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LIMITING DISTRIBUTIONS OF GENERALIZED MONEY EXCHANGE MODELS ON HYPERGRAPHS

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Abstract

The money exchange model is a type of agent-based model used to study how wealth distribution and inequality evolve through monetary exchanges between individuals. The primary focus of this model is to identify the limiting wealth distributions that emerge at the macroscopic level, given the microscopic rules governing the exchanges among agents. In this paper, we formulate generalized versions of the immediate exchange model, the uniform reshuffling model, and the uniform saving model, all of which are types of money exchange model, as discrete-time interacting particle systems and characterize their stationary distributions. Furthermore, we prove that, under appropriate scaling, the asymptotic wealth distribution converges to an exponential distribution for the uniform reshuffling model, and to either an exponential distribution or a gamma distribution depending on the tail behavior of the number of coins given/saved in the immediate exchange model and the random saving model, which generalizes the uniform saving model. In particular, our results provide a mathematically rigorous formulation and generalization of the assertions previously predicted in studies based on numerical simulations and heuristic arguments.

Keywords: Econophysics; interacting particle system; stationary distribution; equivalence of ensembles; local limit theorem

2020 Mathematics Subject Classification: Primary 60K35
Secondary 60F99; 91B80

1. Introduction

1.1. Money exchange models

The *money exchange model* is a type of agent-based model used to study how wealth distribution and inequality evolve through monetary exchanges between individuals. This model has been studied extensively in the field of econophysics, particularly by applying ideas from statistical physics, where the money exchange is viewed as analogous to the transfer of energy between particles in physical systems.

Consider an economy consisting of a finite number of agents. The typical process proceeds as follows.

(i) *Initial conditions*. Assign each agent a random or equal amount of money.

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(ii) *Selecting pairs of agents*. Select uniformly at random a pair of agents who will exchange money.

(iii) *Money exchange*. The selected pair of agents exchange money according to specific rules.

(iv) *Iteration*. Repeat (ii) and (iii) multiple times and observe how the distribution of money in the system changes.

The primary focus of this model is to identify the limiting wealth distributions that emerge at the macroscopic level, given the microscopic rules governing the exchanges among agents. Depending on the rules of exchange, the limiting distribution is often expected to take the form of a gamma or an exponential distribution. In the field of econophysics, various studies have been conducted through numerical simulations and other methods (cf. [2, 10, 26] and references therein). However, mathematically rigorous studies of these models remain relatively few.

We begin by describing several specific models. The first is the *immediate exchange model* proposed in [11]. In this model, the wealth of each agent is represented by a real-valued variable. At each time step, two agents are randomly chosen and give a random fraction of their wealth to each other. The fraction is determined by independent uniformly distributed random variables on the interval [0, 1]. Heinsalu and Patriarca [11] studied the model through numerical simulations, and later, Katriel [12] analytically explored the infinite-population version. As a more realistic microscopic model that includes spatial structure in the form of local interactions representing a social network, Lanchier and Reed [16] formulated the corresponding discrete version as an interacting particle system. We briefly explain their model and result. Consider a finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Each site $x \in \mathcal{V}$ corresponds to an agent, and the economy is represented by the set of agents $\mathcal{V}$. The population size is given by $|\mathcal{V}| = N$. The amount of money each agent holds is represented by the number of coins, where $M_n(x)$ represents the number of coins that agent $x \in \mathcal{V}$ possesses at time n. The edge set $\mathcal{E}$ represents a social network in which only agents connected by an edge can interact to exchange coins. At each time step, an edge $e = \{x, y\} \in \mathcal{E}$ is chosen uniformly at random from $\mathcal{E}$. Agents x and y independently and uniformly select a random number of their coins to give to each other. Thus, $\{(M_n(x))_{x\in\mathcal{V}}\}_{n\geq 0}$ constitutes a time-homogeneous Markov chain. In particular, the total number of coins is conserved in this process, and we denote this total by L. Lanchier and Reed [16] proved that the stationary distribution $\mu_{N,L}$ for this Markov chain uniquely exists and gave the following explicit representation:

$$\begin{aligned}\lim_{n\to\infty} \mathrm{P}(M_n(x) = c) &= \mu_{N,L}\big(\{\xi \in \mathbb{Z}_+^{\mathcal{V}}; \xi(x) = c\}\big) \\ &= \frac{(c+1)\binom{L-c+2N-3}{2N-3}}{\binom{L+2N-1}{2N-1}},\end{aligned} \tag{1}$$

for every $x \in \mathcal{V}$ and $c \in \{0, 1, 2, \ldots, L\}$. The first equality is a consequence of the Markov chain convergence theorem. Then, by applying formal calculations to the right-hand side, the authors obtained the following approximation:

$$\lim_{n\to\infty} \mathrm{P}(M_n(x) = c) \approx \frac{4c}{T^2} \mathrm{e}^{-\frac{2c}{T}}, \tag{2}$$

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for large enough N and $T = L/N$. In this context T represents the average number of coins per agent and is called the *money temperature* by analogy with the temperature in physics, and the limits $N \to \infty$ and $T = L/N \to \infty$ are respectively called the *large-population limit* and the *large-money-temperature limit*. The right-hand side of (2) is the probability density function of the gamma distribution with mean T and shape parameter 2, and this is consistent with the result predicted by [11, 12]. As related results, [4, 9] examined the duality between the real-valued model and the discrete-state version in a continuous-time setting. In addition, [9] considered the case where the exchange fraction is determined by a beta distribution.

We also present two models with different exchange rules that we address in this paper. In the *uniform reshuffling model*, all the coins agents x and y possess are pooled and uniformly redistributed between the two agents. Different from the immediate exchange model, the limiting distribution for this model is predicted to be an exponential distribution with mean T (cf. [6]). Lanchier and Reed [16] obtained the corresponding approximation given by

$$\lim_{n\to\infty} \mathrm{P}(M_n(x) = c) \approx \frac{1}{T} \mathrm{e}^{-\frac{c}{T}}, \tag{3}$$

for large enough N and $T = L/N$.

In the *uniform saving model*, agents x and y independently save a random number of their coins according to a uniform distribution. The remaining coins are then pooled and uniformly redistributed between the two agents. As for the immediate exchange model, the limiting distribution for this model is predicted to be a gamma distribution with shape parameter 2 (cf. [3, 21]), and Lanchier and Reed [16] obtained the same approximation as (2). It is worth noting that the uniform saving model can be interpreted as a combination of the above two models in the sense that it combines choosing a random number of coins like in the immediate exchange model and redistributing the coins left uniformly at random as in the uniform reshuffling model.

Lanchier and Reed [16] formulated several money exchange models as Markov chains with spatial structure, characterized their stationary distributions, and further obtained their formal approximations. Their conclusions explain the distribution of wealth in a sense, as predicted by numerical simulations and other methods. However, the approximations ($\approx$) in (2) and (3) are not mathematically valid. The left-hand sides are defined only for nonnegative integers c, while the right-hand sides represent probability density functions on $\mathbb{R}_+$. The derivations of the approximations (2) and (3), although merely formal calculations, seem to depend on assumptions that are not fully clarified. Specifically, in the proofs of Theorems 1, 2, and 3 in [16], $L+1, L+2, \dots, L+N$ are replaced with L for sufficiently large L and N satisfying $N \ll L$. On the other hand, $L-c+1, L-c+2, \dots, L-c+N$ are replaced with $L-c$, rather than L, even when $c \ll L$. In addition, $c+1$ has been conveniently replaced with c. As a matter of fact, it seems unnatural to consider the approximation of $\mu_{N,L}(\{\xi \in \mathbb{Z}_+^{\mathcal{V}}; \xi(x) = c\})$ for each c since the average number of coins per agent diverges in the limit as $T = L/N \to \infty$. The money exchange model describes the microscopic movement of money, however, our goal is to derive the macroscopic distribution of wealth in the limit as $N \to \infty$ and $L/N \to \infty$. In order to do that, we should analyze the convergence of $\mu_{N,L}$ under appropriate scaling. Therefore, the primary objective of this paper is to provide a mathematically rigorous justification of the approximations (2) and (3), and to clearly demonstrate the convergence of the wealth distribution. Furthermore, we generalize the rules of money exchange as follows.

- Randomly select the number of coins to pass or save based on a probability distribution that depends on the number of coins, rather than using the uniform distribution.

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- Allow the exchange or redistribution of coins among randomly selected groups of three or more agents, thus replacing the underlying complete graph in the original models and the underlying connected graph in [16] with the structure of a hypergraph.

These generalizations appear natural from both mathematical and economic perspectives. We note that as a generalization of the exchange rules in the immediate exchange model, [25] considered a broader class of models where mass is split, exchanged and merged. In [14, 17, 18], the authors formulated other money exchange models as Markov chains and studied their stationary distributions and formal approximations in a manner similar to (1), (2), and (3). Lanchier [15, Section 7.5.3] provides a summary of [14, 16–18] as well as other mathematical papers in econophysics. In addition, the mixing time has recently been studied for the binomial splitting model and the symmetric beta-binomial splitting model, which are variations of the uniform reshuffling model (cf. [23, 24]).

Before introducing our models and results we set out some useful notation. In the following, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ denotes the set of nonnegative integers and $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of positive integers. Here $\lfloor a \rfloor$ denotes the integer part of $a > 0$ and we set $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$ for $a, b \in \mathbb{R}$. For a finite set $A \subset \mathbb{N}$, $|A|$ denotes its cardinality. For two sequences of positive numbers $\{a_n\}$ and $\{b_n\}$, $a_n \sim b_n$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. A function f on $\mathbb{R}^{\mathbb{Z}}$ is called local if it depends only on finitely many coordinates. For a probability measure μ, $\mathbb{E}^{\mu}[\,\cdot\,]$ denotes the expectation with respect to μ. A hypergraph is a pair $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a set of vertices (or nodes), and $\mathcal{E}$ is a set of hyperedges, where each hyperedge is a subset of $\mathcal{V}$ that can contain any number of vertices. Unlike in a standard graph, where edges connect only two vertices, hyperedges in a hypergraph can connect two or more vertices at once.

1.2. Model description and results

Let us state our model precisely. We adopt the standard notation commonly used in the study of interacting particle systems. For $A \subset \mathbb{N}$ and $L \in \mathbb{N}$, we define the configuration space

$$\Omega(A, L) = \left\{ \xi = \{\xi(x)\}_{x\in A} \in \mathbb{Z}_+^A;\ \sum_{x\in A} \xi(x) = L \right\}.$$

When $A = \Lambda_N := \{1, 2, \ldots, N\}$, $\Omega(A, L)$ is denoted by $\Omega_N(L)$. Consider now an economy populated by many agents. Each site $x \in \Lambda_N$ corresponds to an agent and we assume that the economy can be represented by the set of agents Λ_N. N corresponds to the population size. For each $\xi \in \Omega_N(L)$, we interpret $\xi(x)$, $x \in \Lambda_N$, not as the number of particles, but as the number of coins held by agent x. ρ denotes a probability distribution on $\mathcal{D}_N = \{A \subset \Lambda_N; |A| \geq 2\}$, namely $\rho(A) \geq 0$ for every $A \subset \Lambda_N$ with $|A| \geq 2$ and $\sum_{A\in\mathcal{D}_N} \rho(A) = 1$. This represents the distribution that determines which agents handle the money exchange at each time step. We also take a nonnegative function $g(\not\equiv 0)$ defined on $\mathbb{Z}_+$. We now introduce three money exchange models.

Immediate exchange model. Let $\{X_n\}_{n\geq 0}$ be a time-homogeneous Markov chain on the state space $\Omega_N(L)$. For given $X_n = \xi \in \Omega_N(L)$, the configuration X_{n+1} is determined from the following rule.

(i) Choose a hyperedge $A \in \mathcal{D}_N$ according to the distribution ρ.

(ii) For given ξ, let $\{c(x)\}_{x\in A}$ be independent random variables whose distributions are given by

$$\mathrm{P}(c(x) = k) = \frac{1}{G(\xi(x))} g(k), \quad k \in \{0, 1, 2, \ldots, \xi(x)\}, \tag{4}$$

for $x \in A$ where we set $G(k) = \sum_{j=0}^{k} g(j)$, $k \in \mathbb{Z}_+$.

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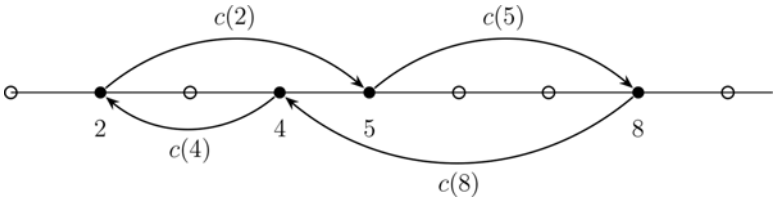


FIGURE 1. Immediate exchange dynamics. In this case, the hyperedge is $A = \{2, 4, 5, 8\}$ and the random permutation is $\sigma(2) = 5$, $\sigma(4) = 2$, $\sigma(5) = 8$, and $\sigma(8) = 4$.

(iii) Choose a permutation $\sigma \in \mathcal{S}_A$ uniformly at random, namely with probability $\frac{1}{|\mathcal{S}_A|} = \frac{1}{|A|!}$ where $\mathcal{S}_A$ denotes the set of all permutations of A.

(iv) For given ξ and realizations A, $\{c(x)\}_{x\in A}$ and σ, define X_{n+1} by

$$X_{n+1}(z) = \begin{cases} \xi(z) - c(z) + c(\sigma^{-1}(z)) & \text{if } z \in A, \\ \xi(z) & \text{if } z \notin A. \end{cases}$$

This dynamics can be interpreted as follows. At each time step, a money exchange occurs between agents in a randomly chosen set A. Here $c(x)$ represents the number of coins that agent x transfers, which is determined by a probability distribution dependent on the weight function g and the number of coins, $\xi(x)$, that agent x possesses. According to a randomly chosen permutation $\sigma \in \mathcal{S}_A$, each agent $x \in A$ passes $c(x)$ coins to agent $\sigma(x) \in A$; see Figure 1. In this process, the total number of coins remains conserved. It is worth noting that the model studied in [16] corresponds to the case where g is a constant function and ρ is the uniform distribution on an edge set of Λ_N. In this case, the distribution (4) matches the uniform distribution on $\{0, 1, 2, \ldots, \xi(x)\}$ and the money exchange occurs between two agents connected by an edge in Λ_N. To be more precise, since the permutation $\sigma \in \mathcal{S}_A$ can include the identity permutation, our model can be viewed as the lazy version of their model in this case.

Uniform reshuffling model. Let $\{Y_n\}_{n\geq 0}$ be a time-homogeneous Markov chain on the state space $\Omega_N(L)$. For given $Y_n = \xi$, the configuration Y_{n+1} is determined from the following rule.

(i) Choose a hyperedge $A \in \mathcal{D}_N$ according to the distribution ρ.

(ii) For given ξ and A, choose a configuration $d = \{d(x)\}_{x\in A} \in \Omega(A, S_A(\xi))$ uniformly at random, namely with probability $\frac{1}{|\Omega(A,S_A(\xi))|}$ where we set $S_A(\xi) = \sum_{x\in A} \xi(x)$ for $\{\xi(x)\}_{x\in A}$.

(iii) For given ξ and realizations A and $\{d(x)\}_{x\in A}$, define Y_{n+1} by

$$Y_{n+1}(z) = \begin{cases} d(z) & \text{if } z \in A, \\ \xi(z) & \text{if } z \notin A. \end{cases}$$

This is an extension of the model studied in [16], which considered the dynamics between two agents, to the dynamics on a hypergraph. At each time step, a redistribution of coins occurs among the agents in a randomly chosen set A. Specifically, the configuration restricted to A is replaced by a configuration chosen uniformly at random among all the configurations with the same number of coins on the set A.

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Random saving model. Let $\{Z_n\}_{n\geq 0}$ be a time-homogeneous Markov chain on the state space $\Omega_N(L)$. For given $Z_n = \xi$, the configuration Z_{n+1} is determined from the following rule.

(i) Choose a hyperedge $A \in \mathcal{D}_N$ according to the distribution ρ.

(ii) For given ξ, let $\{c(x)\}_{x\in A}$ be independent random variables whose distributions are given by

$$\mathrm{P}(c(x) = k) = \frac{1}{G(\xi(x))} g(k), \quad k \in \{0, 1, 2, \dots, \xi(x)\},$$

for $x \in A$.

(iii) For given ξ and $c = \{c(x)\}_{x\in A}$, choose a configuration $d = \{d(x)\}_{x\in A} \in \Omega(A, S_A(\xi) - S_A(c))$ uniformly at random, namely with probability $\frac{1}{|\Omega(A, S_A(\xi)-S_A(c))|}$.

(iv) For given ξ and realizations A, $\{c(x)\}_{x\in A}$ and $\{d(x)\}_{x\in A}$, define Z_{n+1} by

$$Z_{n+1}(z) = \begin{cases} c(z) + d(z) & \text{if } z \in A, \\ \xi(z) & \text{if } z \notin A. \end{cases}$$

Note that in contrast to the immediate exchange model, the random variable $c(x)$ represents how many coins the agent x saves. The random saving model can be interpreted as a combination of the immediate exchange model and the uniform reshuffling model where the agents in the hyperedge choose a random number of coins (as in the immediate exchange model) to save rather to give, and then all the coins left are uniformly redistributed among the agents (like in the uniform reshuffling model). Similarly to the immediate exchange model, [16] studied the case where g is a constant function and ρ is the uniform distribution on an edge set. In that model, $c(x)$ is drawn from the uniform distribution on $\{0, 1, 2, \dots, \xi(x)\}$, and it is referred to as the *uniform saving model*. Since our model considers a more general distribution for $c(x)$, we refer to it as the *random saving model*. As a special case, when the weight function g is defined by $g(k) = \delta_0(k)$ for $k \in \mathbb{Z}_+$, each agent saves no money. When a money exchange occurs within set A, all coins held by the agents in A are pooled and redistributed. This dynamics corresponds to the uniform reshuffling model, and in particular, the uniform reshuffling model can be seen as a special case of the random saving model.

As the first result, we characterize the stationary distributions of these Markov chains.

Proposition 1.1. *Let $N, L \in \mathbb{N}$ be fixed. We assume that the hypergraph $(\Lambda_N, \mathcal{D}_{N,\rho})$ is connected, where the hyperedge set $\mathcal{D}_{N,\rho}$ is defined by $\mathcal{D}_{N,\rho} = \{A \subset \Lambda_N; |A| \geq 2, \rho(A) > 0\}$.*

(i) *Assume that the weight function $g : \mathbb{Z}_+ \to [0, \infty)$ satisfies $g(0) > 0$ and $g(1) > 0$. Then there is a unique stationary distribution $\mu_{N,L}$ for $\{X_n\}_{n\geq 0}$ and it is given by*

$$\mu_{N,L}(\xi) = \frac{1}{Z_{N,L}} \prod_{x\in\Lambda_N} G(\xi(x)), \quad \xi \in \Omega_N(L), \tag{5}$$

where $G(k) = \sum_{j=0}^{k} g(j)$, $k \in \mathbb{Z}_+$, and $Z_{N,L} = \sum_{\xi\in\Omega_N(L)} \prod_{x\in\Lambda_N} G(\xi(x))$ is the normalization factor. In particular, for every $\eta \in \Omega_N(L)$, $x \in \Lambda_N$, and $k \in \{0, 1, 2, \dots, L\}$, we have that

$$\lim_{n\to\infty} \mathrm{P}_\eta\big(X_n(x) = k\big) = \mu_{N,L}\big(\xi(x) = k\big) = \frac{Z_{N-1,L-k}\, G(k)}{Z_{N,L}},$$

where P_η represents the law of $\{X_n\}_{n\geq 0}$ with the initial condition $X_0 = \eta$.

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(ii) *Assume that the weight function* $g : \mathbb{Z}_+ \to [0, \infty)$ *satisfies* $g(0) > 0$. *Then the exact same statement as (i) applies to* $\{Z_n\}_{n\geq 0}$ *instead of* $\{X_n\}_{n\geq 0}$. *In particular, for the uniform reshuffling model* $\{Y_n\}_{n\geq 0}$, *there is a unique stationary distribution* $\pi_{N,L}$ *and it is given by the uniform distribution on* $\Omega_N(L)$, *namely,*

$$\pi_{N,L}(\xi) = \frac{1}{|\Omega_N(L)|}, \quad \xi \in \Omega_N(L).$$

By this proposition we can see that both the immediate exchange model and the random saving model have the same stationary distribution under the conditions $g(0) > 0$ and $g(1) > 0$. In addition, the choice of ρ does not affect the stationary distribution. Lanchier and Reed [16] proved that, for dynamics on a graph, the stationary distribution does not depend on the topology of the underlying connected graph. The present result generalizes this finding to the setting of dynamics on a hypergraph.

Remark 1.1. The condition $g(0) > 0$ is always needed to make the measure (5) well-defined. Furthermore, the conditions $g(0) > 0$ and $g(1) > 0$ for the immediate exchange model, as well as the condition $g(0) > 0$ for the random saving model, ensure that these Markov chains are irreducible and aperiodic.

Remark 1.2. Consider the case where the weight function g is a constant function $g \equiv \gamma > 0$. Then $G(k) = \gamma(k + 1)$, $k \in \mathbb{Z}_+$, and the constant γ is canceled by the normalization factor in the definition of $\mu_{N,L}$. Therefore, we can take G as $G(k) = k + 1$, $k \in \mathbb{Z}_+$, in (5) and in this case the above result matches that of [16]. Lanchier and Reed [16] also gave the explicit representation for $Z_{N,L}$ when $g \equiv 1$ and obtained (1).

Remark 1.3. For the random saving model we can change the role of random variable c(x) to represent the number of coins to offer instead of the number of coins to save. Namely, we modify (iii) and (iv) in the definition of the random saving model as follows.

(iii)′ For given ξ and $c = \{c(x)\}_{x\in A}$, choose a configuration $d = \{d(x)\}_{x\in A} \in \Omega(A, S_A(c))$ uniformly at random, namely with probability $\frac{1}{|\Omega(A,S_A(c))|}$.

(iv)′ For given ξ and realizations A, $\{c(x)\}_{x\in A}$ and $\{d(x)\}_{x\in A}$, define Z_{n+1} by

$$Z_{n+1}(z) = \begin{cases} \xi(z) - c(z) + d(z) & \text{if } z \in A, \\ \xi(z) & \text{if } z \notin A. \end{cases}$$

By symmetry, the exact same proof for Proposition 1.1 below works well in this setting and the same result holds for this modified model.

We are now in a position to state the main result of this paper. To justify the limit $N \to \infty$ and $L/N \to \infty$, we assume that the total number of coins $L = L_N$ satisfies $\lim_{N\to\infty} \frac{L_N}{Na_N} = T$ for some $T > 0$ and divergent sequence $\{a_N\}_{N\geq 1}$. Then we can prove that the law of the scaled field $\left\{\frac{1}{a_N}\eta(x)\right\}_{x\in\Lambda_N}$ under $\mu_{N,L}$ or $\pi_{N,L}$ converges to the independent and identically distributed (i.i.d.) product of probability distributions on $\mathbb{R}_+$. Its marginal distribution depends on the asymptotic behavior of the weight function g.

Theorem 1.1. *Let* $\{L_N\}_{N\geq 1}$ *be a sequence of positive integers that satisfies* $\lim_{N\to\infty} \frac{L_N}{Na_N} = T$ *for some positive constant* $T > 0$ *and a sequence* $\{a_N\}_{N\geq 1}$ *which satisfies* $\lim_{N\to\infty} a_N = \infty$. *Assume that* $g(0) > 0$, *and that there exist* $\alpha \in \mathbb{R}$ *and* $c_\alpha \in (0, \infty)$ *such that* $\lim_{k\to\infty} \frac{g(k)}{k^\alpha} = c_\alpha$.

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Then, for every bounded continuous local function $f : \mathbb{R}^A \to \mathbb{R}$, we have that

$$\lim_{N\to\infty} \mathbb{E}^{\mu_{N,L_N}}\left[f\left(\frac{\cdot}{a_N}\right)\right] = \mathbb{E}^{\overline{\mu}^A_{\alpha,T}}[f(\,\cdot\,)],$$

where A is a finite subset of $\mathbb{N}$ and $\overline{\mu}^A_{\alpha,T}$ denotes the product probability measure on $\mathbb{R}_+^A$ whose one-site marginal distribution on $\mathbb{R}_+$ is given by

$$\mu_{\alpha,T}(dr) = \begin{cases} \dfrac{1}{\Gamma(\alpha+2)}\left(\dfrac{\alpha+2}{T}\right)^{\alpha+2} r^{\alpha+1}\mathrm{e}^{-\frac{\alpha+2}{T}r}\,dr & \text{if } \alpha > -1, \\ \dfrac{1}{T}\mathrm{e}^{-\frac{1}{T}r}\,dr & \text{if } \alpha \le -1, \end{cases} \tag{6}$$

in which $\Gamma(\beta) = \int_0^\infty r^{\beta-1}\mathrm{e}^{-r}\,dr$ is the Gamma function with parameter $\beta > 0$.

In addition, if $g : \mathbb{Z}_+ \to [0,\infty)$ satisfies $g(0) > 0$ and $\sum_{j\ge 0} g(j) < \infty$, then the same conclusion as in the case $\alpha \le -1$ above holds. In particular, for every bounded continuous local function $f : \mathbb{R}^A \to \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{E}^{\pi_{N,L_N}}\left[f\left(\frac{\cdot}{a_N}\right)\right] = \mathbb{E}^{\overline{\pi}^A_T}[f(\,\cdot\,)],$$

where A is a finite subset of $\mathbb{N}$ and $\overline{\pi}^A_T$ denotes the product probability measure on $\mathbb{R}_+^A$ whose one-site marginal distribution on $\mathbb{R}_+$ is given by $\pi_T(dr) = \frac{1}{T}\mathrm{e}^{-\frac{1}{T}r}\,dr$.

As an easy consequence of Proposition 1.1 and Theorem 1.1 we obtain the following result.

Corollary 1.1. *Let $\{X_n^{(N)}\}_{n\ge 0}$, $\{Y_n^{(N)}\}_{n\ge 0}$, and $\{Z_n^{(N)}\}_{n\ge 0}$ denote the immediate exchange model, the uniform reshuffling model and the random saving model on the state space $\Omega_N(L_N)$, respectively. Under the same conditions as in Proposition 1.1 and Theorem 1.1, we have*

$$\lim_{N\to\infty}\lim_{n\to\infty} \mathbb{E}_\eta\left[\frac{1}{N}\left|\left\{x\in\Lambda_N;\, \frac{1}{a_N}X_n^{(N)}(x)\in(b,c)\right\}\right|\right] = \mu_{\alpha,T}((b,c)),$$

$$\lim_{N\to\infty}\lim_{n\to\infty} \mathbb{E}_\eta\left[\frac{1}{N}\left|\left\{x\in\Lambda_N;\, \frac{1}{a_N}Y_n^{(N)}(x)\in(b,c)\right\}\right|\right] = \pi_T((b,c))$$

and

$$\lim_{N\to\infty}\lim_{n\to\infty} \mathbb{E}_\eta\left[\frac{1}{N}\left|\left\{x\in\Lambda_N;\, \frac{1}{a_N}Z_n^{(N)}(x)\in(b,c)\right\}\right|\right] = \mu_{\alpha,T}((b,c)),$$

for every $\eta \in \Omega_N(L_N)$ and every $0 \le b < c \le \infty$, where $\mathbb{E}_\eta[\,\cdot\,]$ denotes the expectation with respect to the law of the Markov chain with the initial condition η.

We note that it is natural to consider the scaling of the process by a factor of $\frac{1}{a_N}$. This is because, under the condition on L_N, we have

$$\frac{1}{N}\sum_{x\in\Lambda_N}\left(\frac{1}{a_N}X_n^{(N)}(x)\right) = \frac{L_N}{Na_N} = T(1+o(1)) \quad \text{as } N\to\infty,$$

which means that the asymptotic average number of coins per agent for the scaled process is given by T. This value corresponds to the money temperature in our model. Corollary 1.1

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provides a precise formulation and generalization of earlier studies in the physics literature, which were based on numerical simulations and heuristic arguments. As time approaches infinity, and in the large-population and large-money-temperature limit, the asymptotic wealth distribution (i.e. the proportion of agents holding a specific number of coins) converges to either a gamma distribution or an exponential distribution for both the immediate exchange model and the random saving model, while it converges to an exponential distribution for the uniform reshuffling model. The parameters of the gamma distribution depend on the asymptotic behavior of the weight function g. When $\alpha > -1$, the limiting wealth distribution is given by a gamma distribution with mean T and shape parameter $\alpha + 2$. When $\alpha \leq -1$, the limiting wealth distribution is given by an exponential distribution with mean T, and this does not depend on the parameter α. In particular, if g is a constant function then $\alpha = 0$ and the limiting distribution is a gamma distribution with mean T and shape parameter 2, which corresponds to (2). In addition, when $g(k) = (k+1)^{\alpha}$, $k \in \mathbb{Z}_+$, the above results are consistent with numerical simulations (see Figures 2 and 3).

Remark 1.4. The limiting distribution (6) for the case $\alpha \leq -1$ can be interpreted as follows: If the weight function g decays rapidly in the random saving model, the probability that each agent saves a large number of coins becomes very small. Moreover, since we are considering the scaled process $\left\{\frac{1}{a_N} Z_n^{(N)}\right\}_{n\geq 0}$, we can assume that each agent offers nearly all the coins they have in each money exchange. Consequently, similarly to the uniform reshuffling model, the limiting distribution becomes an exponential distribution with mean T. In the immediate exchange model, when the weight function g decays rapidly, the probability that each agent exchanges a large number of coins also becomes very small. For the scaled process, this situation can be regarded as similar to the one-coin model, where one agent gives only one coin to another agent at a time. In the one-coin model, the limiting distribution is expected to be exponential (cf. [6, 14]) and the result above aligns with this. Furthermore, a mathematically rigorous justification of the result for the one-coin model can be achieved by formulating and proving it in the same manner as demonstrated in this paper.

1.3. Discussion

Here we examine some further extensions of the model whose stationary distributions are not explicitly known.

Nonuniform permutation. In the immediate exchange model, given a hyperedge $A \in \mathcal{D}_N$, a permutation $\sigma \in \mathcal{S}_A$ is chosen uniformly at random. One may also consider choosing it according to a general probability distribution Q_A on $\mathcal{S}_A$. If Q_A possesses good symmetry properties for every $A \in \mathcal{D}_N$, then the proof of Proposition 1.1 applies, and the stationary distribution is given by (5); see Remark 2.1 following the proof of Proposition 1.1. However, in the general case, the situation remains unclear.

Individual allocation. In the random saving model, after the agents within a hyperedge determine how many coins to save, the remaining coins are pooled and then redistributed uniformly at random among them. Alternatively, one could consider a setting in which each agent individually distributes their remaining coins to the other members of the selected hyperedge. Namely, we modify (iii) and (iv) in the definition of the random saving model as follows.

(iii)″ For given ξ and $c = \{c(x)\}_{x\in A}$, for each $x \in A$, choose a configuration $d^x = \{d^x(y)\}_{y\in A\setminus\{x\}} \in \Omega(A \setminus \{x\}, \xi(x) - c(x))$ uniformly at random, namely with probability $\frac{1}{|\Omega(A\setminus\{x\},\xi(x)-c(x))|}$.

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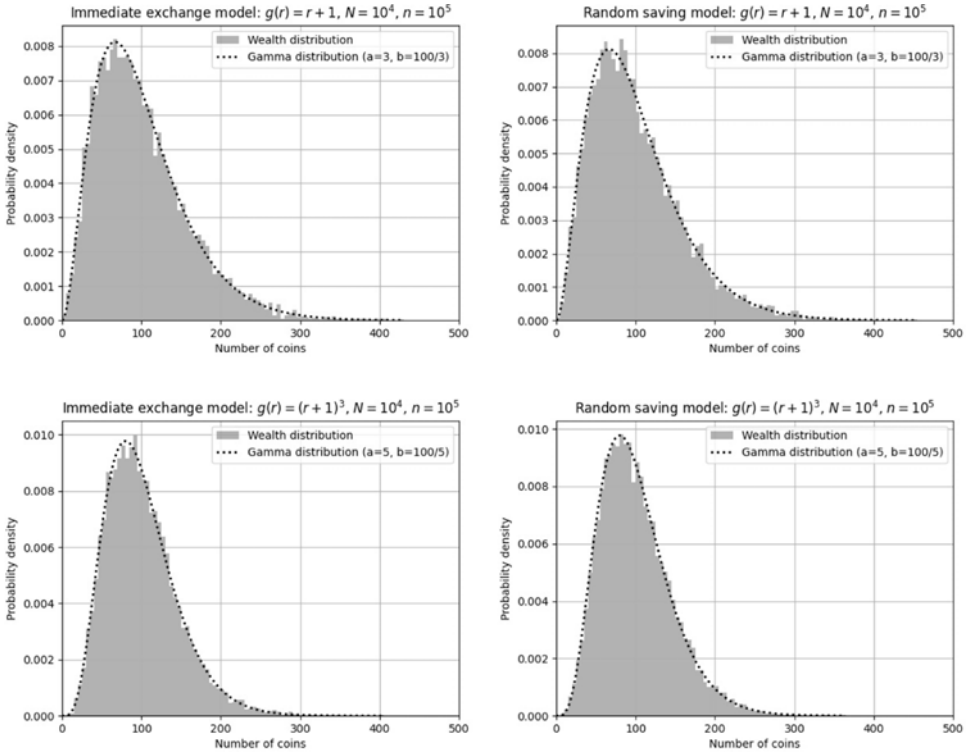


FIGURE 2. Simulation results for a single realization of the immediate exchange model and the random saving model with $g(k) = (k + 1)^\alpha$, where $\alpha = 1, 3$. The number of agents is $N = 10^4$ and the total number of coins is $L = 10^6$, namely the average number of coins per agent is 100. The initial condition is set to a constant configuration $X_0 \equiv 100$ or $Z_0 \equiv 100$. Here ρ is distributed uniformly over the set of hyperedges consisting of all three-element subsets of distinct vertices in Λ_N, and in the immediate exchange model, one of the three-element permutations is selected with probability $\frac{1}{6}$. The gray histograms represent the wealth distribution (i.e. the proportion of agents holding a specific number of coins after $n = 10^5$ updates). The dotted line is the graph of the probability density function of the gamma distribution $f_{a,b}(r) = \frac{1}{\Gamma(a)b^a} r^{a-1} \mathrm{e}^{-\frac{1}{b}r}$, with shape parameter $a = \alpha + 2$ and scale parameter $b = \frac{100}{\alpha+2}$.

(iv)$''$ For given ξ and realizations A, $\{c(x)\}_{x\in A}$ and $\{\{d^x(y)\}_{y\in A\setminus\{x\}}; x \in A\}$, define Z_{n+1} by

$$Z_{n+1}(z) = \begin{cases} c(z) + \displaystyle\sum_{x\in A\setminus\{z\}} d^x(z) & \text{if } z \in A, \\ \xi(z) & \text{if } z \notin A. \end{cases}$$

In this dynamics, each agent $x \in A$ distributes $\xi(x) - c(x)$ coins by giving $d^x(y)$ coins to each other agent $y \in A \setminus \{x\}$. As a result, at time $n + 1$, the number of coins held by each agent equals the sum of the coins they saved and those received from other agents. This model can also be viewed as an extension of the immediate exchange model studied in [16], since, when ρ is supported on the set of edges of Λ_N, the above redistribution rule coincides with the immediate exchange model on a graph. The proof of Proposition 1.1 relies on the fact that the number of coins received can be represented by a single variable. When the received amount

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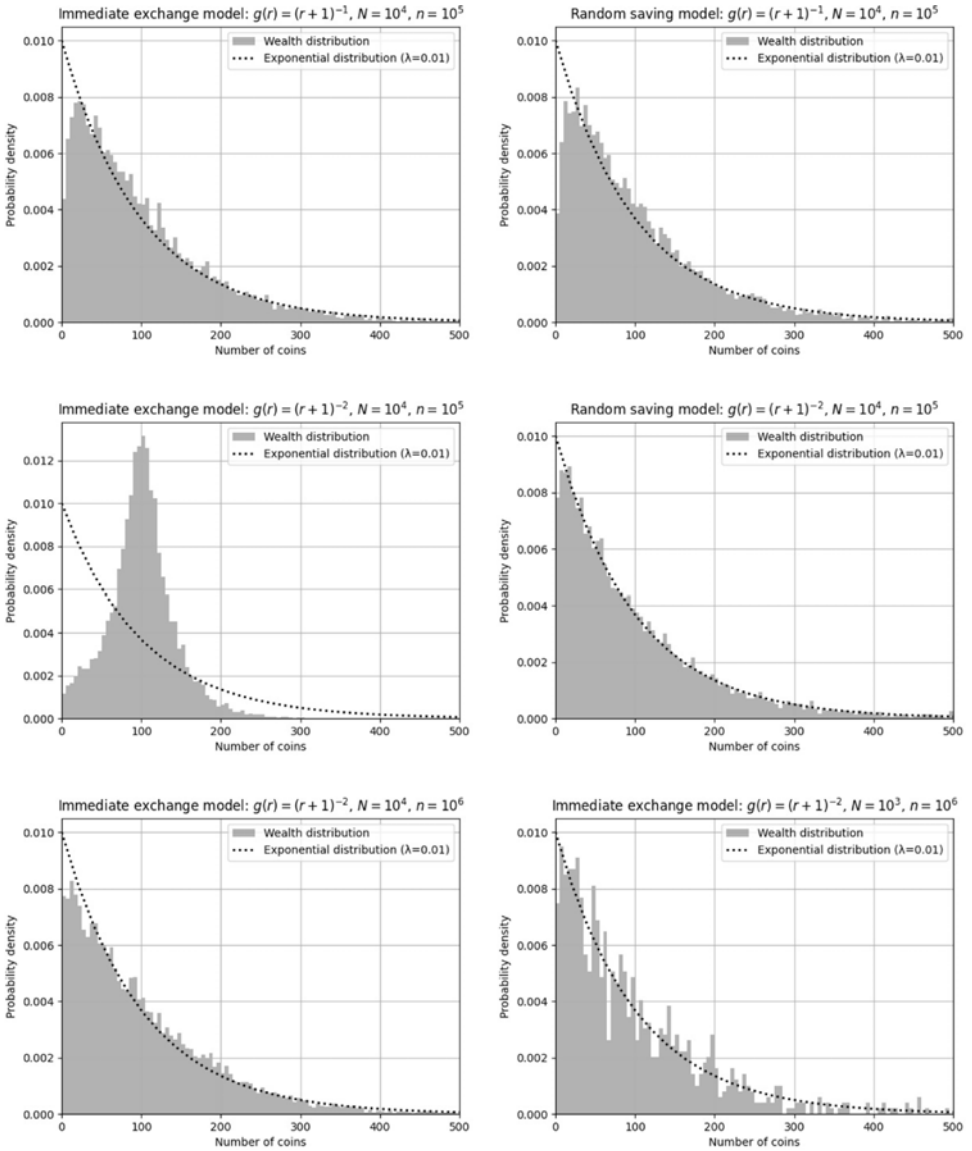


FIGURE 3. Simulation results for a single realization of the immediate exchange model and the random saving model with $g(k) = (k+1)^{\alpha}$, where $\alpha = -1, -2$. The settings of the simulations are the same as in Figure 2. The dotted line is the graph of the probability density function of the exponential distribution $f_\lambda(r) = \lambda e^{-\lambda r}$ with the parameter $\lambda = \frac{1}{100}$. When the weight function g decays rapidly in the immediate exchange model, the probability of each agent exchanging only a small number of coins at a time increases, leading to a longer convergence time to reach a steady state. The simulation result for the case $g(k) = (k+1)^{-2}$ reflects this and $n = 10^5$ updates are not sufficient for convergence (middle left). However, after $n = 10^6$ updates, the histograms approach the limiting probability density function (lower left). In addition, the result for the case $N = 10^3$ and $n = 10^6$ indicates that we need to take a large population N (and, hence, a large L) for convergence (lower right).

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is instead expressed as a sum of multiple variables, the situation becomes considerably more complicated, and the explicit form of the stationary distribution remains unknown.

In the rest of the paper we provide the proofs of Proposition 1.1 and Theorem 1.1 in Sections 2 and 3, respectively. We give some comments about the strategy of the proof. The proof of Proposition 1.1 is standard. It is not difficult to see that our models are irreducible and aperiodic Markov chains on the finite state space $\Omega_N(L)$. All we have to do is to characterize the unique stationary distribution, which is achieved by carefully verifying the detailed balance condition. Namely, we demonstrate that our models are reversible Markov chains, and the reversible distributions for $\{X_n\}_{n\geq 0}$ and $\{Z_n\}_{n\geq 0}$ are given by (5). With respect to the proof of Theorem 1.1, the convergence of the marginal distribution of $\mu_{N,L}$ is closely related to the equivalence of ensembles (cf. [8, 13]). The sequence $\{G(k)\}_{k\in\mathbb{Z}_+}$ is neither a probability distribution nor summable over k. However, multiplying $G(k)$ by the exponential factor s^k, $s \in [0, 1)$, ensures the convergence of $\sum_{k\in\mathbb{Z}_+} s^k G(k)$ under the assumption on g. The stationary distribution (5) can then be interpreted as the microcanonical distribution of an i.i.d. product, where the one-site marginal distribution is of exponential type and proportional to G. Instead of the usual condition $\frac{1}{N}\sum_{x\in\Lambda_N}\eta(x) \to m \in (0, \infty)$ as $N \to \infty$, we consider the condition $\frac{1}{Na_N}\sum_{x\in\Lambda_N}\eta(x) \to T \in (0, \infty)$ and investigate the convergence of the law of the scaled field $\frac{1}{a_N}\eta$ under the corresponding microcanonical distribution. We adapt the proof of the equivalence of ensembles from [13, Appendix 2] to this unusual setting. In particular, the local limit theorem for a triangular array of random variables plays an important role in the argument.

Throughout the paper C, C', and C'' represent positive constants that do not depend on the size of the system N, but may depend on other parameters. These constants in various estimates may change from place to place in the paper.

2. Proof of Proposition 1.1

In this section we prove Proposition 1.1. Under the condition that the hypergraph $(\Lambda_N, \mathcal{D}_{N,\rho})$ is connected, we have that for every $x, y \in \Lambda_N$ there exists a sequence $\{A_k\}_{0\leq k\leq l} \subset \mathcal{D}_{N,\rho}$ such that $x \in A_0$, $y \in A_l$, and $A_k \cap A_{k+1} \neq \emptyset$ for every $0 \leq k \leq l-1$. Combining this fact with the assumption $g(0) > 0$ and $g(1) > 0$, it is easy to see that the following assertions hold.

- For every $\xi, \eta \in \Omega_N(L)$ there exists $m = m(\xi, \eta) \geq 0$ such that $\mathrm{P}(X_m = \eta | X_0 = \xi) > 0$.
- $\mathrm{P}(X_{n+1} = \eta | X_n = \eta) > 0$ for every $\eta \in \Omega_N(L)$ and $n \geq 0$.

The same statements also hold for $\{Z_n\}_{n\geq 0}$ under the condition $g(0) > 0$. Namely, $\{X_n\}_{n\geq 0}$ and $\{Z_n\}_{n\geq 0}$ are irreducible and aperiodic Markov chains on the finite state space $\Omega_N(L)$. Note that for the immediate exchange model, $g(0) > 0$ implies that all the agents keep all their coins with positive probability, which leads to aperiodicity (the second condition above). If in addition $g(1) > 0$, with positive probability, exactly one agent gives exactly one coin while the other agents give no coins, which allows us to move one coin from a given vertex to another given vertex in a finite number of time steps and leads to irreducibility (the first condition above). Similarly, the condition $g(0) > 0$ ensures irreducibility and aperiodicity for the random saving model. Then the stationary distribution of $\{X_n\}_{n\geq 0}$ ($\{Z_n\}_{n\geq 0}$) uniquely exists and the law of X_n (Z_n) converges to it regardless of the initial configuration of coins in the limit $n \to \infty$ by the Markov chain convergence theorem. Therefore, all we have to show is that the measure $\mu_{N,L}$ given by (5) is the stationary distribution for both of $\{X_n\}_{n\geq 0}$ and $\{Z_n\}_{n\geq 0}$. Actually, by verifying the detailed balance condition,

$$\mu_{N,L}(\xi)\mathrm{P}(X_{n+1} = \eta | X_n = \xi) = \mu_{N,L}(\eta)\mathrm{P}(X_{n+1} = \xi | X_n = \eta) \quad \text{for every } \xi, \eta \in \Omega_N(L) \tag{7}$$

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and

$$\mu_{N,L}(\xi)\mathrm{P}(Z_{n+1}=\eta|Z_n=\xi)=\mu_{N,L}(\eta)\mathrm{P}(Z_{n+1}=\xi|Z_n=\eta)\quad\text{for every } \xi,\,\eta\in\Omega_N(L), \tag{8}$$

we prove that (5) is the reversible distribution for both of $\{X_n\}_{n\geq 0}$ and $\{Z_n\}_{n\geq 0}$. In addition, the result for the uniform reshuffling model $\{Y_n\}_{n\geq 0}$ follows from taking the weight function g as $g(k)=\delta_0(k)$ for $k\in\mathbb{Z}_+$ in the random saving model.

Proof for the immediate exchange model. Take arbitrary $\xi,\,\eta\in\Omega_N(L)$ and set $A_0=A_0(\xi,\,\eta):=\{x\in\Lambda_N;\,\xi(x)\neq\eta(x)\}$. We have that

$$\begin{aligned}&\mathrm{P}(X_{n+1}=\eta|X_n=\xi)\\&=\sum_{\substack{A\in\mathcal{D}_N\\A\supset A_0}}\rho(A)\sum_{\sigma\in\mathcal{S}_A}\frac{1}{|\mathcal{S}_A|}\mathrm{P}\big(\eta(z)=\xi(z)-c(z)+c(\sigma^{-1}(z))\text{ for every } z\in A\,\big|\,\xi\big),\end{aligned}$$

where $\mathrm{P}(\,\cdot\,|\xi)$ denotes the law of $\{c(x)\}$ in the dynamics (ii) given the configuration ξ. If $\rho(A)=0$ for every $A\in\mathcal{D}_N$ so that $A\supset A_0$, then the right-hand side is equal to 0. For a finite set A, we label its elements as $A=\{x_1,x_2,\dots,x_{|A|}\}$. Then

$$\begin{aligned}&\mathrm{P}\big(\eta(z)=\xi(z)-c(z)+c(\sigma^{-1}(z))\text{ for every } z\in A\,\big|\,\xi\big)\\&=\sum_{t_1=0}^{\xi(x_1)}\sum_{t_2=0}^{\xi(x_2)}\cdots\sum_{t_{|A|}=0}^{\xi(x_{|A|})}\prod_{i=1}^{|A|}\frac{g(t_i)}{G(\xi(x_i))}\cdot\mathbf{1}\big(\eta(x_i)=\xi(x_i)-t_i+t_{\sigma^{-1}(i)}\text{ for every } 1\leq i\leq|A|\big)\\&=\sum_{t_1=0}^{\xi(x_1)}\sum_{t_2=0}^{\xi(x_2)}\cdots\sum_{t_{|A|}=0}^{\xi(x_{|A|})}\prod_{i=1}^{|A|}\left\{\frac{g(t_i)}{G(\xi(x_i))}\mathbf{1}\big(\eta(x_i)=\xi(x_i)-t_i+t_{\sigma^{-1}(i)}\big)\right\},\end{aligned}$$

where for each $\sigma\in\mathcal{S}_A$ and $1\leq i\leq|A|$, we identify $\sigma^{-1}(i)$ with the label $1\leq k\leq|A|$ which satisfies $x_k=\sigma^{-1}(x_i)$. Therefore,

$$\begin{aligned}&\left\{\prod_{x\in\Lambda_N}G(\xi(x))\right\}\cdot\mathrm{P}(X_{n+1}=\eta|X_n=\xi)\\&=\sum_{\substack{A\in\mathcal{D}_N\\A\supset A_0}}\Bigg[\prod_{x\in\Lambda_N\setminus A}G(\xi(x))\cdot\frac{\rho(A)}{|\mathcal{S}_A|}\\&\qquad\times\sum_{\sigma\in\mathcal{S}_A}\sum_{t_1=0}^{\xi(x_1)}\sum_{t_2=0}^{\xi(x_2)}\cdots\sum_{t_{|A|}=0}^{\xi(x_{|A|})}\prod_{i=1}^{|A|}\Big\{g(t_i)\mathbf{1}\big(\eta(x_i)=\xi(x_i)-t_i+t_{\sigma^{-1}(i)}\big)\Big\}\Bigg]\\&=\sum_{\substack{A\in\mathcal{D}_N\\A\supset A_0}}\Bigg[\prod_{x\in\Lambda_N\setminus A}\left\{\frac{G(\xi(x))+G(\eta(x))}{2}\right\}\cdot\frac{\rho(A)}{|\mathcal{S}_A|}\\&\qquad\times\sum_{\sigma\in\mathcal{S}_A}\sum_{t_1=0}^{\xi(x_1)}\sum_{t_2=0}^{\xi(x_2)}\cdots\sum_{t_{|A|}=0}^{\xi(x_{|A|})}\prod_{i=1}^{|A|}\Big\{g(t_i)\mathbf{1}\big(\eta(x_i)=\xi(x_i)-t_i+t_{\sigma^{-1}(i)}\big)\Big\}\Bigg],\end{aligned} \tag{9}$$

where the last equality follows from $\xi=\eta$ on $\Lambda_N\setminus A_0$. The symmetry of this expression with respect to ξ and η implies the detailed balance condition (7). For $A\subset\Lambda_N$ and $\sigma\in\mathcal{S}_A$, we set

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$$h_A(\sigma,\xi,\eta):=\sum_{t_1=0}^{\xi(x_1)}\sum_{t_2=0}^{\xi(x_2)}\cdots\sum_{t_{|A|}=0}^{\xi(x_{|A|})}\prod_{i=1}^{|A|}\Big\{g(t_i)\mathbf{1}\big(\eta(x_i)=\xi(x_i)-t_i+t_{\sigma^{-1}(i)}\big)\Big\}.$$

To show that (9) is symmetric with respect to ξ and η, it is sufficient to show that $\sum_{\sigma\in\mathcal{S}_A}h_A(\sigma,\xi,\eta)$ is symmetric with respect to ξ and η for every $A\in\mathcal{D}_N$ so that $A\supset A_0$. Now, each permutation $\sigma\in\mathcal{S}_A$ can be decomposed as the product of cyclic permutations and the summand in $h_A(\sigma,\xi,\eta)$ is given by a product form. Accordingly, the following two statements are sufficient for the symmetry of $\sum_{\sigma\in\mathcal{S}_A}h_A(\sigma,\xi,\eta)$.

- When $|A|=2$, $h_A(\sigma,\xi,\eta)$ is symmetric with respect to $\{\xi(x)\}_{x\in A}$ and $\{\eta(x)\}_{x\in A}$ for every $\sigma\in\mathcal{S}_A$.
- When $|A|\geq 3$, $h_A(\sigma,\xi,\eta)+h_A(\sigma^{-1},\xi,\eta)$ is symmetric with respect to $\{\xi(x)\}_{x\in A}$ and $\{\eta(x)\}_{x\in A}$ for every cyclic permutation $\sigma\in\mathcal{S}_A$.

These can be reformulated as follows.

Lemma 2.1. *Let $g:\mathbb{Z}_+\to[0,\infty)$ and $a=\{a_i\}_{i=1}^n, b=\{b_i\}_{i=1}^n\in\mathbb{Z}_+^n$ be two sequences of nonnegative integers which satisfy $\sum_{i=1}^n a_i=\sum_{i=1}^n b_i$.*

(i) *Let $n=2$ and define*

$$S=S(a,b):=\sum_{t_1=0}^{a_1}\sum_{t_2=0}^{a_2}\prod_{i=1}^{2}\{g(t_i)\mathbf{1}(t_i-t_{i+1}=a_i-b_i)\},$$

where we identify t_3 as t_1. Then S is symmetric with respect to a and b.

(ii) *Let $n\geq 3$ and define*

$$\begin{aligned}S=S_+(a,b)+S_-(a,b):=&\sum_{t_1=0}^{a_1}\sum_{t_2=0}^{a_2}\cdots\sum_{t_n=0}^{a_n}\prod_{i=1}^{n}\{g(t_i)\mathbf{1}(t_i-t_{i+1}=a_i-b_i)\}\\&+\sum_{t_1=0}^{a_1}\sum_{t_2=0}^{a_2}\cdots\sum_{t_n=0}^{a_n}\prod_{i=1}^{n}\{g(t_i)\mathbf{1}(t_i-t_{i-1}=a_i-b_i)\},\end{aligned}$$

where we identify t_{n+1} as t_1 and t_0 as t_n. Then S is symmetric with respect to a and b.

Proof.

(i) Under the condition $a_1+a_2=b_1+b_2$, we have

$$\begin{aligned}S(a,b)&=\sum_{t_1=0}^{a_1}\sum_{t_2=0}^{a_2}g(t_1)g(t_1-a_1+b_1)\mathbf{1}(t_2=t_1-a_1+b_1)\\&=\sum_{t_1=0}^{a_1}g(t_1)g(t_1-a_1+b_1)\mathbf{1}(0\leq t_1-a_1+b_1\leq a_2)\\&=\sum_{t_1=0\vee(a_1-b_1)}^{a_1\wedge(a_1+a_2-b_1)}g(t_1)g(t_1-a_1+b_1)\end{aligned}$$

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$$= \sum_{t_1=0\vee(a_1-b_1)}^{a_1\wedge b_2} g(t_1)g(t_1-a_1+b_1)$$
$$= \sum_{t_1=a_1\vee b_1}^{(a_1+b_1)\wedge(b_1+b_2)} g(t_1-b_1)g(t_1-a_1).$$

This is symmetric with respect to a and b because $b_1+b_2=\frac{1}{2}(a_1+a_2+b_1+b_2)$.

(ii) We compute that

$$\begin{aligned}
&S_+(a,b)\\
&= \sum_{t_1\ge 0}\sum_{t_2\ge 0}\cdots\sum_{t_n\ge 0}\left[\prod_{i=1}^{n}\{g(t_i)\mathbf{1}(t_i-t_{i+1}=a_i-b_i)\}\prod_{i=1}^{n}\{\mathbf{1}(t_i\le a_i)\mathbf{1}(t_{i+1}\le b_i)\}\right]\\
&= \sum_{t_1\ge 0}\sum_{t_2\ge 0}\cdots\sum_{t_n\ge 0}\prod_{i=1}^{n} g(t_i)\prod_{i=1}^{n}\{\mathbf{1}(t_i-t_{i+1}=a_i-b_i)\mathbf{1}(t_i\le a_i\wedge b_{i-1})\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&S_-(a,b)\\
&= \sum_{t_1\ge 0}\sum_{t_2\ge 0}\cdots\sum_{t_n\ge 0}\prod_{i=1}^{n} g(t_i)\prod_{i=1}^{n}\{\mathbf{1}(t_i-t_{i-1}=a_i-b_i)\mathbf{1}(t_i\le a_i\wedge b_{i+1})\}\\
&= \sum_{t_1\ge 0}\sum_{t_2\ge 0}\cdots\sum_{t_n\ge 0}\prod_{i=1}^{n} g(t_i)\prod_{i=1}^{n}\{\mathbf{1}(t_{i+1}-t_i=a_i-b_i)\mathbf{1}(t_{i+1}\le a_i\wedge b_{i+1})\}\\
&= \sum_{t_1\ge 0}\sum_{t_2\ge 0}\cdots\sum_{t_n\ge 0}\prod_{i=1}^{n} g(t_i)\prod_{i=1}^{n}\{\mathbf{1}(t_i-t_{i+1}=b_i-a_i)\mathbf{1}(t_i\le a_{i-1}\wedge b_i)\}\\
&= S_+(b,a),
\end{aligned}$$

where the second equality follows from rewriting the variable t_i as t_{i+1}, $1\le i\le n$. Therefore, $S_+(a,b)+S_-(a,b)$ is symmetric with respect to a and b. □

Remark 2.1. Consider the case where, for each $A\in\mathcal{D}_N$, a permutation $\sigma\in\mathcal{S}_A$ is chosen according to a general probability distribution Q_A on $\mathcal{S}_A$, instead of the uniform distribution. Following the argument above, if we can show that $\sum_{\sigma\in\mathcal{S}_A} Q_A(\sigma)h_A(\sigma,\xi,\eta)$ is symmetric with respect to ξ and η for every $A\in\mathcal{D}_N$ so that $A\supset A_0$, then the stationary distribution is given by (5), just as in the case where σ is chosen uniformly at random. As a sufficient condition, suppose that, for every $A\in\mathcal{D}_N$ and every collection of disjoint cyclic permutations that compose a permutation on A, the permutations generated by these cycles and their inverses are assigned equal probabilities under Q_A. Then the symmetry of $\sum_{\sigma\in\mathcal{S}_A} Q_A(\sigma)h_A(\sigma,\xi,\eta)$ holds. As an illustrative example, consider a permutation $\widetilde{\sigma}=\sigma_1\sigma_2\in\mathcal{S}_A$ where σ_1 and σ_2 are two disjoint cyclic permutations. Assume that all permutations on A formed as products of σ_1, σ_2 and their inverses occur with equal probability $Q_A(\widetilde{\sigma})$. Then their total contribution to the sum $\sum_{\sigma\in\mathcal{S}_A} Q_A(\sigma)h_A(\sigma,\xi,\eta)$ is given by

$$Q_A(\widetilde{\sigma})\big\{h_A(\sigma_1\sigma_2,\xi,\eta)+h_A(\sigma_1\sigma_2^{-1},\xi,\eta)+h_A(\sigma_1^{-1}\sigma_2,\xi,\eta)+h_A(\sigma_1^{-1}\sigma_2^{-1},\xi,\eta)\big\}.$$

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The symmetry of the expression

$$h_A(\sigma_1\sigma_2, \xi, \eta) + h_A(\sigma_1\sigma_2^{-1}, \xi, \eta) + h_A(\sigma_1^{-1}\sigma_2, \xi, \eta) + h_A(\sigma_1^{-1}\sigma_2^{-1}, \xi, \eta)$$

with respect to ξ and η then follows from the preceding proof.

Proof for the random saving model. We use similar notation to the proof for the immediate exchange model. Take arbitrary $\xi, \eta \in \Omega_N(L)$. We have

$$\begin{aligned}
&\mathrm{P}(Z_{n+1} = \eta | Z_n = \xi) \\
&= \sum_{\substack{A \in \mathcal{D}_N \\ A \supset A_0}} \rho(A) \mathrm{P}\big(\eta(z) = c(z) + d(z) \text{ for every } z \in A \,\big|\, \xi\big) \\
&= \sum_{\substack{A \in \mathcal{D}_N \\ A \supset A_0}} \rho(A) \sum_{t_1=0}^{\xi(x_1)} \sum_{t_2=0}^{\xi(x_2)} \cdots \sum_{t_{|A|}=0}^{\xi(x_{|A|})} \Bigg[\prod_{i=1}^{|A|} \frac{g(t_i)}{G(\xi(x_i))} \sum_{\zeta \in \Omega(A, S_A(\xi) - S_A(t))} \frac{1}{|\Omega(A, S_A(\xi) - S_A(t))|} \\
&\qquad\qquad \times \mathbf{1}\big(\eta(x_i) = t_i + \zeta(x_i) \text{ for every } 1 \le i \le |A|\big) \Bigg] \\
&= \sum_{\substack{A \in \mathcal{D}_N \\ A \supset A_0}} \rho(A) \sum_{t_1=0}^{\xi(x_1)\wedge\eta(x_1)} \sum_{t_2=0}^{\xi(x_2)\wedge\eta(x_2)} \cdots \sum_{t_{|A|}=0}^{\xi(x_{|A|})\wedge\eta(x_{|A|})} \Bigg[\prod_{i=1}^{|A|} \frac{g(t_i)}{G(\xi(x_i))} \times \\
&\qquad \sum_{\zeta \in \Omega(A, S_A(\xi) - S_A(t))} \frac{1}{|\Omega(A, S_A(\xi) - S_A(t))|} \mathbf{1}\big(\eta(x_i) = t_i + \zeta(x_i) \text{ for every } 1 \le i \le |A|\big) \Bigg],
\end{aligned}$$

where $S_A(t) = \sum_{i=1}^{|A|} t_i$ for $t = \{t_i\}_{i=1}^{|A|}$. We have $S_A(\xi) = S_A(\eta)$ for $A \supset A_0$. In addition, for given $t = \{t_i\}_{i=1}^{|A|}$ so that $0 \le t_i \le \xi(x_i) \wedge \eta(x_i)$ for every $1 \le i \le |A|$, there exists a unique $\zeta \in \Omega(A, S_A(\xi) - S_A(t))$ which satisfies $\eta(x_i) = t_i + \zeta(x_i)$ for every $1 \le i \le |A|$. Therefore,

$$\begin{aligned}
&\left\{ \prod_{x \in \Lambda_N} G(\xi(x)) \right\} \cdot \mathrm{P}(Z_{n+1} = \eta | Z_n = \xi) \\
&\quad = \sum_{\substack{A \in \mathcal{D}_N \\ A \supset A_0}} \Bigg[\prod_{x \in \Lambda_N \setminus A} \left\{ \frac{G(\xi(x)) + G(\eta(x))}{2} \right\} \cdot \rho(A) \\
&\qquad \times \sum_{t_1=0}^{\xi(x_1)\wedge\eta(x_1)} \sum_{t_2=0}^{\xi(x_2)\wedge\eta(x_2)} \cdots \sum_{t_{|A|}=0}^{\xi(x_{|A|})\wedge\eta(x_{|A|})} \prod_{i=1}^{|A|} g(t_i) \cdot \frac{1}{|\Omega(A, \frac{S_A(\xi)+S_A(\eta)}{2} - S_A(t))|} \Bigg].
\end{aligned}$$

Since the expression is symmetric with respect to ξ and η, we obtain (8).

If we define g as $g(k) = \delta_0(k)$, $k \in \mathbb{Z}_+$, then $G \equiv 1$ and the above argument yields that $\mathrm{P}(Z_{n+1} = \eta | Z_n = \xi)$ is symmetric with respect to ξ and η. Therefore, the uniform reshuffling model is doubly stochastic and its unique stationary distribution is given by the uniform distribution on $\Omega_N(L)$. Actually, this matches when $G \equiv 1$ is set in (5). □

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3. Proof of Theorem 1.1

For the proof of Theorem 1.1, we adapt the proof of the equivalence of ensembles for the i.i.d. product measure (cf. [13, Appendix 2]). In the following we assume that $g(0) > 0$ and there exist $\alpha \in \mathbb{R}$ and $c_\alpha \in (0, \infty)$ such that $\lim_{k\to\infty} \frac{g(k)}{k^\alpha} = c_\alpha$. We set out the required notation. Define $G(k) = \sum_{j=0}^{k} g(j)$, $k \in \mathbb{Z}_+$, and $Q_n(s) = \sum_{k\geq 0} k^n s^k G(k)$, $n \in \mathbb{Z}_+$. By the assumption on g, we have the following asymptotics for $G(k)$ as $k \to \infty$:

$$G(k) \sim \begin{cases} \dfrac{c_\alpha}{\alpha+1} k^{\alpha+1} & \text{if } \alpha > -1, \\ c_\alpha \log k & \text{if } \alpha = -1, \\ C_0 & \text{if } \alpha < -1, \end{cases} \tag{10}$$

where $C_0 = C_0(g) > 0$ is a constant which depends on g. In particular, the radius of convergence of $Q_n(s)$ is 1 and we have that $\lim_{s\uparrow 1} Q_n(s) = \infty$ for every $n \in \mathbb{Z}_+$ and $\alpha \in \mathbb{R}$. We define the exponential family of distributions $\{\nu_s(\,\cdot\,); s \in [0, 1)\}$ on $\mathbb{Z}_+$ by $\nu_s(k) = \frac{s^k G(k)}{Q_0(s)}$, $k \in \mathbb{Z}_+$. It is easy to see that $\mathbb{E}^{\nu_s}[\eta(0)] = \frac{Q_1(s)}{Q_0(s)}$ is continuous, increasing in s and diverges to infinity as $s \uparrow 1$. Hence, for every $K > 0$ there exists a unique $s^* = s^*(K) \in (0, 1)$ such that $\mathbb{E}^{\nu_{s^*}}[\eta(0)] = K$. To examine the asymptotic behavior of $s^*(K)$ as $K \to \infty$, we use a Tauberian theorem of the following form (cf. [1, Corollary 1.7.3]).

Theorem 3.1. *Let $\{a_k\}_{k\geq 0}$ be a sequence of nonnegative numbers and assume that $A(s) = \sum_{k=0}^{\infty} a_k s^k$ converges for $s \in [0, 1)$ and $\{a_k\}_{k\geq 0}$ is monotone. Then the following assertions are equivalent:*

- *$A(s) \sim \Gamma(\beta+1)(1-s)^{-\beta} h(\frac{1}{1-s})$ as $s \uparrow 1$ for $\beta > 0$ and slowly varying function h;*
- *$a_k \sim \beta k^{\beta-1} h(k)$ as $k \to \infty$ for $\beta > 0$ and slowly varying function h.*

By this theorem and (10), the following asymptotics holds in the limit as $s \uparrow 1$:

$$Q_n(s) \sim \begin{cases} \dfrac{c_\alpha \Gamma(n+\alpha+2)}{\alpha+1} (1-s)^{-(\alpha+n+2)} & \text{if } \alpha > -1, \\ c_\alpha \Gamma(n+1)(1-s)^{-(n+1)} \log \dfrac{1}{1-s} & \text{if } \alpha = -1, \\ C_0 \Gamma(n+1)(1-s)^{-(n+1)} & \text{if } \alpha < -1, \end{cases} \tag{11}$$

where we used the relation $\Gamma(\beta+1) = \beta\Gamma(\beta)$ for every $\beta > 0$. Therefore,

$$\mathbb{E}^{\nu_s}[\eta(0)] = \frac{Q_1(s)}{Q_0(s)} \sim \begin{cases} \dfrac{\alpha+2}{1-s} & \text{if } \alpha > -1, \\ \dfrac{1}{1-s} & \text{if } \alpha \leq -1, \end{cases}$$

and this yields that

$$s^*(K) = \begin{cases} 1 - \dfrac{\alpha+2}{K}(1+o(1)) & \text{if } \alpha > -1, \\ 1 - \dfrac{1}{K}(1+o(1)) & \text{if } \alpha \leq -1, \end{cases} \tag{12}$$

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as $K \to \infty$. By these asymptotics we also have

$$\mathrm{Var}_{\nu_{s^*(K)}}(\eta(0)) = \frac{Q_2(s^*(K))}{Q_0(s^*(K))} - K^2 \tag{13}$$
$$= \begin{cases} \dfrac{1}{\alpha+2}K^2(1+o(1)) & \text{if } \alpha > -1, \\ K^2(1+o(1)) & \text{if } \alpha \le -1, \end{cases}$$

as $K \to \infty$.

Next, for each $s \in [0, 1)$ and $B \subset \mathbb{N}$, let $\overline{\nu}_s^B$ be the product measure on $\mathbb{Z}_+^B$ whose one-site marginal distribution equals ν_s. $\overline{\nu}_s^B\big(\,\cdot\,\big|\,\Omega(B, L)\big)$ denotes the conditioned probability of $\overline{\nu}_s^B$ on the event that the total number of coins on B equals to L. Then we have the following key identity for (5):

$$\mu_{N,L}(\,\cdot\,) = \overline{\nu}_s^{\Lambda_N}\big(\,\cdot\,\big|\,\Omega_N(L)\big).$$

Note that the right-hand side does not depend on the choice of the parameter s. Let $f : \mathbb{R}^A \to \mathbb{R}$ be a bounded continuous local function where A is a finite subset of $\mathbb{N}$. For every $N \in \mathbb{N}$ such that $\Lambda_N \supset A$, we have

$$\begin{aligned}
&\mathbb{E}^{\mu_{N,L}}\left[f\left(\frac{\cdot}{a_N}\right)\right] \\
&= \sum_{\eta \in \mathbb{Z}_+^A} f\left(\frac{\eta}{a_N}\right) \overline{\nu}_s^{\Lambda_N}\left(\xi_A = \eta \,\Big|\, \sum_{x\in\Lambda_N} \xi(x) = L\right) \\
&= \sum_{\eta \in \mathbb{Z}_+^A} f\left(\frac{\eta}{a_N}\right) \overline{\nu}_s^{A}(\eta) + \sum_{\eta \in \mathbb{Z}_+^A} f\left(\frac{\eta}{a_N}\right) \left\{ \frac{\overline{\nu}_s^{\Lambda_N\setminus A}\left(\sum_{x\in\Lambda_N\setminus A} \xi(x) = L - S_A(\eta)\right)}{\overline{\nu}_s^{\Lambda_N}\left(\sum_{x\in\Lambda_N} \xi(x) = L\right)} - 1 \right\} \overline{\nu}_s^{A}(\eta) \\
&=: I_1 + I_2,
\end{aligned} \tag{14}$$

where ξ_A denotes the configurations ξ restricted to the set A. Now, we set $L = L_N$ where $\{L_N\}_{N\ge1}$ is a sequence of positive integers that satisfies $\lim_{N\to\infty} \frac{L_N}{Na_N} = T$ for some constant $T > 0$ and divergent sequence $\{a_N\}_{N\ge1}$. We also take s on the right-hand side of (14) as $s_N^* := s^*(\frac{L_N}{N})$. For this choice of s we show that $I_1 \to \int_{\mathbb{R}_+^A} f(r)\overline{\mu}_{\alpha,T}^A(dr)$ and $I_2 \to 0$ as $N \to \infty$.

For the proof of the convergence of I_1, we assume that f is a function of one variable for notational simplicity. The general case can be proven in a similar manner since $\overline{\nu}_s^A$ is a product measure with the same marginal distribution. First, we consider the case $\alpha > -1$. Let $R > 0$. By (10) and (11),

$$\begin{aligned}
I_1 &= \sum_{k\ge0} f\left(\frac{k}{a_N}\right) \frac{1}{Q_0(s_N^*)} (s_N^*)^k G(k) \\
&= \frac{1}{\Gamma(\alpha+2)}(1-s_N^*) \sum_{k=0}^{\lfloor Ra_N\rfloor} f\left(\frac{k}{a_N}\right)(s_N^*)^k\big((1-s_N^*)k\big)^{\alpha+1} \\
&\qquad + \frac{1}{\Gamma(\alpha+2)}(1-s_N^*) \sum_{k=\lfloor Ra_N\rfloor+1}^{\infty} f\left(\frac{k}{a_N}\right)(s_N^*)^k\big((1-s_N^*)k\big)^{\alpha+1} + o(1) \\
&=: I_3 + I_4 + o(1),
\end{aligned}$$

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as $N \to \infty$. We note that since $Q_0(s_N^*) \to \infty$ in the limit $N \to \infty$, a finite sum in I_1 is negligible and we can replace G with the right-hand side of (10) with an error of $o(1)$. Then, by (12) and the condition on L_N,

$$I_3 = \frac{1}{\Gamma(\alpha+2)} \frac{\alpha+2}{T} \frac{1}{a_N} \sum_{k=0}^{\lfloor Ra_N \rfloor} f\left(\frac{k}{a_N}\right) \left\{ \left(1 - \frac{\alpha+2}{T} \frac{1}{a_N}\right)^{a_N} \right\}^{\frac{k}{a_N}} \left(\frac{\alpha+2}{T} \frac{k}{a_N}\right)^{\alpha+1} + o(1)$$

$$\to \frac{1}{\Gamma(\alpha+2)} \left(\frac{\alpha+2}{T}\right)^{\alpha+2} \int_0^R f(r) \mathrm{e}^{-\frac{\alpha+2}{T} r} r^{\alpha+1} \, dr,$$

as $N \to \infty$ where the convergence follows from the definition of the Riemann integral. By taking the limit $R \to \infty$, the right-hand side converges to $\int_0^\infty f(r) \mu_{\alpha,T}(dr)$. For I_4, we have

$$|I_4| \le \frac{1}{\Gamma(\alpha+2)} \frac{\alpha+2}{T} \frac{1}{a_N} \sum_{k=\lfloor Ra_N \rfloor + 1}^{\infty} \left| f\left(\frac{k}{a_N}\right) \right| \left(1 - \frac{\alpha+2}{T} \frac{1}{a_N}\right)^k \left(\frac{\alpha+2}{T} \frac{k}{a_N}\right)^{\alpha+1} + o(1)$$

$$\le C \frac{1}{a_N} \sum_{k=\lfloor Ra_N \rfloor + 1}^{\infty} \mathrm{e}^{-\frac{\alpha+2}{T} \frac{k}{a_N}} \left(\frac{k}{a_N}\right)^{\alpha+1} + o(1)$$

$$\le C' \sum_{j=\lfloor R \rfloor}^{\infty} \mathrm{e}^{-\frac{\alpha+2}{T} j} j^{\alpha+1} + o(1),$$

for every N large enough where C, C' are positive constants independent of N. By taking the limits $N \to \infty$ and $R \to \infty$, we obtain $I_4 \to 0$.

Second, we consider the case $\alpha = -1$. By (10) and (11) again,

$$I_1 = \frac{1 - s_N^*}{-\log(1 - s_N^*)} \sum_{k=0}^{\lfloor Ra_N \rfloor} f\left(\frac{k}{a_N}\right) (s_N^*)^k \log k$$

$$+ \frac{1 - s_N^*}{-\log(1 - s_N^*)} \sum_{k=\lfloor Ra_N \rfloor + 1}^{\infty} f\left(\frac{k}{a_N}\right) (s_N^*)^k \log k + o(1)$$

$$=: I_3' + I_4' + o(1),$$

as $N \to \infty$. We have $-\log(1 - s_N^*) = (1 + o(1)) \log a_N$, and this yields that

$$I_3' = \frac{1}{Ta_N \log a_N} \sum_{k=0}^{\lfloor Ra_N \rfloor} f\left(\frac{k}{a_N}\right) \left\{ \left(1 - \frac{1}{Ta_N}\right)^{a_N} \right\}^{\frac{k}{a_N}} \log k + o(1)$$

$$= \frac{1}{Ta_N} \sum_{k=0}^{\lfloor Ra_N \rfloor} f\left(\frac{k}{a_N}\right) \left\{ \left(1 - \frac{1}{Ta_N}\right)^{a_N} \right\}^{\frac{k}{a_N}}$$

$$+ \frac{1}{T \log a_N} \frac{1}{a_N} \sum_{k=0}^{\lfloor Ra_N \rfloor} f\left(\frac{k}{a_N}\right) \left\{ \left(1 - \frac{1}{Ta_N}\right)^{a_N} \right\}^{\frac{k}{a_N}} \log\left(\frac{k}{a_N}\right) + o(1).$$

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In the limits $N\to\infty$ and $R\to\infty$, the first term of the right-hand side converges to $\frac{1}{T}\int_0^\infty f(r)\mathrm{e}^{-\frac{r}{T}}\,dr$ and the second term vanishes due to the extra factor $\frac{1}{\log a_N}$. I_4' goes to 0 in the same way as I_4. The case $\alpha<-1$ also follows from a similar argument.

Next, for the convergence of I_2, we use the following local limit theorem.

Theorem 3.2. *Let $\{b_N\}_{N\ge1}$ be a sequence of positive numbers which satisfies $\lim_{N\to\infty}b_N=\infty$ and set $s_N^*:=s^*(b_N)$. For each $N\in\mathbb{N}$, $\{X_j^{(N)}\}_{j\in\Lambda_N}$ denotes a family of i.i.d. $\mathbb{Z}_+$-valued random variables with common distribution $\nu_{s_N^*}$. Then, for every finite set $B\subset\mathbb{N}$,*

$$\lim_{N\to\infty}\sup_{L\ge0}\left|\sqrt{\sigma_N^2(N-|B|)}\mathrm{P}\left(\sum_{j\in\Lambda_N\setminus B}X_j^{(N)}=L\right)-\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(L-(N-|B|)b_N)^2}{2\sigma_N^2(N-|B|)}\right\}\right|=0, \tag{15}$$

where $\sigma_N^2=\mathrm{Var}(X_1^{(N)})$.

The proof of this theorem is given later. By applying this theorem for $L=L_N$ and $b_N=\frac{L_N}{N}$, we have

$$\begin{aligned}&\frac{\overline{\nu}_{s_N^*}^{\Lambda_N\setminus A}\left(\sum_{x\in\Lambda_N\setminus A}\xi(x)=L_N-S_A(\eta)\right)}{\overline{\nu}_{s_N^*}^{\Lambda_N}\left(\sum_{x\in\Lambda_N}\xi(x)=L_N\right)}\\&\qquad=\frac{\sqrt{N}}{\sqrt{N-|A|}}\frac{\exp\left\{-\frac{(L_N-S_A(\eta)-(N-|A|)\frac{L_N}{N})^2}{2\sigma_N^2(N-|A|)}\right\}+o(1)}{1+o(1)},\end{aligned}$$

as $N\to\infty$ for every $\eta\in\mathbb{Z}_+^A$. Note that the $o(1)$ terms do not depend on η. Set $D_A(k)=\{0,1,\dots,k\}^A$, $k\in\mathbb{Z}_+$. Then the above asymptotics and (13) yield that

$$\lambda_{N,R}:=\sup_{\eta\in D_A(\lfloor Ra_N\rfloor)}\left|\frac{\overline{\nu}_{s_N^*}^{\Lambda_N\setminus A}\left(\sum_{x\in\Lambda_N\setminus A}\xi(x)=L_N-S_A(\eta)\right)}{\overline{\nu}_{s_N^*}^{\Lambda_N}\left(\sum_{x\in\Lambda_N}\xi(x)=L_N\right)}-1\right|\to0$$

as $N\to\infty$ for every $R>0$, and

$$\sup_{N\ge1}\sup_{\eta\in\mathbb{Z}_+^A}\left|\frac{\overline{\nu}_{s_N^*}^{\Lambda_N\setminus A}\left(\sum_{x\in\Lambda_N\setminus A}\xi(x)=L_N-S_A(\eta)\right)}{\overline{\nu}_{s_N^*}^{\Lambda_N}\left(\sum_{x\in\Lambda_N}\xi(x)=L_N\right)}-1\right|\le C''$$

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for some constant $C'' > 0$. Therefore,

$$|I_2| \le \sum_{\eta \in D_A(\lfloor Ra_N \rfloor)} \left| f\left(\frac{\eta}{a_N}\right) \right| \left| \frac{\overline{\nu}_{s_N^*}^{\Lambda_N \setminus A}\left(\sum_{x \in \Lambda_N \setminus A} \xi(x) = L_N - S_A(\eta) \right)}{\overline{\nu}_{s_N^*}^{\Lambda_N}\left(\sum_{x \in \Lambda_N} \xi(x) = L_N \right)} - 1 \right| \overline{\nu}_{s_N^*}^{A}(\eta)$$

$$+ \sum_{\eta \in \mathbb{Z}_+^A \setminus D_A(\lfloor Ra_N \rfloor)} \left| f\left(\frac{\eta}{a_N}\right) \right| \left| \frac{\overline{\nu}_{s_N^*}^{\Lambda_N \setminus A}\left(\sum_{x \in \Lambda_N \setminus A} \xi(x) = L_N - S_A(\eta) \right)}{\overline{\nu}_{s_N^*}^{\Lambda_N}\left(\sum_{x \in \Lambda_N} \xi(x) = L_N \right)} - 1 \right| \overline{\nu}_{s_N^*}^{A}(\eta)$$

$$\le \|f\|_\infty \lambda_{N,R} + C'' \sum_{\eta \in \mathbb{Z}_+^A \setminus D_A(\lfloor Ra_N \rfloor)} \left| f\left(\frac{\eta}{a_N}\right) \right| \overline{\nu}_{s_N^*}^{A}(\eta).$$

The first term on the right-hand side goes to 0 as $N \to \infty$ for every $R > 0$, and the second term goes to 0 as $N \to \infty$ and $R \to \infty$ by a similar computation to the estimate of I_4 above. Hence, we obtain $I_2 \to 0$ and this completes the proof.

If we assume the condition $g(0) > 0$ and $\sum_{j \ge 0} g(j) < \infty$ for $g : \mathbb{Z}_+ \to [0, \infty)$ instead, then the proof for the case $\alpha < -1$ above can be applied as it is. □

Remark 3.1. If $(L - (N - |B|)b_N)^2 \gg 2\sigma_N^2(N - |B|)$ in (15) then the exponential term converges to 0, rendering the local limit theorem ineffective. For this reason, the local limit theorem in the form of Theorem 3.2 was insufficient to prove the equivalence of ensembles in general settings, and a more refined version such as the Edgeworth expansion of at least the second order was necessary (cf. the proofs of Corollaries 1.4 and 1.7 in [13, Appendix 2]). On the other hand, such an expansion is not necessary and Theorem 3.2 is sufficient in our case. Because we divided the summation $\sum_{\eta \in \mathbb{Z}_+^A}$ of I_2 into $\sum_{\eta \in D_A(\lfloor Ra_N \rfloor)}$ and $\sum_{\eta \in \mathbb{Z}_+^A \setminus D_A(\lfloor Ra_N \rfloor)}$, the estimate for the latter part was reduced to the estimate of $\overline{\nu}_{s_N^*}^{A}(\mathbb{Z}_+^A \setminus D_A(\lfloor Ra_N \rfloor))$ which can be managed because we know the explicit form of $\nu_{s_N^*}$.

Proof of Theorem 3.2. First of all, we note that Theorem 3.2 corresponds to the local limit theorem for a triangular array of random variables since $\nu_{s_N^*}$ depends on the number of random variables N. Combining this with the fact that $s_N^* := s^*(b_N) \to 1$ as $N \to \infty$, we cannot apply Theorem 1.3 or 1.5 directly in [13, Appendix 2] which studied the refined version of the local limit theorem for the i.i.d. random variables with common distribution ν_s, $s \in (0, 1)$. In addition, the known criteria of the local limit theorem for a triangular array of integer-valued random variables (e.g. [5, 20]) do not hold in our setting since $\sigma_N^2 \to \infty$ as $N \to \infty$. Therefore, we give the proof of the theorem according to the classical argument [22, Chapter VII]. For notational simplicity we only consider the case $B = \emptyset$. The modification for the general finite set $B \subset \mathbb{N}$ is straightforward.

Set $\widetilde{X}_j^{(N)} := \frac{1}{\sqrt{N\sigma_N^2}}(X_j^{(N)} - b_N)$, $j \in \Lambda_N$, and $t_N(L) := \frac{1}{\sqrt{N\sigma_N^2}}(L - Nb_N)$. We define $\phi_N(\theta) = \mathbb{E}\left[\exp\{i\theta X_1^{(N)}\}\right]$ and $\psi_N(\theta) = \mathbb{E}\left[\exp\left\{i\theta\left(\sum_{j \in \Lambda_N} \widetilde{X}_j^{(N)}\right)\right\}\right]$, $\theta \in \mathbb{R}$, where $i = \sqrt{-1}$. By the

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inversion formula, we have

$$\mathrm{P}\left(\sum_{j\in\Lambda_N} X_j^{(N)} = L\right) = \frac{1}{2\pi\sqrt{N\sigma_N^2}} \int_{-\pi\sqrt{N\sigma_N^2}}^{\pi\sqrt{N\sigma_N^2}} \mathrm{e}^{-\mathrm{i}\theta t_N(L)} \psi_N(\theta)\, d\theta.$$

Therefore,

$$\begin{aligned}
&2\pi \left| \sqrt{N\sigma_N^2}\mathrm{P}\left(\sum_{j\in\Lambda_N} X_j^{(N)} = L\right) - \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}t_N(L)^2} \right| \\
&= \left| \int_{-\pi\sqrt{N\sigma_N^2}}^{\pi\sqrt{N\sigma_N^2}} \mathrm{e}^{-\mathrm{i}\theta t_N(L)} \psi_N(\theta)\, d\theta - \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\theta t_N(L)} \mathrm{e}^{-\frac{1}{2}\theta^2}\, d\theta \right| \\
&\le \int_{|\theta|\le R} |\psi_N(\theta) - \mathrm{e}^{-\frac{1}{2}\theta^2}|\, d\theta + \int_{R\le|\theta|\le\gamma\sqrt{N\sigma_N^2}} |\psi_N(\theta)|\, d\theta \\
&\qquad + \int_{\gamma\sqrt{N\sigma_N^2}\le|\theta|\le\pi\sqrt{N\sigma_N^2}} |\psi_N(\theta)|\, d\theta + \int_{|\theta|\ge R} \mathrm{e}^{-\frac{1}{2}\theta^2}\, d\theta \\
&=: I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

for every $R > 0$ and $0 < \gamma < \pi$. We show that the right-hand side converges to 0 as $N \to \infty$ and $R \to \infty$.

For I_1, assume that the law of $\sum_{j\in\Lambda_N} \widetilde{X}_j^{(N)}$ converges to the standard normal distribution. Then we have $\lim_{N\to\infty} \psi_N(\theta) = \mathrm{e}^{-\frac{1}{2}\theta^2}$ for every $\theta \in \mathbb{R}$ and we obtain $I_1 \to 0$ as $N \to \infty$ by the bounded convergence theorem. For the convergence of the law of $\sum_{j\in\Lambda_N} \widetilde{X}_j^{(N)}$, we have only to show that $\sum_{j\in\Lambda_N} \mathbb{E}\big[(\widetilde{X}_j^{(N)})^2; |\widetilde{X}_j^{(N)}| \ge \varepsilon\big] \to 0$ as $N \to \infty$ for every $\varepsilon > 0$ by Lindberg's central limit theorem (cf. [7, Theorem 3.4.10]). By (13), $|\widetilde{X}_j^{(N)}| \ge \varepsilon$ implies that $X_j^{(N)} \ge \frac{1}{2}\varepsilon\sqrt{N\sigma_N^2}$ for every N large enough. Therefore,

$$\begin{aligned}
&\sum_{j\in\Lambda_N} \mathbb{E}\Big[\big(\widetilde{X}_j^{(N)}\big)^2; |\widetilde{X}_j^{(N)}| \ge \varepsilon\Big] \\
&\le \frac{1}{\sigma_N^2} \mathbb{E}\Big[\big(X_1^{(N)} - b_N\big)^2; X_1^{(N)} \ge \frac{1}{2}\varepsilon\sqrt{N\sigma_N^2}\Big] \\
&\le \frac{2}{\sigma_N^2} \mathbb{E}\Big[\big(X_1^{(N)}\big)^2; X_1^{(N)} \ge \frac{1}{2}\varepsilon\sqrt{N\sigma_N^2}\Big] + \frac{2b_N^2}{\sigma_N^2} \mathrm{P}\left(X_1^{(N)} \ge \frac{1}{2}\varepsilon\sqrt{N\sigma_N^2}\right) \\
&=: J_1 + J_2,
\end{aligned}$$

where the second inequality follows from the fact that $(a+b)^2 \le 2a^2 + 2b^2$ for every $a, b \in \mathbb{R}$. We first consider the case $\alpha > -1$ for the estimate of J_1. We have

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$$J_1 = \frac{2}{\sigma_N^2} \sum_{k \geq \frac{1}{2}\varepsilon\sqrt{N\sigma_N^2}} k^2 \frac{1}{Q_0(s_N^*)} (s_N^*)^k G(k)$$

$$\leq \frac{C}{b_N^2} \sum_{k \geq \frac{\varepsilon}{4\sqrt{\alpha+2}}\sqrt{Nb_N^2}} k^2 \left(\frac{\alpha+2}{b_N}\right)^{\alpha+2} \left\{\left(1 - \frac{\alpha+2}{b_N}\right)^{b_N}\right\}^{\frac{k}{b_N}} k^{\alpha+1}$$

$$\leq C' \frac{1}{b_N} \sum_{\frac{k}{b_N} \geq \frac{\varepsilon}{4\sqrt{\alpha+2}}\sqrt{N}} \left(\frac{k}{b_N}\right)^{\alpha+3} \left\{\left(1 - \frac{\alpha+2}{b_N}\right)^{b_N}\right\}^{\frac{k}{b_N}},$$

for some constants $C, C' > 0$ and every N large enough, where we used (11), (12), and (13) for the first inequality. The right-hand side goes to 0 as $N \to \infty$ because

$$\frac{1}{b_N} \sum_{\frac{k}{b_N} \geq a} \left(\frac{k}{b_N}\right)^{\alpha+3} \left\{\left(1 - \frac{\alpha+2}{b_N}\right)^{b_N}\right\}^{\frac{k}{b_N}} \to \int_a^\infty r^{\alpha+3} \mathrm{e}^{-(\alpha+2)r}\, dr < \infty,$$

as $N \to \infty$ for every $a \geq 0$. By a similar computation we obtain $J_1 \to 0$ when $\alpha \leq -1$. For the estimate of J_2, we have

$$J_2 \leq C\mathrm{P}\left(X_1^{(N)} \geq \frac{1}{2}\varepsilon\sqrt{N\sigma_N^2}\right) \leq C\frac{2}{\varepsilon\sqrt{N\sigma_N^2}}\mathbb{E}\left[X_1^{(N)}\right] \leq \frac{C'}{\sqrt{N}} \to 0,$$

as $N \to \infty$ where we used Markov's inequality and (13).

Next, we consider I_2. By Taylor's theorem, there exists $\gamma_0 > 0$ such that for every $\theta \in \mathbb{R}$ which satisfies $\left|\frac{\theta}{\sqrt{N\sigma_N^2}}\right| \leq \gamma_0$, we have

$$\left|\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\theta\widetilde{X}_1^{(N)}}\right]\right| \leq 1 - \frac{1}{4}\left(\frac{\theta}{\sqrt{N\sigma_N^2}}\right)^2 \mathbb{E}\left[(X_1^{(N)} - b_N)^2\right] = 1 - \frac{\theta^2}{4N} \leq \mathrm{e}^{-\frac{\theta^2}{4N}}.$$

Therefore, $|\psi_N(\theta)| \leq \mathrm{e}^{-\frac{\theta^2}{4}}$ for every $\theta \in \mathbb{R}$ which satisfies $|\theta| \leq \gamma_0\sqrt{N\sigma_N^2}$. Taking γ in the definition of I_2 as γ_0, we obtain

$$I_2 \leq \int_{R \leq |\theta| \leq \gamma_0\sqrt{N\sigma_N^2}} \mathrm{e}^{-\frac{\theta^2}{4}}\, d\theta \leq 2\int_R^\infty \mathrm{e}^{-\frac{\theta^2}{4}}\, d\theta \to 0,$$

as $R \to \infty$. Similarly, we have $I_4 \to 0$ as $R \to \infty$.

The final task is the estimate of I_3. We take γ in the definition of I_3 as γ_0 above. Let $0 < \delta < 1$ be fixed and define $\phi_N^\delta(\theta) := \mathbb{E}\left[(\delta\mathrm{e}^{\mathrm{i}\theta})^{X_1^{(N)}}\right] = \sum_{k\geq 0} \delta^k \mathrm{e}^{\mathrm{i}\theta k} \nu_{s_N^*}(k)$. By the proof of [19, Lemma 5.4], we know that

$$|\phi_N^\delta(\theta)| \leq \frac{1}{|1 - \delta\mathrm{e}^{\mathrm{i}\theta}|}\left\{\nu_{s_N^*}(0) + \sum_{k\geq 0}\left|\nu_{s_N^*}(k+1) - \nu_{s_N^*}(k)\right|\right\}.$$

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We have $\nu_{s_N^*}(0) = \frac{G(0)}{Q_0(s_N^*)} \to 0$ as $N \to \infty$ by (11) and (12). In addition,

$$\begin{aligned}
&|\nu_{s_N^*}(k+1) - \nu_{s_N^*}(k)| \\
&= \nu_{s_N^*}(k) \left| \frac{\nu_{s_N^*}(k+1)}{\nu_{s_N^*}(k)} - 1 \right| \\
&= \nu_{s_N^*}(k) \left| s_N^* \left\{ \frac{g(k+1)}{G(k)} + 1 \right\} - 1 \right| \le \nu_{s_N^*}(k) \left\{ (1 - s_N^*) + \frac{g(k+1)}{G(k)} \right\}.
\end{aligned}$$

By the assumption on g, (10), and (12), for every $\varepsilon > 0$ there exist $N_0 \ge 1$ and $C_1 > 0$ such that $(1 - s_N^*) + \frac{g(k+1)}{G(k)} \le \frac{C_1}{b_N}$ for every $N \ge N_0$ and $k \ge \varepsilon b_N$. Moreover, $(1 - s_N^*) + \frac{g(k+1)}{G(k)} \le C_2$ for every $k \in \mathbb{Z}_+$ and $N \in \mathbb{N}$, where $C_2 > 0$ is a constant independent of N and k. Therefore,

$$\sum_{k \ge 0} |\nu_{s_N^*}(k+1) - \nu_{s_N^*}(k)| \le \sum_{k=0}^{\varepsilon b_N} C_2 \nu_{s_N^*}(k) + \sum_{k \ge \varepsilon b_N} \frac{C_1}{b_N} \nu_{s_N^*}(k).$$

The first term on the right-hand side converges to $C_3 \int_0^\varepsilon r^{\alpha+1} \mathrm{e}^{-(\alpha+2)r}\, dr$ if $\alpha > -1$ and $C_3 \int_0^\varepsilon \mathrm{e}^{-r}\, dr$ if $\alpha \le -1$ for some $C_3 > 0$ by a similar computation as before. The second term is less than $\frac{C_1}{b_N}$ and this goes to 0 as $N \to \infty$. As a result, for every $\varepsilon > 0$ there exists $C_4(\varepsilon) > 0$ such that $C_4(\varepsilon) \to 0$ as $\varepsilon \to 0$ and

$$|\phi_N^\delta(\theta)| \le \frac{1}{|1 - \delta \mathrm{e}^{\mathrm{i}\theta}|} \left\{ C_4(\varepsilon) + \frac{1}{\pi} \gamma_0 \right\},$$

for every N large enough and every $\theta \in \mathbb{R}$. By taking the limit $\delta \uparrow 1$ and using the estimate $|1 - \mathrm{e}^{\mathrm{i}\theta}| \ge \frac{2}{\pi} |\theta|$ for every $|\theta| \le \pi$, we obtain

$$\begin{aligned}
\left| \phi_N \left(\frac{\theta}{\sqrt{N\sigma_N^2}} \right) \right| &\le \frac{\sqrt{N\sigma_N^2}}{|\theta|} \frac{\pi}{2} \left\{ C_4(\varepsilon) + \frac{1}{\pi} \gamma_0 \right\} \\
&\le \frac{1}{\gamma_0} \frac{\pi}{2} \left\{ C_4(\varepsilon) + \frac{1}{\pi} \gamma_0 \right\} = \frac{\pi}{2\gamma_0} C_4(\varepsilon) + \frac{1}{2},
\end{aligned}$$

for every $\theta \in \mathbb{R}$ so that $\gamma_0 \sqrt{N\sigma_N^2} \le |\theta| \le \pi \sqrt{N\sigma_N^2}$. Hence, by taking $\varepsilon > 0$ small enough, there exists $r < 1$ such that $\left| \phi_N \left(\frac{\theta}{\sqrt{N\sigma_N^2}} \right) \right| \le r$ for every N large enough and $\theta \in \mathbb{R}$ so that $\gamma_0 \sqrt{N\sigma_N^2} \le |\theta| \le \pi \sqrt{N\sigma_N^2}$. This yields that

$$\begin{aligned}
I_3 = \int_{\gamma_0 \sqrt{N\sigma_N^2} \le |\theta| \le \pi \sqrt{N\sigma_N^2}} |\psi_N(\theta)|\, d\theta &= \int_{\gamma_0 \sqrt{N\sigma_N^2} \le |\theta| \le \pi \sqrt{N\sigma_N^2}} \left| \phi_N \left(\frac{\theta}{\sqrt{N\sigma_N^2}} \right) \right|^N d\theta \\
&\le 2\pi \sqrt{N\sigma_N^2}\, r^N \to 0,
\end{aligned}$$

as $N \to \infty$ and we can complete the proof of Theorem 3.2. □

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Proof of Corollary 1.1. By Proposition 1.1, we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}_\eta \left[\frac{1}{N} \left| \left\{ x \in \Lambda_N ; \frac{1}{a_N} X_n^{(N)}(x) \in (b, c) \right\} \right| \right] \\
&\quad = \lim_{n\to\infty} \frac{1}{N} \sum_{x\in\Lambda_N} \mathbb{E}_\eta \left[\mathbf{1}\left(\frac{1}{a_N} X_n^{(N)}(x) \in (b, c) \right) \right] \\
&\quad = \frac{1}{N} \sum_{x\in\Lambda_N} \mu_{N,L_N}\left(\frac{1}{a_N} \xi(x) \in (b, c) \right) \\
&\quad = \mu_{N,L_N}\left(\frac{1}{a_N} \xi(1) \in (b, c) \right).
\end{aligned}$$

Therefore, it is sufficient to show that $\lim_{N\to\infty} \mu_{N,L_N}\left(\frac{1}{a_N} \xi(1) \in (b, c) \right) = \mu_{\alpha,T}((b, c))$. This follows from Theorem 1.1 and the basic facts about the weak convergence of probability measures. The same is true for $\{Y_n^{(N)}\}_{n\geq 0}$ and $\{Z_n^{(N)}\}_{n\geq 0}$. □

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