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On finite groups with bounded conjugacy classes of commutators

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Abstract
In 1954, B. H. Neumann discovered that if G is a group in which all conjugacy classes have finite cardinality at most m, then the derived group G' is finite of m-bounded order. In 2018, G. Dierings and P. Shumyatsky showed that if $|x^G| \leq m$ for any commutator $x \in G$, then the second derived group G'' is finite and has m-bounded order. This paper deals with finite groups in which $|x^G| \leq m$ whenever $x \in G$ is a commutator of prime power order. The main result is that G'' has m-bounded order.

1. Introduction

Given a group G and an element $x \in G$, we write x^G for the conjugacy class containing x. Of course, if the number of elements in x^G is finite, we have $|x^G| = [G : C_G(x)]$. B. H. Neumann discovered that if all conjugacy classes in a group G are finite of size at most m, then the order of the commutator subgroup G' is finite and bounded in terms of m only [10]. A first explicit bound for the order of G' was found by J. Wiegold [15], and the best known was obtained in [8] (see also [11, 13]).

There are several recent publications where Neumann's theorem has been extended in various directions (see in particular [1–3]). Here, we are interested in groups G such that $|x^G| \leq m$ whenever x is a commutator, that is, there are elements $a, b \in G$ for which $x = a^{-1}b^{-1}ab$. The following results are known.

Theorem 1.1. *Let m be a positive integer and G a group in which $|x^G| \leq m$ whenever x is a commutator. Then, we have*

(1) The second commutator subgroup G'' is finite of m-bounded order.
(2) G has a normal subgroup H, such that the index $[G : H]$ and the order of $\gamma_4(H)$ are both m-bounded.

Throughout, we say that a quantity is $(a, b, c \ldots)$-bounded to mean that it is bounded by a constant depending only on the parameters $a, b, c \ldots$.

Part 1 of the above theorem was established in [3]. Part 2 is immediate from the main result of [5].

In the case of finite groups, Neumann's theorem admits an interesting generalization:

Proposition 1.2. *Let m be a positive integer and G a finite group in which $|x^G| \leq m$ whenever x is an element of prime power order. Then, G' has m-bounded order.*

A proof of the above statement is given in the next section. In this article, we examine the question whether, in the case of finite groups, Theorem 1.1 can be generalized in a similar manner.

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More precisely, we concentrate on finite groups G in which $|x^G| \leq m$ whenever x is a commutator of prime power order. We show by means of examples that the Fitting subgroup of such a group can have arbitrarily large index. Thus, Part 2 of Theorem 1.1 cannot be extended to the groups in question.

The main purpose of this paper is to show that, in the case of Part 1, the answer to the above question is positive. We establish the following theorem.

Theorem 1.3. *Let m be a positive integer and G a finite group in which $|x^G| \leq m$ whenever x is a commutator of prime power order. Then, G'' has m-bounded order.*

In view of Theorem 1.3, it is natural to raise the question whether a similar result holds for other commutator words. A theorem obtained in [2] says that if w is a multilinear commutator word of weight n and G a group in which $|x^G| \leq m$ whenever x is a w-value, then the commutator subgroup of the verbal subgroup $w(G)$ has finite (m, n)-bounded order. It would be interesting to see whether, given a finite group G, the hypothesis that $|x^G| \leq m$ for any w-value x of prime power order implies that the commutator subgroup of $w(G)$ has (m, n)-bounded order. At present, we are unable to answer the question.

The notation used in this paper is mostly standard.

2. Preliminaries

We start this section with a proof of Proposition 1.2. Throughout, we write $\pi(G)$ to denote the set of prime divisors of the order of a finite group G.

Proof of Proposition 1.2. Recall that G is a finite group in which $|x^G| \leq m$ whenever x is an element of prime power order. We wish to show that G' has m-bounded order.

Let p be a prime bigger than m and assume that $p \in \pi(G)$. Let P be a Sylow p-subgroup of G. Since $|x^G| \leq m$ whenever x is an element of prime power order in G, it follows that P commutes with any element of prime power order and so $P \leq Z(G)$.

Pick an arbitrary element $g \in G$ and suppose that the order of g is $p_1^{\alpha_1} \dots p_s^{\alpha_s}$, where p_i are distinct primes in $\pi(G)$. Write $g = g_1 g_2 \dots g_s$, where the element g_i is of order $p_i^{\alpha_i}$. This can be done, for example, choosing g_i as a generator of the Sylow p_i-subgroup of the cyclic group $\langle g \rangle$. In view of the above, $g_i \in Z(G)$ whenever $p_i > m$. It follows that there are less than m non-central elements in the list $g_1, \dots, g_s$. Each of these non-central elements has centralizer of index at most m. Therefore $|g^G| \leq m^m$. This holds for an arbitrary element $g \in G$ so the claim follows from Neumann's theorem. □

We now provide an example of a finite group G in which $|x^G| \leq m$ whenever x is a commutator of prime power order and the Fitting subgroup has arbitrarily large index.

Example 2.1. *Let $p_1, p_2, \dots, p_s$ be distinct odd primes, and for every i let D_i be the dihedral group of order $2p_i$. Let $G = \prod D_i$ be the direct product of the groups D_i. It is easy to see that $|x^G| \leq 2$ whenever x is a commutator of prime power order. It is also clear that the Fitting subgroup of G has index 2^s.*

The above example shows that there is no obvious extension of Part 2 of Theorem 1.1 for finite groups in which commutators of prime power order lie in conjugacy classes of size at most m.

In what follows, we will require the concept of system normalizers. Let G be a finite soluble group. Recall that G has a Sylow system. This is a family of pairwise permutable Sylow p_i-subgroups P_i of G, exactly one for each prime in $\pi(G)$, and any two Sylow systems are conjugate. The system normalizer (also known as the basis normalizer) of such a Sylow system in G is $T = \cap_i N_G(P_i)$. The reader can consult [4, pp. 235–240] or [12, pp. 262–269] for general information on system normalizers. In particular, we will use the facts that the system normalizers are conjugate, nilpotent, and $G = T\gamma_\infty(G)$ where $\gamma_\infty(G)$ denotes the intersection of the lower central series of G. Moreover, the group G is generated by its system

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normalizers. If N is a normal subgroup of G, then the image of a system normalizer in the quotient G/N is a system normalizer of G/N.

Lemma 2.2. *Let G be a finite soluble group and T a system normalizer in G. Then $[G, T] = G'$.*

Proof. We just need to show that $G' \leq [G, T]$. We can pass to the quotient $G/[G, T]$ and assume that $[G, T] = 1$. Then $T \leq Z(G)$. Since G is generated by the conjugates of T, we deduce that $G = T$ is abelian. Hence, $G' \leq [G, T]$ and the lemma follows. □

We will require the following theorem, which is a particular case of [14, Theorem 1.2].

Theorem 2.3. *Let n be a positive integer and G a group containing a subgroup A such that $|[g, a]^G| \leq n$ for all $g \in G$ and $a \in A$. Then the commutator subgroup of the subgroup $[G, A]$ has finite n-bounded order.*

Suppose that G is a group generated by a symmetrical set X (here symmetrical means that $X = X^{-1}$). For any $g \in G$, we write $l_X(g)$ to denote the minimal number l such that g is a product of l elements from X.

The next lemma is taken from [3].

Lemma 2.4. *Let H be a group generated by a symmetrical set X and let K be a subgroup of finite index m in H. Then each coset Kb contains an element g such that $l_X(g) \leq m - 1$.*

The following lemma is similar to Lemma 2.3 of [3]. For the reader's convenience, we provide a proof.

Lemma 2.5. *Let m be a positive integer, and let X be the set of all elements g of a group G such that $|g^G| \leq m$. Assume that G is generated by X and let $x \in X$. Then $[G, x]$ has m-bounded order.*

Proof. Since $C_G(x)$ has index at most m in G, by Lemma 2.4 we can choose elements $y_1, \ldots, y_m$ such that $l_X(y_i) \leq m - 1$ and $[G, x]$ is generated by the commutators $[y_i, x]$. Here, the elements $y_1, \ldots, y_m$ are not necessarily pairwise distinct. For each $i = 1, \ldots, m$, write $y_i = y_{i1} \ldots y_{i(m-1)}$ where $y_{ij} \in X$. We can write $[y_i, x]$ as a product of conjugates of $[y_{ij}, x]$. Let $h_1, \ldots, h_s$ be the conjugates in G of elements from the set $K = \{x, y_{ij}; 1 \leq i \leq m,\ 1 \leq j \leq m - 1\}$. Notice that the size of each conjugacy class containing an element of K is less than or equal to m, and K has at most $m(m - 1) + 1$ elements. Therefore, s is m-bounded. Let $H = \langle h_1, \ldots, h_s \rangle$. It is clear that $[G, x] \leq H'$, so it is sufficient to show that H' has finite m-bounded order. Observe that $C_G(h_i)$ has index at most m in G for each $i = 1, \ldots, s$. It follows that the center $Z(H)$ has index at most m^s in H. Thus, Schur's theorem [9, 2.4.1] tells us that H' has finite m-bounded order. □

3. Theorem 1.3

A difficulty that one encounters while looking for an approach to a proof of Theorem 1.3 is that the hypothesis may fail in a homomorphic image of G. More precisely, if $Y(G)$ denotes the set of commutators of prime power order of the finite group G and if $\overline{G}$ is a homomorphic image of G, then, in general, $Y(\overline{G})$ is not necessarily the image of $Y(G)$.

To circumvent the obstacle, we will work with the subset $X(G)$ of $Y(G)$ that consists of all commutators $[x, y]$ for which there exists a Sylow subgroup $P \leq G$ such that x and $[x, y]$ both lie in P while y can be any element of G. Thus,

$$X(G) = \{[x, y] \mid \text{there is } P \in Syl(G) \text{ such that } x, [x, y] \in P, y \in G\}.$$

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We observe that the set $X(G)$ is characteristic in G, that is, invariant under any automorphism of G. The main advantage of working with the set $X(G)$ rather than $Y(G)$ is that $X(G)$ survives taking quotients. More precisely, we have the following lemma.

Lemma 3.1. *Let G be a finite group and N a normal subgroup of G. Set $\overline{G} = G/N$. Then $\overline{X(G)} = X(\overline{G})$.*

Proof. Of course, $\overline{X(G)} \subseteq X(\overline{G})$. We need to establish the inverse containment $X(\overline{G}) \subseteq \overline{X(G)}$.

Pick $[\overline{x}, \overline{y}] \in X(\overline{G})$. There is a Sylow p-subgroup $\overline{P}$ of $\overline{G}$ such that $[\overline{x}, \overline{y}] \in \overline{P}$ and $\overline{x} \in \overline{P}$. The full preimage of $\overline{P}$ is PN for some Sylow p-subgroup P of G. Choose x and y such that $\overline{x} = xN$ and $\overline{y} = yN$. Both elements x and x^y lie in PN.

Write $x = x'n$, where $x' \in P$ and $n \in N$. Since $x^y \in PN$, we also have $x'^y \in PN$. Thus, there is a Sylow p-subgroup P_0 in PN such that $x'^y \in P_0$. Since P_0 is conjugate to P in PN, there is an element $a \in N$ such that $x'^{ya} \in P$.

So, we have

$$[\overline{x}, \overline{y}] = [x', y]N = [x', ya]N.$$

Since $[x', ya] \in P$ and $x' \in P$, it follows that $[x', ya] \in X(G)$ and, therefore, $[x, y] \in X(G)N$. This completes the proof. □

Let G be a finite group and P a Sylow p-subgroup of G. An immediate corollary of the focal subgroup theorem [7, Theorem 7.3.4] is that $P \cap G'$ is generated by $P \cap X(G)$. Consequently, G' is generated by $X(G)$.

Lemma 3.2. *Let G be a finite group such that $|x^G| \leq m$ whenever $x \in X(G)$. Let P be a Sylow p-subgroup of G for some prime $p > m$. Then $[G', P] = 1$.*

Proof. For $x \in X(G)$, let N_x be the normal core of $C_G(x)$, that is, $N_x = \bigcap_{g \in G} C_G(x)^g$. As $[G : C_G(x)] \leq m$, the index $[G : N_x]$ divides $m!$.

Taking into account that $p > m$ we conclude that $P \leq N_x$. Since this happens for all $x \in X(G)$ and since G' is generated by $X(G)$, it follows that $[G', P] = 1$. □

In the next lemma, we will use the well-known fact that if A is a group of coprime automorphisms of a finite group H such that $[H, A, A] = 1$, then $A = 1$ (see e.g., [7, Theorem 5.3.6]).

Lemma 3.3. *Let G be a finite soluble group such that $|x^G| \leq m$ whenever $x \in X(G)$. Then G'' has m-bounded order.*

Proof. Let $\pi_1 = \{p_1, \ldots, p_k\}$ the set of prime divisors of $|G|$ which are less than or equal to m and $\pi_2 = \pi(G) \setminus \pi_1$. Let H_1 be a π_1-Hall subgroup and H_2 a π_2-Hall subgroup of G.

By Lemma 3.2, H_2 centralizes G'. Therefore, $G' = K_1 \times K_2$, where $K_1 = G' \cap H_1$ and $K_2 = G' \cap H_2$. Obviously, G'' has trivial intersection with K_2, so we pass to the quotient G/K_2 and simply assume that $G' \leq H_1$. In particular, H_1 is normal and $[H_1, H_2, H_2] = 1$. It follows that $G = H_1 \times H_2$. Therefore, $G'' = H_1''$ and so without loss of generality, we can assume that $G = H_1$.

Let $\{P_1, \ldots, P_k\}$ be a Sylow basis of G and $T = \bigcap_{i=1}^{k} N_G(P_i)$ the system normalizer. Since $G = P_1 P_2 \ldots P_k$, any element $x \in G$ can be written as a product $x = x_1 x_2 \ldots x_k$, where $x_i \in P_i$. For $a \in G$, write

$$[x, a] = [x_1 x_2 \ldots x_k, a] = [x_1, a]^{x_2 \ldots x_k} [x_2, a]^{x_3 \ldots x_k} \ldots [x_k, a].$$

If $a \in T$, we have $[x_i, a] \in P_i$ and, therefore, $[x, a]$ is a product of at most k elements from $X(G)$. So, the index $[G : C_G([x, a])]$ is at most m^k. Obviously, k here is less than m. Thus, for any $x \in G$, $a \in T$, we

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have $|[x,a]^G| \leq m^k$, which is m-bounded. The combination of Lemma 2.2 and Theorem 2.3 now implies that $[G,T]' = G''$ has m-bounded order. □

We say that a group is semisimple if it is a direct product of non-abelian simple groups.

Lemma 3.4. *Let G be a finite semisimple group such that $|x^G| \leq m$ whenever $x \in X(G)$. Then G has m-bounded order.*

Proof. Write $G = S_1 \times \ldots \times S_k$, where S_i are the simple factors. Each of these factors contain a subgroup of index at most m whence, because of the simplicity, it follows that $|S_i| \leq m!$. Therefore, we only need to show that k is m-bounded.

Recall that simple groups have even order [6]. For every $i = 1, \ldots, k$, choose a Sylow 2-subgroup P_i of S_i and elements $x_i \in P_i$ and $y_i \in S_i$ such that $1 \neq [x_i, y_i] \in Z(P_i)$. Indeed, if the subgroup P_i is non-abelian, both elements x_i and y_i can be chosen inside P_i. If P_i is abelian, then the existence of x_i and y_i is immediate from the theorem of Frobenius [7, Theorem 7.4.5]. For every $i = 1, \ldots, k$ set $g_i = [x_i, y_i]$. Since the index of the centralizer of an element in a simple group is not a prime power [7, Lemma 4.3.2], it follows that $|g_i^{S_i}| \geq 15$. Let $g = \prod_i g_i$. Observe that $g = [\prod x_i, \prod y_i] \in \prod P_i$. We deduce that $g \in X(G)$ and $|g^G| \geq 15^k$. Hence, $15^k \leq m$ and so $k \leq \log_{15}(m)$. □

Recall that the generalized Fitting subgroup $F^*(G)$ of a finite group G is the product of the Fitting subgroup $F(G)$ and all subnormal quasisimple subgroups; here, a group is quasisimple if it is perfect and its quotient by the center is a non-abelian simple group. In any finite group G, we have $C_G(F^*(G)) \leq F^*(G)$. Since $G/C_G(F^*(G))$ embeds in $Aut(F^*(G))$, it follows that if $|F^*(G)| = t$, then $|G/F^*(G)| \leq (t-1)!$ and therefore $|G| \leq t!$.

Lemma 3.5. *Let G be a finite group such that $|x^G| \leq m$ whenever $x \in X(G)$, and let $R = R(G)$ be the soluble radical of G. Then, the quotient G/R has m-bounded order.*

Proof. Without loss of generality, we may assume that $R = 1$. Then, $F^*(G)$ is semisimple and by Lemma 3.4 the order of $F^*(G)$ is m-bounded. Since $|G| \leq |F^*(G)|!$, we immediately deduce that the order of G is m-bounded. This proves the lemma. □

We are now ready to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Recall that G is a finite group in which $|x^G| \leq m$ whenever x is a commutator of prime power order. We wish to prove that G'' has m-bounded order.

We will prove the stronger result that if G is a finite group in which $|x^G| \leq m$ whenever $x \in X(G)$, then G'' has m-bounded order. We observe that the condition is inherited by subgroups and quotients of G.

Let $H = G^{(\infty)}$ be the last term of the derived series of G. Of course, $H = H'$. Note that G/H is soluble and so by Lemma 3.3, the second commutator subgroup of G/H has m-bounded order. Therefore, it is sufficient to show that H has m-bounded order. Hence, without loss of generality, we assume that $G = H$, that is, $G = G'$.

Let $R = R(G)$ be the soluble radical of G. Lemma 3.5 tells us that R has m-bounded index in G. Therefore, we can use induction on the index $[G:R]$.

If $[G:R] = 1$, then G is soluble. Since $G = G'$, it follows that $G = 1$ and we are done. So we assume that $[G:R] \geq 2$. Suppose that G contains a normal subgroup M such that $R \leq M < G$. It is clear that $[M:R(M)] < [G:R]$, so by induction $|M''|$ is m-bounded. Certainly, $M/M'' \leq R(G/M'')$. Therefore, the index of the soluble radical in G/M'' is strictly less than $[G:R]$ and the result follows by induction.

Hence, we can assume that such subgroups M do not exist and so G/R is simple. Set $X = X(G)$ and observe that, since $G = G'$, it follows that G is generated by X. By Lemma 2.5, the order of the subgroup $[G,x]$ is m-bounded whenever $x \in X$.

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Taking into account that G/R is a nontrivial simple group, we conclude that there is $x_0 \in X$ such that $[G, x_0]R = G$. Observe that the quotient $G/[G, x_0]$ is soluble. Since $G = G'$, we deduce that $G = [G, x_0]$. Hence, the order of G is m-bounded, as required. □

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