^{\infty} f(r) \, dr \right]^{\lambda} \left[\int_0^{\infty} g(r) \, dr \right]^{1-\lambda}$$
$$\geqslant \left[\int_K \mathrm{e}^{-a_1 \varphi(x)} \, d\mu(x) \right]^{\lambda} \left[\int_L \mathrm{e}^{-a_2 \varphi(x)} \, d\mu(x) \right]^{1-\lambda}. \qquad \square$$

Proof of Theorem 3.1. It is clearly sufficient (by approximation) to prove the log-Brunn–Minkowski inequality for absolutely continuous measures with densities of the form $\mathrm{e}^{-\varphi_{s,t}}$, where φ is any even convex function defined in $\mathbb{R}^n$, $t > s > \varphi(0)$ and $\varphi_{s,t}$ is given by

$$\varphi_{s,t}(x) = \begin{cases} \varphi(x), & s \leqslant \varphi(x) \leqslant t, \\ s, & \varphi(x) < s, \\ \infty, & \varphi(x) > t. \end{cases}$$

Note that $\varphi_{s,t}$ satisfies the conditions of Lemma 3.5, for all choices of s and t. Fix s, t and let $c > 0$ and $\overline{\varphi}$ be as in Lemma 3.6, i.e. $\overline{\varphi}(x) = \varphi_{s,t}(x) + c$. Then,

$$\int_{\lambda \cdot K + 0 (1-\lambda) \cdot L} \mathrm{e}^{-\varphi_{s,t}} \, dx = \mathrm{e}^{c} \int_{\lambda \cdot K + 0 (1-\lambda) \cdot L} \mathrm{e}^{-\overline{\varphi}} \, dx.$$

Therefore, we need to prove the log-Brunn–Minkowski inequality for the even log-concave density $\mathrm{e}^{-\overline{\varphi}}$. Note that $\overline{\varphi}$ is non-negative. We need to show that the assumption of Lemma 3.7 is satisfied. Let $r_1, r_2 > 0$. By Lemma 3.6, we have

$$[\lambda \cdot K +_0 (1-\lambda) \cdot L] \cap \{\overline{\varphi} \leqslant r_1^{\lambda} r_2^{1-\lambda}\}$$
$$\supseteq [\lambda \cdot K +_0 (1-\lambda) \cdot L] \cap [\lambda \cdot \{\overline{\varphi} \leqslant r_1\} +_0 (1-\lambda) \cdot \{\overline{\varphi} \leqslant r_2\}]$$
$$\supseteq \lambda \cdot (K \cap \{\overline{\varphi} \leqslant r_1\}) +_0 (1-\lambda) \cdot (L \cap \{\overline{\varphi} \leqslant r_2\}).$$

Since we assumed that the log-Brunn–Minkowski inequality holds for the Lebesgue measure, the assertion follows by taking volumes in the previous inclusion and by Lemma 3.7 (used with $\mu = |\cdot|$ and $a_1 = a_2 = 1$). $\square$

§4. *Log-concavity properties for dual bodies.* We start this section with a simple observation.

LEMMA 4.1. *Let M be a symmetric convex body and μ be a measure, such that the function*

$$\mathbb{R} \ni t \mapsto \mu(\mathrm{e}^t M)$$

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is log-concave. Then, for $p \geqslant 1$, $a \in \mathbb{R}$, the function

$$\mathbb{R} \ni t \mapsto \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{e}^{at}\|x\|_M^p}\, d\mu(x)$$

is also log-concave.

Proof. This is a special case of Lemma 3.7. Indeed, set $K = \mathbb{R}^n = L$ and $\varphi(x) = \|x\|_M^p$. For $\lambda \in (0,1)$, $r_1, r_2 > 0$, we have

$$\begin{aligned}
&\mu([\lambda \cdot K +_0 (1-\lambda)\cdot L] \cap \{\varphi \leqslant r_1^\lambda r_2^{1-\lambda}\}) = \mu(\{\|\cdot\|_M^p \leqslant r_1^\lambda r_2^{1-\lambda}\}) \\
&\quad = \mu(r_1^{\lambda/p} r_2^{(1-\lambda)/p} M) \\
&\quad \geqslant \mu(r_1^{1/p} M)^\lambda \mu(r_2^{1/p} M)^{1-\lambda} \\
&\quad = \mu(\{\|\cdot\|_M \leqslant r_1^{1/p}\})^\lambda \mu(\{\|\cdot\|_M \leqslant r_2^{1/p}\})^{1-\lambda} \\
&\quad = \mu(K \cap \{\varphi \leqslant r_1\})^\lambda \mu(L \cap \{\varphi \leqslant r_2\})^{1-\lambda}.
\end{aligned}$$

Therefore, by Lemma 3.7, if $a_1, a_2 > 0$,

$$\int_{\mathbb{R}^n} \mathrm{e}^{-a_1^\lambda a_2^{1-\lambda}\|x\|_M^p}\, d\mu(x) \geqslant \left[\int_{\mathbb{R}^n} \mathrm{e}^{-a_1\|x\|_M^p}\, d\mu(x)\right]^\lambda \left[\int_{\mathbb{R}^n} \mathrm{e}^{-a_2\|x\|_M^p}\, d\mu(x)\right]^{1-\lambda},$$

proving our claim. □

The following is well known.

LEMMA 4.2. *Let n be an integer and $p > 0$. There exists a constant $c_{n,p} > 0$, that depends only on n and p, such that for any convex body M that contains the origin in its interior, the following is true:*

$$|M| = c_{n,p} \int_{\mathbb{R}^n} \mathrm{e}^{-\|x\|_M^p}\, dx.$$

PROPOSITION 4.3. *Let $p \geqslant 1$ and L be a symmetric convex body which has one of the following two properties.*

(i) *The measure with density $\mathrm{e}^{-\|\cdot\|_L^p}$ satisfies the B-theorem.*
(ii) *The uniform measure of L satisfies the B-theorem.*

Then, for any symmetric convex body K, the function

$$\mathbb{R} \ni t \mapsto |(K^\circ +_p \mathrm{e}^t \cdot L^\circ)^\circ|$$

is log-concave.

Proof. Suppose that (i) holds. Using Lemma 4.1 with $d\mu = \mathrm{e}^{-\|x\|_L^p}\, dx$, we obtain that the function

$$\mathbb{R} \ni t \mapsto \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{e}^{-t}\|x\|_K^p}\, d\mu(x) = \int_{\mathbb{R}^n} \mathrm{e}^{-\|x\|_L^p - \mathrm{e}^{-t}\|x\|_K^p}\, dx =: \phi(t).$$

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is log-concave. Note that $(\|\cdot\|_L^p + \mathrm{e}^{-t}\|\cdot\|_K^p)^{1/p}$ is the support function (= dual norm) of the convex body $L^\circ +_p \mathrm{e}^{-t}\cdot K^\circ$. Therefore, by Lemma 4.2, $\phi(t) = c_{n,p}^{-1}|(K^\circ +_p \mathrm{e}^t L^\circ)^\circ|$ and the function

$$|(K^\circ +_p \cdot \mathrm{e}^t L^\circ)^\circ| = \mathrm{e}^{nt/p}|(L^\circ +_p \mathrm{e}^{-t}\cdot K^\circ)^\circ|$$

is log-concave.

Assume now that (ii) holds. Use again Lemma 4.1 with $d\mu = \mathbf{1}_K(x)\,dx$ to get that the function

$$\mathbb{R} \ni t \mapsto \int_K \mathrm{e}^{-\mathrm{e}^t\|x\|_L^p}\,dx$$

is log-concave. Write

$$\begin{aligned}\int_K \mathrm{e}^{-\mathrm{e}^t\|x\|_L^p}\,dx &= \int_K \mathrm{e}^{-\|\mathrm{e}^{t/p}x\|_L^p}\,dx\\ &= \mathrm{e}^{-nt/p}\int_{\mathrm{e}^{-t/p}K} \mathrm{e}^{\|x\|_L^p}\,dx\\ &=: \mathrm{e}^{-nt/p}\mu'(\mathrm{e}^{-t/p}K),\end{aligned}$$

where μ' is the measure with density $\mathrm{e}^{-\|x\|_L^p}$. Since the function $\mathrm{e}^{-nt/p}$ is log-affine, it follows that the assumption of Lemma 4.1 holds with μ' instead of μ, thus the function

$$\mathbb{R} \ni t \mapsto \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{e}^t\|x\|_K^p}\,d\mu'(x) = \phi(t)$$

is log-concave, where $\phi(t)$ was defined previously and was proven to be proportional to $|(K^\circ +_p \mathrm{e}^t\cdot L^\circ)^\circ|$. This proves our claim. □

Remark 4.4. Proposition 4.3 asserts that the B-conjecture for uniform measures implies the log-concavity of the function $t \mapsto |(K^\circ +_p \mathrm{e}^t\cdot L^\circ)^\circ|$. The opposite is also true, since the limiting case $p = \infty$ is just the B-conjecture for uniform measures.

Next, let us confirm the B-conjecture for uniform measures in its most simple case: the case of the symmetric strips.

THEOREM 4.5. *Let $u \in S^{n-1}$, $a > 0$. Set $E = \{x \in \mathbb{R}^n \mid |x\cdot u| \leqslant a\}$. Then, for every symmetric convex body K, the function $\mathbb{R} \ni t \mapsto |E \cap \mathrm{e}^t K|$ is log-concave.*

Proof. We need to prove that for every $\lambda \in (0,1)$, $t_1, t_2 \in \mathbb{R}$, then

$$|E \cap \mathrm{e}^{\lambda t_1 + (1-\lambda)t_2}K| \geqslant |E \cap \mathrm{e}^{t_1}K|^\lambda |E \cap \mathrm{e}^{t_2}K|^{1-\lambda}. \tag{6}$$

One can easily verify that for each $b > 0$,

$$S_u(E \cap bK) = E \cap bS_u K,$$

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hence nothing changes in (6) in terms of volumes if we replace K with the Schwarz symmetrization $S_u K$. But then, E, $S_u K$ are unconditional with respect to some (any) orthonormal basis that contains u. Now, Theorem C proves our claim. □

The next corollary follows immediately by Proposition 4.3 and Theorems 4.5, A, B and C.

COROLLARY 4.6. *Let K, L be symmetric convex bodies, $p \geqslant 1$ and u be a unit vector. The function $\mathbb{R} \ni t \mapsto |(K^\circ +_p \mathrm{e}^t \cdot L)^\circ|$ is log-concave (at least) in the following cases:*

(i) *$p = 2$ and $L = B_2^n$;*
(ii) *L is an origin symmetric line segment;*
(iii) *K and L are unconditional, with respect to the same orthonormal basis;*
(iv) *K and L are planar.*

The variance conjecture [1, 3] states that if X is a random vector with log concave probability density f, whose barycenter is at the origin and its covariance matrix is the identity (i.e. X is isotropic), then the variance of $\|X\|_2^2$ satisfies

$$\mathrm{Var}(\|X\|_2^2) \leqslant Cn,$$

where $C > 0$ is an absolute constant. The variance conjecture plays a central role in modern convex geometry. Surprisingly, it implies other major conjectures (see [12, 13]), such as the slicing problem and the KLS conjecture [25] up to a logarithmic factor. The best general known estimate to date is of order $n^{5/3}$, due to Guedon and Milman [22] (see also [17]). It has been confirmed for random vectors with unconditional log-concave densities [27] (see also [11, 16]). We refer to [23] for more information and references.

We would like to restrict our attention to the class of random vectors whose densities are indicator functions of symmetric convex bodies. In this case the variance conjecture becomes: let K be a symmetric isotropic convex body. Then,

$$\sigma^2(K) := \frac{|K| \int_K \|x\|_2^4 \, dx - \left[\int_K \|x\|_2^2 \, dx\right]^2}{\frac{1}{n}\left[\int_K \|x\|^2 \, dx\right]^2} \leqslant C.$$

LEMMA 4.7. *Let K, L be symmetric convex bodies, $p \geqslant 1$, $a > 0$. Set $P := (K^\circ +_p a \cdot L^\circ)^\circ$. If the function*

$$\mathbb{R} \ni t \mapsto |(K^\circ +_p \mathrm{e}^t a \cdot L^\circ)^\circ|$$

is log-concave, then

$$|P| \int_P \|x\|_L^{2p} \, dx - \left[\int_P \|x\|_L^p \, dx\right]^2 \leqslant \frac{p}{a(n+p)} |P| \int_P \|x\|_L^p \, dx.$$

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Proof. Set $f(t)=|(K^\circ +_p \mathrm{e}^t a\cdot L^\circ)^\circ|$. Then, $f(0)=|P|$ and f is log-concave. Integrating in polar coordinates we obtain

$$f(t)=\frac{1}{n}\int_{S^{n-1}}(\|x\|_K^p+\mathrm{e}^t a\|x\|_L^p)^{-n/p}\,dx,$$

thus

$$f'(t)=\frac{1}{n}\int_{S^{n-1}}\frac{-n}{p}\mathrm{e}^t a\|x\|_L^p(\|x\|_K^p+\mathrm{e}^t a\|x\|_L^p)^{-(n+p)/p}\,dx.$$

So,

$$f'(0)=-\frac{1}{p}\int_{S^{n-1}}a\|x\|_L^p\|x\|_T^{-(n+p)}\,dx=-a\frac{n+p}{p}\int_T\|x\|_L^p\,dx.$$

Also,

$$f''(t)=f'(t)+\frac{1}{n}\int_{S^{n-1}}\frac{n(n+p)}{p^2}\mathrm{e}^{2t}a^2\|x\|_L^{2p}(\|x\|_K^p+\mathrm{e}^t a\|x\|_L^p)^{-(n+2p)/p}\,dx.$$

Therefore,

$$\begin{aligned}f''(0)&=-\frac{a(n+p)}{p}\int_P\|x\|_L^p\,dx\\&\quad+\frac{a^2(n+p)(n+2p)}{p^2}\int_{S^{n-1}}\frac{1}{n+2p}\|x\|_L^{2p}\|x\|_P^{-(n+2p)}\,dx\\&=-\frac{a(n+p)}{p}\int_P\|x\|_L^p\,dx+\frac{a^2(n+p)(n+2p)}{p^2}\int_P\|x\|_L^{2p}\,dx\\&\geqslant-\frac{a(n+p)}{p}\int_P\|x\|_L^p\,dx+\left[\frac{a(n+p)}{p}\right]^2\int_P\|x\|_L^{2p}\,dx.\end{aligned}$$

Now, the log-concavity of f implies $f''(0)f(0)\leqslant[f'(0)]^2$ and the assertion follows. □

For $a>0$, define the class of convex bodies $\mathcal{C}_a$ as follows:

$$\begin{aligned}\mathcal{C}_a=\{&(K^\circ +_2 a\cdot B_2^n)^\circ \mid K \text{ is a symmetric convex body,}\\&(K^\circ +_2 a\cdot B_2^n)^\circ \text{ is isotropic}\}.\end{aligned}$$

Combining Lemma 4.7 with Corollary 4.6, we immediately obtain the following result.

THEOREM 4.8. *Let $P\in\mathcal{C}_a$, with $|P|=1$, for some $a>0$. Then,*

$$\sigma^2(P)\leqslant\frac{2}{a(n+2)L_P^2}.$$

In particular, if $a>c/n$, for some absolute constant $c>0$, then P satisfies the variance conjecture.

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Before ending this section, we would like to give an alternative description of the class $\mathcal{C}_a$.

LEMMA 4.9. *Let K be a symmetric convex body and a be a positive number. Then, $(K +_2 a \cdot B_2^n)^\circ$ is isotropic if and only if*

$$|(K +_2 a \cdot B_2^n)^\circ| = \max_{T \in SL_n} |(TK +_2 a \cdot B_2^n)^\circ|. \tag{7}$$

Proof. It is easy to check that the quantity $|(TK +_2 a \cdot B_2^n)^\circ|$ indeed attains a maximum, among $T \in SL(n)$. Let $v \in S^{n-1}$, $t \in \mathbb{R}$, $|t| < 1$. Define the linear map

$$T_t(x) = \left(\frac{1}{1+t}\right)^{1/n} (x + t(x \cdot v)v).$$

Then, $T_t \in SL_n$. Using polar coordinates, one may compute

$$\begin{aligned}
&\frac{\partial}{\partial t}\bigg|_{t=0} |(T_t^{-1}K +_2 a \cdot B_2^n)^\circ| \\
&\quad = \frac{\partial}{\partial t}\bigg|_{t=0} |(K +_2 a \cdot T_t B_2^n)^\circ| \\
&\quad = \frac{\partial}{\partial t}\bigg|_{t=0} \frac{1}{n} \int_{S^{n-1}} (h_K(x)^2 + a\|T_t x\|_2^2)^{-n/2}\, dx \\
&\quad = \frac{\partial}{\partial t}\bigg|_{t=0} \frac{1}{n} \int_{S^{n-1}} (h_K(x)^2 + a(1+t)^{-2/n}\|x + t(x \cdot v)\|_2^2)^{-n/2}\, dx \\
&\quad = \frac{1}{n} \int_{S^{n-1}} \frac{n}{2}((2a/n)\|x\|_2^2 - 2a(x \cdot v)^2)(h_K(x)^2 + a\|x\|_2^2)^{-(n+2)/2}\, dx \\
&\quad = (n+2)a\left[-\int_{(K+_2 aB_2^n)^\circ} (x \cdot v)^2\, dx + \frac{1}{n}\int_{(K+_2 aB_2^n)^\circ} \|x\|_2^2\, dx\right].
\end{aligned} \tag{8}$$

Therefore, if (7) holds, then the derivative at $t = 0$ of the volume of $(T_t^{-1}K +_2 a \cdot B_2^n)^\circ$ equals zero, so by (8),

$$\int_{(K+_2 aB_2^n)^\circ} (x \cdot v)^2\, dx = \frac{1}{n}\int_{(K+_2 aB_2^n)^\circ} \|x\|_2^2\, dx.$$

Since this is true for all $v \in S^{n-1}$, it follows that $(K +_2 a \cdot B_2^n)^\circ$ is isotropic. On the other hand, if T_0 is a critical point of the function $SL_n \ni T \mapsto |(TK +_2 a \cdot B_2^n)^\circ|$, we have proved that $(T_0K +_2 a \cdot B_2^n)^\circ$ is isotropic. By the uniqueness up to isometry of the isotropic position, it follows that this critical point is unique, thus if $(K +_2 a \cdot B_2^n)^\circ$ is isotropic, then (7) holds. □

§5. *Reduction to the log-Brunn–Minkowski inequality for coordinate parallelepipeds.* The goal of this section is to establish the following.

THEOREM 5.1. (i) *Assume that for all $n \in \mathbb{N}$, there exists an even function $f_n : \mathbb{R}^n \to \mathbb{R}$, whose restriction to any subspace of $\mathbb{R}^n$ is integrable, with the*

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following property: for all $T \in GL_n$ and for all diagonal $n \times n$ matrices A, the function

$$\mathbb{R} \ni t \mapsto \int_{\mathrm{e}^{tA}C_n} f_n(Tx)\,dx$$

is log-concave. Then, the log-Brunn–Minkowski inequality holds for all even log-concave densities $g : \mathbb{R}^n \to \mathbb{R}$, for all $n \in \mathbb{N}$.

(ii) *Assume that for all $n \in \mathbb{N}$, there exists an even function $f_n : \mathbb{R}^n \to \mathbb{R}$, whose restriction in any subspace of $\mathbb{R}^n$ is integrable, with the following property: the log-Brunn–Minkowski inequality holds for the density f_n, for any two parallelepipeds with parallel facets. Then, the log-Brunn–Minkowski inequality holds for all even log-concave densities $g : \mathbb{R}^n \to \mathbb{R}$, for all $n \in \mathbb{N}$.*

(iii) *Fix $n \in \mathbb{N}$. If the log-Brunn–Minkowski inequality holds for some density $f : \mathbb{R}^n \to \mathbb{R}$, then for all $T \in GL_n$ and for all diagonal $n \times n$-matrices A, the function*

$$\mathbb{R} \ni t \mapsto \int_{\mathrm{e}^{tA}C_n} f(Tx)\,dx$$

is log-concave.

Proof. Let us first prove (iii). It is easily verified (see [5, 41]) that if $s, t \in \mathbb{R}$ and A is a diagonal $n \times n$ matrix, then for $\lambda \in (0, 1)$,

$$\mathrm{e}^{[\lambda s+(1-\lambda)t]A}C_n = \lambda \cdot (\mathrm{e}^{sA}C_n) +_0 (1-\lambda) \cdot (\mathrm{e}^{tA}C_n).$$

Therefore, for $T \in GL_n$,

$$\begin{aligned}\lambda \cdot (T\mathrm{e}^{sA}C_n) +_0 (1-\lambda) \cdot (T\mathrm{e}^{tA}C_n) &= T[\lambda \cdot (\mathrm{e}^{sA}C_n) +_0 (1-\lambda) \cdot (\mathrm{e}^{tA}C_n)] \\ &= T\mathrm{e}^{[\lambda s+(1-\lambda)t]A}C_n. \qquad (9)\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\mathrm{e}^{[\lambda s+(1-\lambda)t]A}C_n} f(Tx)\,dx &= |\det T| \int_{T\mathrm{e}^{[\lambda s+(1-\lambda)t]A}C_n} f(x)\,dx \\ &= |\det T| \int_{\lambda\cdot(T\mathrm{e}^{sA}C_n)+_0(1-\lambda)\cdot(T\mathrm{e}^{tA}C_n)} f(x)\,dx \\ &\geqslant |\det T| \left[\int_{T\mathrm{e}^{sA}C_n} f(x)\,dx\right]^{\lambda} \left[\int_{T\mathrm{e}^{tA}C_n} f(x)\,dx\right]^{1-\lambda} \\ &= \left[\int_{\mathrm{e}^{sA}C_n} f(Tx)\,dx\right]^{\lambda} \left[\int_{\mathrm{e}^{tA}C_n} f(Tx)\,dx\right]^{1-\lambda}.\end{aligned}$$

Assertion (ii) is just a reformulation of (i). Indeed, one can check that if P_1, P_2 are two parallelepipeds with parallel facets, then there exist $s_1, s_2 \in \mathbb{R}$, a diagonal matrix A and a $GL(n)$-map T, such that $P_i = T\mathrm{e}^{s_i A}C_n$, $i = 1, 2$. Thus, by (9),

$$\int_{\lambda\cdot P_1+_0(1-\lambda)\cdot P_2} f_n(x)\,dx = |\det T| \int_{\mathrm{e}^{[\lambda s+(1-\lambda)t]A}C_n} f_n(Tx)\,dx.$$

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It remains to prove (i). Let K, L be symmetric convex bodies in $\mathbb{R}^n$ and $\lambda \in (0,1)$. As in [41, Theorem 1.5], consider the following discretized version of the logarithmic sum of K and L: let $v_1, \dots, v_m$ be unit vectors in $\mathbb{R}^n$, $m \geqslant n$. Set

$$R_\lambda := \{x \in \mathbb{R}^n \mid |x \cdot v_i| \leqslant r_i^\lambda s_i^{1-\lambda}, i = 1, \dots, m\},$$

where $r_i = h_K(v_i)$, $s_i = h_L(v_i)$, $i = 1, \dots, m$. We will prove that under the assumption of (i),

$$|R_\lambda| \geqslant |R_0|^{1-\lambda} |R_1|^\lambda. \tag{10}$$

Since R_λ can be chosen to be arbitrarily close to $\lambda \cdot K +_0 (1-\lambda) \cdot L$, as $m \to \infty$, if (10) is proved for any choice of the v_i, r_i, s_i, then the log-Brunn–Minkowski inequality will be established for the Lebesgue measure. But then, by Theorem 3.1, the log-Brunn–Minkowski inequality for any log-concave measure will follow. Therefore, it suffices to prove (10) for any choice of $m, r_i > 0, s_i > 0$, $v_i \in S^{n-1}$, $i = 1, \dots, m$. As in [41], write

$$|R_\lambda| = \int_{\mathbb{R}^n} \prod_{i=1}^m \mathbf{1}_{[-r_i^\lambda s_i^{1-\lambda}, r_i^\lambda s_i^{1-\lambda}]}(x \cdot v_i)\, dx.$$

Set also,

$$G_\lambda(\varepsilon) := \int_{x \in \mathbb{R}^n} \int_{u \in \mathbb{R}^m} \prod_{i=1}^m \mathbf{1}_{[-r_i^\lambda s_i^{1-\lambda}, r_i^\lambda s_i^{1-\lambda}]}(x \cdot v_i + u_i) \varepsilon^{-n} f_{m+n}(u/\varepsilon)\, du\, dx,$$

where $\varepsilon > 0$, $u_i = u \cdot e_i$, $i = 1, \dots, m$ and $\{e_1, \dots, e_m\}$ is an orthonormal basis in $\mathbb{R}^m$. It follows by the change of variables $U := u/\varepsilon$ that

$$\begin{aligned} G_\lambda(\varepsilon) &:= \int_{x \in \mathbb{R}^n} \int_{U \in \mathbb{R}^m} \prod_{i=1}^m \mathbf{1}_{[-r_i^\lambda s_i^{1-\lambda}, r_i^\lambda s_i^{1-\lambda}]}(x \cdot v_i + \varepsilon U_i) f_{m+n}(U)\, dU\, dx \\ &\xrightarrow{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \prod_{i=1}^m \mathbf{1}_{[-r_i^\lambda s_i^{1-\lambda}, r_i^\lambda s_i^{1-\lambda}]}(x \cdot v_i)\, dx \int_{U \in \mathbb{R}^m} f_{m+n}(U)\, dU \\ &= \|f_{m+n}|\mathbb{R}^m\|_1 \cdot |R_\lambda|. \end{aligned}$$

Thus, it suffices to prove that, for $\varepsilon > 0$,

$$G_\lambda(\varepsilon) \geqslant G_1(\varepsilon)^\lambda G_0(\varepsilon)^{1-\lambda}.$$

Using the change of variables $w_i := (u_i + x \cdot v_i)/s_i$, $i = 1, \dots, m$, we obtain

$$\begin{aligned} G_\lambda(\varepsilon) = A \int_{w \in \mathbb{R}^m} \int_{x \in \mathbb{R}^n} \prod_{i=1}^m \mathbf{1}_{[-r_i^\lambda s_i^{1-\lambda}, r_i^\lambda s_i^{1-\lambda}]}(s_i w_i) \\ \times f_{m+n}\left(\varepsilon^{-1} \sum_{i=1}^m (s_i w_i - x \cdot v_i) e_i\right) dx\, dw \end{aligned}$$

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$$= \lim_{a\to\infty} A \int_{w\in\mathbb{R}^m}\int_{x\in\mathbb{R}^n} \prod_{i=1}^m \mathbf{1}_{[-r_i^\lambda s_i^{1-\lambda}, r_i^\lambda s_i^{1-\lambda}]}(s_i w_i)\mathbf{1}_{aC_n}(x)$$
$$\times f_{m+n}\left(\varepsilon^{-1}\sum_{i=1}^m (s_i w_i - x\cdot v_i)e_i\right) dx\, dw,$$

where $A = \varepsilon^{-n} s_1 \dots s_n$. Define the (singular) linear map $T : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$, with $T(w,x) = \varepsilon^{-1}\sum_{i=1}^m (s_i w_i - x\cdot v_i)e_i$. Note that the linear map $T_\delta := T + \delta \mathrm{Id}_{\mathbb{R}^{m+n}}$ becomes invertible, for $\delta > 0$, small enough. Therefore, if

$$A_a := \mathrm{diag}(\log(r_1 s_1^{-1}), \dots, \log(r_m s_m^{-1}), \log a, \dots, \log a), \quad a > 1,$$

then

$$G_\lambda(\varepsilon) = A \lim_{a\to\infty}\lim_{\delta\to 0^+} \int_{\mathrm{e}^{[\lambda\cdot 1+(1-\lambda)\cdot 0]A_a} C_{m+n}} f_{m+n}(T_\delta z)\, dz$$
$$=: A \lim_{a\to\infty}\lim_{\delta\to 0^+} F(a,\delta,\lambda).$$

Using our assumption,

$$F(a,\delta,\lambda) \geqslant F(a,\delta,1)^\lambda F(a,\delta,0)^{1-\lambda},$$

for all $a > 1$, $\delta > 0$ (δ small enough). This proves our claim. □

§6. *The dual log-Brunn–Minkowski inequality.* The main goal of this section is to establish the following dual log-Brunn–Minkowski inequality (see Corollary 6.6 below).

THEOREM 6.1. *Let K, L be two convex bodies in $\mathbb{R}^n$ and $\lambda \in (0,1)$. Then,*

$$|(\lambda\cdot K +_0 (1-\lambda)\cdot L)^\circ| \leqslant |K^\circ|^\lambda |L^\circ|^{1-\lambda}. \tag{11}$$

Once Theorem 6.1 is established, one can follow Firey's argument [**15**] to prove Corollary 6.6 (see below), where the volume is replaced by the other quermassintegrals. Since the dual L^0-sum contains the dual L^p-sum, for $p \geqslant 0$, Theorem 6.1 extends immediately to the L^p-setting, for all $p \geqslant 0$. Therefore, it is stronger than Firey's [**14**] dual Brunn–Minkowski inequality. It is also stronger than the dual Brunn–Minkowski inequality with respect to L^0-radial sums, established in [**21**]. It seems plausible that the equality cases in (11) are exactly the equality cases in Conjecture 1.1 (here of course non-symmetric bodies are allowed); we do not address this here.

Without loss of generality, we may assume that K and L contain zero in their interiors. Otherwise, the assertion would be trivial. As in the previous section, we will prove our claim for the (asymmetric) discrete approximations of the logarithmic sum K and L. The rest of the proof will follow by compactness. Set

$$AR_\lambda = \{x\in\mathbb{R}^n \mid x\cdot v_i \leqslant r_i^\lambda s_i^{1-\lambda},\ i = 1,\dots,m\},$$

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where $v_1, \dots, v_m \in S^{n-1}$, $r_i = h_K(v_i)$, $s_i = h_L(v_i)$, $i = 1, \dots, m$. Since K and L contain zero in their interiors, it is true that $r_i, s_i > 0$, $i = 1, \dots, m$, thus AR_λ is well defined. One, then, needs to prove that the function $(0, 1) \ni \lambda \mapsto |(AR_\lambda)^\circ|$ is log-convex (i.e. its logarithm is convex). On the other hand,

$$(AR_\lambda)^\circ = \mathrm{conv}\{(r_i^{-1})^\lambda (s_i^{-1})^{1-\lambda} v_i \mid i = 1, \dots, m\},$$

therefore the proof of Theorem 6.1 reduces to the proof of the following result.

THEOREM 6.2. *Let $x_1, \dots, x_m \in \mathbb{R}^n$, $a_1, \dots, a_m \in \mathbb{R}$ and consider the family of polytopes*

$$P_t = \mathrm{conv}\{e^{a_i t} x_i \mid i = 1, \dots, m\}, \quad t \in (t_1, t_2),$$

for some $t_1 < t_2$. If P_t contains the origin in its interior for all $t \in (t_1, t_2)$, then the function

$$(t_1, t_2) \ni t \mapsto |P_t|$$

is log-convex.

We remark here that Theorem 6.2 may be viewed as the dual version of the B-conjecture for uniform measures in the following sense.

COROLLARY 6.3. *If K, L are convex bodies that contain the origin in their interiors, then the function $t \mapsto |((e^t K) \cap L)^\circ|$ is log-convex.*

The idea for the proof of Theorem 6.2 is taken from Saroglou [**40**, Theorem 3.1]. First we will need two easy lemmas.

LEMMA 6.4. *Let $f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R}_+$ be log-convex functions. Then, their sum is log-convex.*

Proof. For $\lambda \in (0, 1)$, $t_1, t_2 \in \mathbb{R}^n$, we have

$$\sum_{i=1}^m f_i(\lambda t_1 + (1-\lambda) t_2) \leqslant \sum_{i=1}^m f_i^\lambda(t_1) f_i^{1-\lambda}(t_2) \leqslant \left[\sum_{i=1}^m f_i(t_1)\right]^\lambda \left[\sum_{i=1}^m f_i(t_2)\right]^{1-\lambda}.$$

This proves our assertion. □

LEMMA 6.5. *Let $x_1, \dots, x_n \in \mathbb{R}^n$, $a_1, \dots, a_n \in \mathbb{R}$. Then, the function*

$$t \mapsto |\mathrm{conv}\{0, e^{a_1 t} x_1, \dots, e^{a_n t} x_n\}|$$

is log-affine and therefore log-convex.

Proof. We have

$$|\mathrm{conv}\{0, e^{a_1 t} x_1, \dots, e^{a_n t} x_n\}| = \frac{1}{n!} |\det(e^{a_1 t} x_1, \dots, e^{a_n t} x_n)| = \frac{e^{a_1 t} \dots e^{a_n t}}{n!},$$

proving our claim. □

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Proof of Theorem 6.2. We need to prove that the function $(t_1, t_2) \ni t \mapsto |P_t|$ is log-convex. Actually, we need to prove that for any $s_1, s_2 \in (t_1, t_2)$, $s_1 < s_2$,

$$|P_{(s_1+s_2)/2}| \leqslant |P_{s_1}|^{1/2}|P_{s_2}|^{1/2}.$$

Set $s = (s_1 + s_2)/2$, $p = (s_2 - s_1)/2$. Let $\{T_1, \dots, T_k\}$ be a triangulation of the boundary of P_s; that is a subdivision of the boundary of P_s into non-overlapping ($(n-1)$-dimensional) simplices, whose vertices are vertices of P_s. Set

$$\Delta_i := \operatorname{conv}(\{0\} \cup T_i), \quad i = 1, \dots, k.$$

Then, the family $\{\Delta_1, \dots, \Delta_k\}$ is a triangulation of P_s. For $i = 1, \dots, k$, consider the following transformation of Δ_i: if $\Delta_i = \operatorname{conv}\{0, \mathrm{e}^{a_{j_1}s}x_{j_1}, \dots, \mathrm{e}^{a_{j_n}s}x_{j_n}\}$, for some $1 \leqslant j_1 < \cdots < j_n \leqslant m$, set

$$\Delta_{i,r} = \operatorname{conv}\{0, \mathrm{e}^{a_{j_1}(r+s)}x_{j_1}, \dots, \mathrm{e}^{a_{j_n}(r+s)}x_{j_n}\}, \quad r \in [-p, p].$$

It is clear that the $\Delta_{i,r}$ are non-overlapping, for $r \in [-p, p]$. This is because, for $i = 1, \dots, k$, $r \in [-p, p]$, $\Delta_{i,r}$ is contained in the positive cone spanned by Δ_i and every two such cones are, by construction, non-overlapping. Now, it is clear that

$$|P_{s+r}| = \left|\operatorname{conv}\left(\bigcup_{i=1}^{k} \Delta_{i,r}\right)\right| \geqslant \sum_{i=1}^{k} |\Delta_{i,r}|, \quad r \in [-p, p]. \tag{12}$$

By Lemmas 6.4 and 6.5, the function $[-p, p] \ni r \mapsto \sum_{i=1}^{k} |\Delta_{i,r}|$ is log-convex. Thus,

$$\begin{aligned} |P_s| &= \sum_{i=1}^{k} |\Delta_i| = \sum_{i=1}^{k} |\Delta_{i,0}| \leqslant \left(\sum_{i=1}^{k} |\Delta_{i,-p}|\right)^{1/2} \left(\sum_{i=1}^{k} |\Delta_{i,p}|\right)^{1/2} \\ &\leqslant |P_{s-p}|^{1/2}|P_{s+p}|^{1/2} = |P_{s_1}|^{1/2}|P_{s_2}|^{1/2}, \end{aligned}$$

as required. □

COROLLARY 6.6. *Let K, L be two convex bodies that contain zero in their interior. For $i = 0, \dots, n-1$, $p \geqslant 0$ and $\lambda \in (0, 1)$, the following is true:*

$$W_i([\lambda \cdot K +_p (1-\lambda) \cdot L]^\circ) \leqslant W_i(K^\circ)^\lambda W_i(L^\circ)^{1-\lambda}.$$

Corollary 6.6 also generalizes a result of Firey [**15**], who proved this for the L^1-sum. This, was recently extended in the L^p-case, for $p \geqslant 1$ in [**24**], where it was explained that by the homogeneity of the quermassintegrals, dual Brunn–Minkowski inequalities have dimension-dependent equivalent forms (in the same manner as the original Brunn–Minkowski inequality does; see e.g. [**19**]). Adopting the same argument we obtain the following corollary.

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COROLLARY 6.7. *Let K, L be two convex bodies that contain the origin in their interior. For $i = 0, \dots, n-1$, $p \geqslant 0$ and $\lambda \in (0,1)$, the following is true:*

$$W_i([\lambda \cdot K +_p (1-\lambda) \cdot L]^\circ)^{-p/(n-i)} \geqslant \lambda W_i(K^\circ)^{-p/(n-i)} + (1-\lambda) W_i(L^\circ)^{-p/(n-i)}.$$

Proof. Use Corollary 6.6 with $\overline{K} = W_i(K^\circ)^{1/(n-i)} K$, $\overline{L} = W_i(L^\circ)^{1/(n-i)} L$, $\overline{\lambda} = \lambda W_i(L^\circ)^{-p/(n-i)} \cdot \mu^{-1}$, where $\mu = \lambda W_i(K^\circ)^{-p/(n-i)} + (1-\lambda) W_i(L^\circ)^{-p/(n-i)}$, in the place of K, L, λ respectively. □

For the proof of Corollary 6.6, the following (contained in an earlier version of [**41**]) is required.

LEMMA 6.8. *Let $\lambda \in [0,1]$, K, L be convex bodies in $\mathbb{R}^n$ and H be a subspace of $\mathbb{R}^n$. Then,*

$$\lambda \cdot (K \cap H) +_0 (1-\lambda) \cdot (L \cap H) \subseteq [\lambda \cdot K +_0 (1-\lambda) \cdot L] \cap H,$$

where the logarithmic sum in the left-hand side of the previous inclusion is considered with respect to the subspace H.

Proof. Note that if x, $u \in H$ and $y \in H^\perp$, then $x \cdot (u+y) = x \cdot u$,

$$h_{K \cap H}(u+y) = \max_{z \in K \cap H} z \cdot (u+y) = \max_{z \in K \cap H} z \cdot u = h_{K \cap H}(u)$$

and, similarly, $h_{L \cap H}(u+y) = h_{L \cap H}(u)$. Thus,

$$\begin{aligned}
&\lambda \cdot (K \cap H) +_0 (1-\lambda) \cdot (L \cap H) \\
&\quad = \{x \in H \mid x \cdot (u+y) \leqslant h^\lambda_{K \cap H}(u+y) h^{1-\lambda}_{L \cap H}(u+y), \forall u \in H, y \in H^\perp\} \\
&\quad = \{x \in H \mid x \cdot w \leqslant h^\lambda_{K \cap H}(w) h^{1-\lambda}_{L \cap H}(w), \forall w \in \mathbb{R}^n\} \\
&\quad \subseteq \{x \in H \mid x \cdot w \leqslant h^\lambda_K(w) h^{1-\lambda}_L(w), \forall w \in \mathbb{R}^n\} \\
&\quad = [\lambda \cdot K +_0 (1-\lambda) \cdot L] \cap H.
\end{aligned}$$ □

Proof of Corollary 6.6. First of all, by Theorem 6.1, we may assume that $i > 0$. We will make use of Firey's argument for passing from the volume to other quermassintegrals (see [**15**]) and the fact that the p-convex combination of convex bodies contains the logarithmic convex combination, $p > 0$. It follows immediately by Lemma 6.8, that

$$[\lambda \cdot K +_0 (1-\lambda) \cdot L]^o \mid H \subseteq [\lambda \cdot (K \cap H) +_0 (1-\lambda) \cdot (L \cap H)]^\circ.$$

Therefore, by the Kubota formula and Theorem 6.1, we obtain

$$\begin{aligned}
W_i([\lambda \cdot K +_p (1-\lambda) \cdot L]^\circ) &\leqslant W_i([\lambda \cdot K +_0 (1-\lambda) \cdot L]^\circ) \\
&= \frac{\omega_n}{\omega_{n-i}} \int_{\mathcal{G}_{n,n-i}} |[\lambda \cdot K +_0 (1-\lambda) \cdot L]^o \mid H|_{n-i} \, dH
\end{aligned}$$

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$$\leqslant \frac{\omega_n}{\omega_{n-i}} \int_{\mathcal{G}_{n,n-i}} |[\lambda \cdot (K \cap H) +_0 (1-\lambda) \cdot (L \cap H)]^\circ|_{n-i}\, dH$$
$$\leqslant \frac{\omega_n}{\omega_{n-i}} \int_{\mathcal{G}_{n,n-i}} |(K \cap H)^\circ|_{n-i}^{\lambda} |(L \cap H)^\circ|_{n-i}^{1-\lambda}\, dH$$
$$= \frac{\omega_n}{\omega_{n-i}} \int_{\mathcal{G}_{n,n-i}} |K^\circ|H|_{n-i}^{\lambda} |L^\circ|H|_{n-i}^{1-\lambda}\, dH$$
$$\leqslant \left[\frac{\omega_n}{\omega_{n-i}} \int_{\mathcal{G}_{n,n-i}} |K^\circ|H|_{n-i}\, dH\right]^{\lambda} \left[\frac{\omega_n}{\omega_{n-i}} \int_{\mathcal{G}_{n,n-i}} |L^\circ|H|_{n-i}\, dH\right]^{1-\lambda}$$
$$= W_i(K^\circ)^{\lambda} W_i(L^\circ)^{1-\lambda}, \quad i = 1, \ldots, n-1. \qquad \square$$

Before ending this note, we would like to state a consequence of Theorem 6.1 that concerns the logarithmic sum itself, rather than its dual.

COROLLARY 6.9. *Let $\Delta_1, \Delta_2 \subseteq \mathbb{R}^2$ be two triangles whose centroids are at the origin. Then,*

$$|\lambda \cdot \Delta_1 +_0 (1-\lambda) \cdot \Delta_2| \geqslant |\Delta_1|^{\lambda} |\Delta_2|^{1-\lambda}. \tag{13}$$

Proof. It is well known (see [37]) that if K is any planar convex body, then

$$|K| \cdot |K^\circ| \geqslant |\Delta_1| \cdot |\Delta_1^\circ|. \tag{14}$$

Now, if (13) is not true, then by Theorem 6.1 and (14) we obtain

$$|\lambda \cdot \Delta_1 +_0 (1-\lambda) \cdot \Delta_2| \cdot |(\lambda \cdot \Delta_1 +_0 (1-\lambda) \cdot \Delta_2)^\circ|$$
$$< |\Delta_1|^{\lambda} |\Delta_2|^{1-\lambda} |\Delta_1^\circ|^{\lambda} |\Delta_2^\circ|^{1-\lambda} = |\Delta_1| \cdot |\Delta_1^\circ|,$$

which contradicts (14). $\square$

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