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Integrable Hamiltonian flows with positive Lebesgue-measure entropy

LEO T. BUTLER

Department of Mathematics, Northwestern University, Evanston, IL, USA
(e-mail: lbutler@math.northwestern.edu)

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Abstract. This paper shows that any symplectic diffeomorphism $\mathbf{F} : (\Sigma^{2n}, \eta) \to (\Sigma^{2n}, \eta)$ can be embedded as a subsystem of a Liouville-integrable Hamiltonian flow on some symplectic manifold. If $\mathbf{F}$ is real-analytic, then the flow can be chosen to be real-analytic, but it is Liouville integrable with smooth first integrals. Examples are constructed of integrable, volume-preserving Hamiltonian flows on Poisson manifolds whose metric entropy with respect to the volume form is positive. Completely integrable Hamiltonian flows on a symplectic manifold are constructed which have positive metric entropy with respect to an invariant probability measure that is absolutely continuous with respect to the canonical volume form.

1. *Introduction*

Part of mathematical folklore is that integrable dynamical systems cannot exhibit complicated dynamics. This belief is largely due to Poincaré's ground-breaking discovery of hyperbolic dynamics in certain classical Hamiltonian systems and to the immense amount of literature that is devoted to demonstrating the real-analytic non-integrability of Hamiltonian dynamical systems. In this paper, we construct smoothly integrable dynamical systems which exhibit very complicated dynamics. In addition, we construct smoothly integrable dynamical systems in which the complicated dynamics dominate the integrable dynamics, from a probabilistic point-of-view.

We say that a flow is integrable if its *generic* trajectory is a conditionally periodic winding on an invariant torus†. In this paper, we study volume-preserving (Hamiltonian) flows, so there is an ambiguity in the meaning of *generic*: is it to be understood in the sense of topology or measure theory? Examples can be constructed in the smooth category where this difference is meaningful, but these examples have no dynamics on

† No standard definition of integrability exists; cf. [**1, 6, 12**] which are standard sources for the proof of the Liouville–Arnold theorem. Let us also remark that Bogoyavlenskij [**2, 3**] and Fassò [**7**] have independently studied this paper's definition of integrability and have each demonstrated that it is more general than the conventional notion of Liouville or non-commutative integrability in the setting of Hamiltonian mechanics.

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the set complementary to the set of invariant tori [**8**]. The central result of this paper is the construction of Hamiltonian dynamical systems whose flow is, on an open, dense, invariant subset, locally conjugate to a translation-type flow on a toroidal ball $\mathbb{T}^k \times \mathbb{D}^l_r$, where $\mathbb{D}^l_r$ is the open l-dimensional disc of radius r, while on a closed, invariant, full-measure subset of the complement, the flow has a Poincaré return map that is partially hyperbolic. Indeed, we construct a family of Hamiltonian flows that preserve a fixed smooth, positive probability measure m (a Lebesgue measure) and the *singular* set on which the Poincaré return map is partially hyperbolic can be made to have m-measure $1 - \epsilon$ for any $\epsilon > 0$. In other words, we construct a family of Hamiltonian flows whose topologically generic trajectory is a translation on an invariant, embedded torus, but for which the probability of finding a trajectory whose Lyapunov exponents all vanish can be made arbitrarily close to zero.

Let us state the results more formally. Let (Σ^{2n}, η) be a compact symplectic manifold, $\mathbf{F} : (\Sigma^{2n}, \eta) \to (\Sigma^{2n}, \eta)$ be a symplectic diffeomorphism and $M_{\mathbf{F}}$ be the suspension manifold of $\mathbf{F}$. The suspension manifold comes equipped with a flow $\phi^{\mathbf{F}}_t$, called the suspension flow, for which Σ is a global cross-section and the Poincaré return map is $\mathbf{F}$. Let $Q := \mathbb{T}^2 \times M_{\mathbf{F}}$ and equip Q with the natural product probability measure m and Poisson structure $\mathcal{P}$ (see Theorem 2.1).

THEOREM 1.1. *Let $0 < \epsilon \leq 1$ be given. Then there exists a locally-Hamiltonian, m-volume-preserving flow $\phi^\epsilon_t : Q \to Q$ such that ϕ^ϵ_t is Liouville integrable. In addition, there exists a closed ϕ^ϵ_t-invariant set $\hat{\mathrm{C}} \subset Q$ such that $m(\hat{\mathrm{C}}) \geq 1 - \epsilon$ and the Poincaré return map of $\phi^\epsilon_t|\hat{\mathrm{C}}$ equals the restriction of $id_{\mathbb{T}^2} \times \phi^{\mathbf{F}}_1$ to a set $K \times M_{\mathbf{F}} \subset Q$, where $K \subset \mathbb{T}^2$. The measure theoretic entropy of ϕ^ϵ_t satisfies*

$$h_m(\phi^\epsilon_1) = \frac{2m(C)}{\pi} h_{m_\eta}(\mathbf{F}),$$

where m_η is the natural probability measure on (Σ, η).

A byproduct of this construction is that any symplectic map is a factor of a Poincaré return map of an integrable system restricted to a singular subset.

The flow ϕ^ϵ_t is only locally-Hamiltonian, so we also show that by taking $Q' = \mathbb{C} \times \mathbb{T}^2 \times M_{\mathbf{F}}$, we can obtain the same conclusions with a Hamiltonian flow. In addition, the manifolds Q and Q' are odd dimensional and so not symplectic, so we also prove that the example can be made both Hamiltonian and symplectic. Let $M := \mathbb{R} \times \mathbb{T}^4 \times M_{\mathbf{F}}$ and equip it with the natural symplectic structure and measure m (see Theorem 4.1).

THEOREM 1.2. *Let $0 < \epsilon < 1$ be given. Then there exists a Liouville-integrable Hamiltonian flow ϕ^ϵ_t on the symplectic manifold M with a closed invariant set $\hat{\mathrm{C}}$ such that the Poincaré return map of $\phi^\epsilon_t|\hat{\mathrm{C}}$ equals the restriction of $id_{\mathbb{R}\times\mathbb{T}^4} \times \phi^{\mathbf{F}}_1$ to a set $K \times M_{\mathbf{F}} \subset M$, where $K \subset \mathbb{R} \times \mathbb{T}^4$ is a closed set. In addition, let $K_r := (-r, r) \times \mathbb{T}^4 \times M_{\mathbf{F}}$ and $C_r := K_r \cap \hat{\mathrm{C}}$. Then the conditional probability measure m_{C_r} is ϕ^ϵ_t-invariant and absolutely continuous with respect to m, and*

$$h_{m_{C_r}}(\phi_1) = 4\bar{f} h_{m_\eta}(\mathbf{F}),$$

where $\bar{f}$ is the mean of $|f|$ over the (positive-measure) critical-point set of f (see (14)).

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In the trivial case where $\epsilon = 1$, if $\mathbf{F}$ is real-analytic, then we can choose ϕ_t^1 to be a real-analytic Hamiltonian flow, but m_{C_r} is then singular.

The picture that emerges from this paper is that smooth (C^∞) integrable dynamical systems can imitate some of the properties of 'near' integrable systems. In these systems, due to Kolmogorov-Arnold-Moser (KAM) theory, there exists a positive-measure subset of invariant tori on which the flow is conjugate to an irrational translation-type flow. On a complementary set, it is generally possible to prove the flow possesses an invariant (partially) hyperbolic set. However, in the near-integrable case, both invariant sets are typically Cantor sets. In the examples constructed here, the set of invariant tori is open and dense (hence the justification for calling the system integrable) while a complementary, partially-hyperbolic set is a closed set (of arbitrarily large measure).

This paper stems from an examination of the work of Bolsinov and Taĭmanov [**4, 5**] and Paternain [**14–16**]. Paternain studied the topological implications of Liouville integrability of geodesic flows by pursuing the *entropy* approach. Namely, Paternain has studied smooth integrability by finding a number of conditions which imply that the topological entropy of the flow must vanish. Colloquially, in light of Paternain's results, we might conjecture that the dynamics of an integrable system should be tame everywhere, not just on the regular Liouville tori [**14–16**]. This conjecture was disproven in [**5**], where Bolsinov and Taĭmanov produce a Liouville integrable geodesic flow with positive topological entropy. The geodesic flow in their example, however, has a set of invariant Liouville tori of full measure, so its measure theoretic entropy vanishes relative to the canonical Lebesgue measure on S^*M. This paper utilizes the basic mechanism in Bolsinov and Taĭmanov's example to construct Hamiltonian dynamical systems which are both Liouville integrable with C^∞ first integrals and which have very large-in-measure singular sets with partially-hyperbolic dynamics similar to their geodesic flow.

The outline of the paper is as follows. Section 2 sets up the locally-Hamiltonian example of Theorem 1.1, §3 extends this to a globally Hamiltonian example, §4 constructs the Hamiltonian-symplectic example of Theorem 1.2 and §§5–7 contain the proofs. Section 8 concludes with a comparison with the work of Bolsinov and Taimanov and contains suggestions for further work.

Figure 1 depicts typical solutions to the system in equation (5), projected onto the $\mathbb{T}^1[z] \times \mathbb{R}[x] \times \mathbb{T}^1[\theta]$ factor; the motion in the Σ^2 fibre is trivial. The solutions execute simple harmonic motion in the planes $z =$ constant when $z \neq 0$ and when $z = 0$ the solution (a straightline) diverges to infinity. On the compact quotient, the former solutions lie on invariant Liouville tori, while the latter solution lies in the singular set and generates the chaotic dynamics.

1.1. *Terminology and definitions.* Let us now introduce some terminology.

A *Poisson structure* $\mathcal{P}$ on the smooth manifold M is an element of $\mathrm{Hom}(T^*M, TM)$ such that for any $f \in C^\infty(M)$, the vector field $X_f = \mathcal{P}\,df$ preserves both $\mathcal{P}$ and f. This condition is equivalent to the statement that the bracket $\{f, g\} := \mathcal{P}(df, dg)$ is a Lie bracket on $C^\infty(M)$. The Poisson structure is the basic geometrical object in classical mechanics.

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A *Casimir* of a Poisson structure $\mathcal{P}$ is an element $f \in C^\infty(M)$ such that $X_f = \mathcal{P}\,df \equiv 0$. Two functions $f, g \in C^\infty(M)$ are said to be in involution (or to Poisson commute) if $\{f, g\} \equiv 0$. A Casimir Poisson commutes with all smooth functions. The rank of the Poisson structure $\mathcal{P}$ at $x \in M$ is the rank of the linear map $\mathcal{P}_x : T_x^*M \to T_xM$; since $\mathcal{P}$ is skew symmetric the rank is even and $\operatorname{im}\mathcal{P}_x$ is a symplectic subspace of T_xM. If we assume that rank $\mathcal{P}_x$ is constant, then $\operatorname{im}\mathcal{P}$ generates a distribution that is involutive and, thus, Frobenius's theorem implies that through each $x \in M$ there is a maximal integral submanifold of this distribution, M_x, and this distribution comes equipped with a symplectic form. These integral submanifolds are called *symplectic leaves*, locally the foliation of M by symplectic leaves is fibration and the components of the fibration map are Casimirs of $\mathcal{P}$. In fact, we will assume that the symplectic leaves of $\mathcal{P}$ are globally the fibres of a fibrating map—although many examples exist where this is not true [**18**].

Definition 1.3. (Liouville integrability) Let $(M^n, \mathcal{P})$ be a Poisson manifold and let $\mathcal{P}$ have a constant rank $2k$. Suppose that there exist $n - 2k$ Casimirs of $\mathcal{P}$, $f_1, \ldots, f_{n-2k} \in C^\infty(M)$, and k smooth functions $g_1, \ldots, g_k \in C^\infty(M)$ such that:

(i) $\{g_i, g_j\} \equiv 0$ for all i, j;

(ii) the common level sets $\{f_1 = c_1, \ldots, f_{n-2k} = c_{n-2k}, g_1 = d_1, \ldots, g_k = d_k\}$ are compact for all $c_1, \ldots, c_{n-2k}, d_1, \ldots, d_k \in \mathbb{R}$; and

(iii) $f_1, \ldots, g_k$ are functionally independent on an open, dense subset of M.

Then any $H = H(f_1, \ldots, f_{n-2k}, g_1, \ldots, g_k)$ will be said to be Liouville integrable [**1**].

Definition 1.4. (Non-commutative integrability) Suppose that there exist smooth functions $g_1, \ldots, g_l, h_1, \ldots, h_{2k-l}$ such that:

(i) $\{g_i, g_j\} = \{g_i, h_m\} \equiv 0$ for all i, j, m;

(ii) the common level sets $\{f_1 = c_1, \ldots, f_{n-2k} = c_{n-2k}, g_1 = d_1, \ldots, g_l = d_l, h_1 = e_1, \ldots, h_{2k-l} = e_{2k-l}\}$ are compact for all $c_1, \ldots, c_{n-2k}, d_l, \ldots, d_l, e_1, \ldots, e_{2k-l} \in \mathbb{R}$; and

(iii) $f_1, \ldots, f_{n-2k}, g_1, \ldots, g_l, h_1, \ldots, h_{2k-l}$ are functionally independent on an open, dense subset of M.

Then any $H = H(f_1, \ldots, f_{n-2k}, g_1, \ldots, g_l)$ will be said to be *non-commutatively integrable* [**13**].

Note that we have chosen the 'topological' definition of integrability, in the sense that we do not require the measure of the singular set where the integrals are dependent to be zero.

Definition 1.5. Let $(M, \mathcal{P})$ be a Poisson manifold. The smooth 1-form θ is closed with respect to $\mathcal{P}$ if $\theta|L$ is closed for all $\mathcal{P}$-symplectic leaves L.

Observe that θ is closed with respect to $\mathcal{P}$ iff $X_\theta = \mathcal{P}\theta$ preserves $\mathcal{P}$: $\mathcal{L}_{X_\theta}\mathcal{P} = 0$. In this case, the vector field X_θ will be called *locally Hamiltonian*. If $\theta = dH$ where $H \in C^\infty(M)$, then X will be said to be *Hamiltonian*. Definitions (1.3, 1.4) also make sense for a locally Hamiltonian vector field: if X_H is integrable in either sense, and X_θ is

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a locally Hamiltonian vector field such that X_H and X_θ are everywhere collinear, then we will say that X_θ is integrable.

1.2. *Notation and terminology.*

- A subset of a topological space is *generic* if it is a residual set, i.e. it is a countable intersection of open, dense subsets.
- If $\mathcal{P} \in \mathrm{Hom}(T^*M, TM)$ is a Poisson structure, then we will abuse notation and denote by $\mathcal{P} \in \Gamma(\Lambda^2(M))$ the induced section of the bundle of alternating $(2,0)$ tensors on M; $\ker \mathcal{P} \subset T^*M$ is the singular co-distribution that annihilates the image of $\mathcal{P}$, $\mathrm{im}\,\mathcal{P} \subset TM$.
- A measure m on a smooth manifold is a *Lebesgue* measure if its restriction to any coordinate chart is absolutely continuous with respect to the measure induced by the Lebesgue measure on $\mathbb{R}^n$.
- If M is a smooth manifold with local coordinates x^i, then we will denote this by $M[x]$.
- If a group G acts on a set X, then X^G will denote the elements of X fixed by G.
- If $(X, \mathcal{B}, m)$ is a probability space and $\phi_t : X \to X$ is a one-parameter group of automorphisms, then $h_m(\phi_t)$ will denote the m-measure theoretic entropy of the time-1 map ϕ_1.
- $\mathrm{Diff}_\omega(M)$ denotes the set of C^∞ diffeomorphisms of M that preserve the tensor field ω.

Let us now turn to a statement of the main theorems of the present paper. We only deal with the case where $\mathbf{F}$ is an area-preserving diffeomorphism of a surface. The higher-dimensional case is no more difficult, but is notationally more cumbersome.

2. *An integrable, locally-Hamiltonian vector field with positive measure entropy on a five-dimensional Poisson manifold*

Let (Σ^2, η) be a smooth-oriented, compact surface and $\mathbf{F} \in \mathrm{Diff}_\eta(\Sigma^2)$ be a C^∞ η-preserving diffeomorphism. We have in mind that $\mathbf{F}$ will be a hyperbolic toral automorphism or a Bernoulli diffeomorphism such as that constructed in [**10**]; however, this additional hypothesis is not necessary.

2.1. *The manifold.* Let $M_\mathbf{F}$ denote the suspension manifold of $\mathbf{F}$: it is the quotient of $\tilde{M}_\mathbf{F} := \mathbb{R}[x] \times \Sigma^2[v]$ by the action of $\Gamma \simeq \mathbb{Z}$ that is given by $1 * (x, v) = (x + 1, \mathbf{F}(v))$.

Let $\tilde{Q} = \mathbb{T}^1[z] \times \mathbb{T}^1[\theta] \times \mathbb{R}[x] \times \Sigma^2[v]$. Let $D \simeq \mathbb{Z}$ act on $\tilde{Q}$ by

$$1 * (z, \theta, x, v) := (z, \theta, x + 1, \mathbf{F}(v)) \tag{1}$$

and let $Q := D\backslash\tilde{Q}$. Clearly $Q = \mathbb{T}^2 \times M_\mathbf{F}$ and the following diagram commutes:

$$\begin{array}{ccc} \tilde{Q} & \xrightarrow{\tilde{\pi}} & \tilde{M}_\mathbf{F} \\ \Pi \downarrow & & \downarrow \Pi_\mathbf{F} \\ Q & \xrightarrow{\pi} & M_\mathbf{F} \end{array} \tag{2}$$

where $\Pi_\mathbf{F}(x, v) = \Gamma(x, v)$, $\Pi(z, \theta, x, v) = D(z, \theta, x, v)$ and $\tilde{\pi}(z, \theta, x, v) = (x, v)$.

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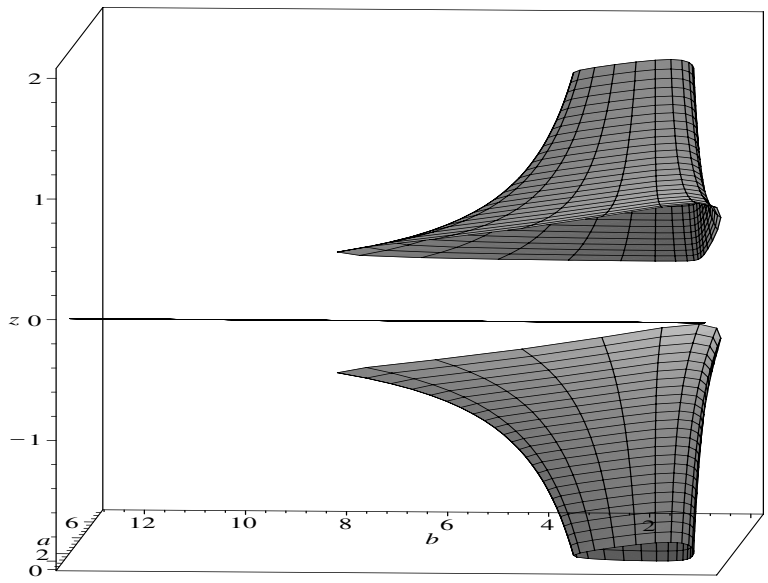


FIGURE 1. Solutions to (5) in the $z - a - b$ plane, $a + ib = e^{x+2\pi i\theta}$.

2.2. *The locally-Hamiltonian flow.* Let v be Darboux coordinates for η, so that $\eta = dv^1 \wedge dv^2$. Let $dz, d\theta, dx$ denote the obvious smooth forms on $\tilde{Q}$. Let $\varphi \in C^\infty(\mathbb{T}^1)$ be a smooth function and define the 1-form

$$\tilde{\alpha}_\varphi(z, \theta, x, v) := \varphi(z)\, dx - \cos 2\pi\theta\, d\theta, \tag{3}$$

on $\tilde{Q}$. Define the skew-symmetric bivector

$$\widetilde{\mathcal{P}} := \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \theta} + \frac{\partial}{\partial v^1} \wedge \frac{\partial}{\partial v^2} \tag{4}$$

on $\tilde{Q}$. Then $\widetilde{\mathcal{P}}$ is a D-invariant smooth Poisson structure, $\tilde{\alpha}$ is a D-invariant smooth 1-form and

$$X_{\tilde{\alpha}} = \widetilde{\mathcal{P}}\,\tilde{\alpha}_\varphi = \varphi(z)\frac{\partial}{\partial \theta} + \cos 2\pi\theta \frac{\partial}{\partial x} \tag{5}$$

is a D-invariant, locally-Hamiltonian vector field. Let $\tilde{m}_\xi$ denote the smooth measure induced by the form ξ, $\xi \in \{dz, d\theta, dx, \eta\}$ and let $\tilde{m} = \tilde{m}_{dz}\tilde{m}_{d\theta}\tilde{m}_{dx}\tilde{m}_\eta$ be the product measure on $\tilde{Q}$. Let $\mathcal{P}$, $\alpha = \alpha_\varphi$, $X = X_\alpha$ and m denote the induced smooth objects on Q and let ϕ_t denote the flow of X on Q. It is clear that m is a ϕ_t-invariant probability measure on Q. We abuse notation and view φ as a function on Q. Let $C := \varphi^{-1}(0)$; clearly C is a closed, ϕ_t-invariant subset of Q.

THEOREM 2.1. *If C is nowhere dense and $\mathbf{F} \in \mathrm{Diff}_\eta(\Sigma^2)$, then the locally-Hamiltonian flow $\phi_t : Q^5 \to Q^5$ satisfies:*

(1) *ϕ_t is C^∞ Liouville integrable;*
(2) *ϕ_t is C^∞ non-commutatively integrable;*
(3) *there is an open, dense subset of Q on which the orbits of ϕ_t are all periodic;*

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(4) $h_m(\phi_1) = (2m(C)/\pi)h_{m_\eta}(\mathbf{F})$*; and*
(5) *if* $\mathbf{F}$ *is a Bernoulli diffeomorphism, there is a full-measure subset* $K \subset C$ *such that* ϕ_t *has non-zero Lyapunov exponents everywhere in* K.

This theorem immediately yields the following corollary (see Theorem 1.1).

COROLLARY 2.2. *Let* $0 < \epsilon \leq 1$ *be given. Then there exists an* m*-measure preserving, locally-Hamiltonian flow* ϕ_t^ϵ *on the compact Poisson manifold* $(Q, \mathcal{P})$ *such that:*
(1) ϕ_t^ϵ *is Liouville-integrable;*
(2) ϕ_t^ϵ *is non-commutatively integrable;*
(3) *the* m*-measure of the set of invariant Liouille tori is* $\leq \epsilon$*;*
(4) *the* m*-measure entropy of* ϕ_t^ϵ *is* $h_m(\phi_1^\epsilon) \geq (2(1-\epsilon)/\pi)h_{m_\eta}(\mathbf{F})$*;*
(5) *the* m*-measure of the set where the Lyapunov exponents of* ϕ_t^ϵ *are nowhere zero is greater than or equal to* $1 - \epsilon$.

The dynamics of the flow ϕ_t of X_α are particularly degenerate and so it is desirable to know whether a locally Hamiltonian vector field exists whose generic trajectories are dense windings on the invariant two-dimensional Liouville tori.

Definition 2.3. (Non-degeneracy) Let $U \subset \mathbb{R}^a$, $V \subset \mathbb{R}^k$ be open subsets and $M = U[J] \times V[I] \times \mathbb{T}^k[\theta]$ with the Poisson structure

$$\mathcal{P} = \sum_{i=1}^{k} \frac{\partial}{\partial I_i} \wedge \frac{\partial}{\partial \theta^i}$$

and the Liouville-integrable Hamiltonian vector field X_H, $H = H(J, I)$. The Hamiltonian H is *non-degenerate* if

$$\det \left[\frac{\partial^2 H(J, I)}{\partial I_i \partial I_j} \right] \neq 0 \tag{6}$$

for (J, I) in an open, dense subset of $U \times V$.

If X is an integrable Hamiltonian vector field on the Poisson manifold $(M, \mathcal{P})$, then X is non-degenerate if, in every action-angle coordinate system, X is non-degenerate in the sense of the previous paragraph.

Non-degeneracy implies that the generic trajectory of X_H is a dense winding on the invariant torus $\{J = \text{constant}, I = \text{constant}\}$; it is also a sufficient condition for the application of KAM theory. If X_α is Liouville integrable and only *locally* Hamiltonian, but Hamiltonian in an open neighbourhood of every invariant Liouville torus, then the preceding definition also makes sense. In particular, this means that 'non-degenerate' is well defined for certain Liouville-integrable locally-Hamiltonian vector fields.

COROLLARY 2.4. *For each* $0 < \epsilon \leq 1$*, there exists a Liouville-integrable, non-degenerate locally Hamiltonian vector field* X^ϵ *on the Poisson manifold* $(Q, \mathcal{P})$ *such that the conclusions (3)–(5) of Corollary 2.2 are true.*

With this basic example in hand, we can now show that all manifolds of dimension $n \geq 5$ possess integrable flows with similar properties.

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Definition 2.5. A Borel measure μ on a topological space is *positive* if $\mu(E) > 0$ for all non-empty open sets E.

COROLLARY 2.6. *Let M^n be a smooth manifold of dimension $n \geq 5$. Then, on M, there exist a smooth Poisson structure $\mathcal{Q}$ and a smooth 1-form ϑ that is closed with respect to $\mathcal{Q}$ and a locally Hamiltonian vector field $V = \mathcal{Q}\vartheta$ with flow $f_t : M \to M$ such that either:*

(1) *f_t is C^∞ non-commutatively integrable and there is an open, dense subset of M on which the orbits of f_t are all periodic; or*

(1′) *f_t is C^∞ Liouville integrable and non-degenerate.*

In both cases:

(2) *there is a closed f_t-invariant set K such that if μ is a positive, f_t-invariant Borel probability measure, then $\mu(K) > 0$ and f_t has non-zero Lyapunov exponents everywhere on K and $h_\mu(f_1) > 0$.*

3. *An integrable, Hamiltonian vector field with positive measure entropy on a seven-dimensional Poisson manifold*

3.1. *The Manifold.* Let $(\Sigma^2, \eta, \mathbf{F})$ be as in the previous section. Let $\tilde{M}^7 = \mathbb{T}^1[z] \times \mathbb{C}[p] \times \mathbb{C}[u] \times \Sigma^2[v]$ and let $D = \mathbb{Z} \oplus i\mathbb{Z}$ act on $\tilde{M}$ by

$$m * (z, p, u, v) := (z, p, u + m, \mathbf{F}^{\operatorname{Re} m}(v)) \tag{7}$$

for all $m \in D$ and all $(z, p, u, v) \in \tilde{M}$. Let $M := D\backslash\tilde{M}$. We naturally have the following commutative diagram:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\pi}} & \tilde{M}_{\mathbf{F}} \\ \Pi \downarrow & & \downarrow \Pi_{\mathbf{F}} \\ M & \xrightarrow{\pi} & M_{\mathbf{F}} \end{array} \tag{8}$$

where $\Pi_{\mathbf{F}}(x, v) = \Gamma(x, v)$, $\Pi(z, p, u, v) = D(z, p, u, v)$ and $\tilde{\pi}(z, p, u, v) = (\operatorname{Re} u, v)$.

3.2. *The Hamiltonian flow.* Let $\varphi \in C^\infty(\mathbb{T}^1)$ and define the Poisson structure

$$\widetilde{\mathcal{P}}_\varphi := \frac{\partial}{\partial p_x} \wedge \frac{\partial}{\partial x} + \frac{\partial}{\partial p_w} \wedge \frac{\partial}{\partial w} + \varphi(z) \frac{\partial}{\partial p_x} \wedge \frac{\partial}{\partial p_w} + \frac{\partial}{\partial v^1} \wedge \frac{\partial}{\partial v^2} \tag{9}$$

and the Hamiltonian

$$\tilde{H}(z, p, u, v) := \tfrac{1}{2}(p_x^2 + p_w^2), \tag{10}$$

for all $(z, p, u, v) \in \mathbb{T}^1 \times \mathbb{C} \times \mathbb{C} \times \mathbb{T}^2$ with $u = x + iw$, $p = p_x + ip_w$. Then $\widetilde{\mathcal{P}}$ and $\tilde{H}$ are D-invariant C^∞ tensor fields on $\tilde{M}$ and induce C^∞ tensor fields $\mathcal{P}$ and H on M. Let $\tilde{m}$ denote the measure induced by the volume form

$$\tilde{\Omega} = \left(\frac{\exp(-\tilde{H})}{2\pi} \right) dz \wedge dp_x \wedge dp_w \wedge dx \wedge dw \wedge dv^1 \wedge dv^2.$$

$\tilde{\Omega}$ ($\tilde{m}$) is D-invariant and induces a smooth volume form Ω (probability measure m) on M. Let $a > 0$ and let

$$\tilde{\Omega}_a = \left(\frac{e^{-a}}{2\pi} \right) dz \wedge d\theta \wedge dx \wedge dw \wedge dv^1 \wedge dv^2$$

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denote the Liouville volume form on $\tilde{H}^{-1}(2a)$. Let $\tilde{m}_a$ denote the induced Liouville measure and let m_a denote the induced probability measure on $H^{-1}(2a)$. Let $C := \varphi^{-1}(0)$, where we abuse notation and view φ a function on M.

THEOREM 3.1. *If C is nowhere dense and $\mathbf{F} \in \mathrm{Diff}_\eta(\Sigma^2)$, then the flow $\phi_t : M^7 \to M^7$ of the Hamiltonian vector field $X_{H,\varphi} = \mathcal{P}_\varphi\, dH$ on M satisfies:*

(1) *ϕ_t is C^∞ Liouville integrable;*
(2) *ϕ_t is C^∞ non-commutatively integrable;*
(3) *there is an open, dense subset of Q on which the orbits of ϕ_t are all periodic;*
(4) *$h_m(\phi_1) = \sqrt{2/\pi}\, m(C) h_{m_\eta}(\mathbf{F})$ and $h_{m_a}(\phi_1) = (2\sqrt{2a}/\pi) m(C) h_{m_\eta}(\mathbf{F})$; and*
(5) *if $\mathbf{F}$ is Bernoulli, there is a full-measure subset $K \subset C$ such that ϕ_t has non-zero Lyapunov exponents everywhere in K.*

Theorem 3.1 is valid for any Hamiltonian H of the form $H = H(p)$, provided that the level sets of a generic level set of H (thought of as a function only of p mod $\mathbb{Z}^2$) are contractible, with a suitable modification of conclusion (4). In particular, the Hamiltonian H might be taken to be of the form $H = \sin(p_x) + \sin(p_w)\sin(p_x) + \sin(p_w)$. In this case the manifold M is compact.

COROLLARY 3.2. *Let $0 < \epsilon \leq 1$ be given. Then there exists a Poisson structure $\mathcal{P}_\epsilon$ and an m-measure-preserving, $\mathcal{P}_\epsilon$-Hamiltonian flow $\phi_t^\epsilon : M^7 \to M^7$ on the Poisson manifold $(M, \mathcal{P}_\epsilon)$ such that:*

(1) *ϕ_t^ϵ is Liouville-integrable;*
(2) *ϕ_t^ϵ is non-commutatively integrable;*
(3) *the m-measure of the set of invariant Liouille tori is $\leq \epsilon$;*
(4) *the m-measure entropy of ϕ_t^ϵ is $h_m(\phi_1^\epsilon) \geq \sqrt{2/\pi}(1-\epsilon) h_{m_\eta}(\mathbf{F})$;*
(4′) *the m_a-measure entropy of $\phi_t^\epsilon | H^{-1}(2a)$ is $h_{m_a}(\phi_1^\epsilon) \geq (2\sqrt{2a}/\pi)(1-\epsilon) h_{m_\eta}(\mathbf{F})$;*
(5) *the m-measure of the set where the Lyapunov exponents of ϕ_t^ϵ are nowhere zero is $\geq 1 - \epsilon$.*

We can also construct a Hamiltonian flow on M^7 whose generic trajectories are dense windings on the invariant three-dimensional Liouville tori and which has an arbitrarily large measure set on which it is partially hyperbolic.

COROLLARY 3.3. *Let $0 < \epsilon \leq 1$ be given. There exists a non-degenerately Liouville-integrable Hamiltonian flow ϕ_t^ϵ on the Poisson manifold $(M, \mathcal{P})$ such that conclusions (3)–(5) of Corollary 3.2 are true.*

COROLLARY 3.4. *Let N^k be a smooth manifold of dimension $k \geq 7$. Then, on N, there exists a smooth Poisson structure $\mathcal{Q}$, a smooth function $F \in C^\infty(N)$ and a Hamiltonian vector field $V = \mathcal{Q}\, dF$ with flow f_t such that:*

(1) *f_t is C^∞ Liouville integrable;*
(2) *f_t is C^∞ non-commutatively integrable;*
(3) *there is an open, dense subset of N on which the orbits of f_t are all periodic;*
(4) *there is a closed f_t-invariant set K and if μ is a positive, f_t-invariant Borel probability measure, then $\mu(K) > 0$ and on this set the flow f_t has non-zero Lyapunov exponents everywhere.*

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4. *An integrable, Hamiltonian vector field with positive measure entropy on an eight-dimensional symplectic manifold*

Let $(\Sigma^2, \eta, \mathbf{F})$ be as in the previous section.

4.1. *The manifold.* Let $\tilde{M} = T^*\mathbb{T}^1[p,\theta] \times \mathbb{T}^2[z,w] \times T^*\mathbb{T}^1[x,\phi] \times \Sigma^2[v]$ and let $D = \mathbb{Z}$ act on $\tilde{M}$ by

$$1 * (p,\theta,z,w,x,\phi,v) := (p,\theta,z,w,x+1,\phi,\mathbf{F}(v)) \tag{11}$$

for all $(p,\theta,z,w,x,\phi,v) \in \tilde{M}$. Let $M := D\backslash\tilde{M}$, so $M = \mathbb{R} \times \mathbb{T}^4 \times M_{\mathbf{F}}$. We naturally have the following commutative diagram:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\pi}} & \tilde{M}_{\mathbf{F}} \\ {\scriptstyle\Pi}\downarrow & & \downarrow{\scriptstyle\Pi_{\mathbf{F}}} \\ M & \xrightarrow{\pi} & M_{\mathbf{F}} \end{array} \tag{12}$$

where $\Pi_{\mathbf{F}}(x,v) = \Gamma(x,v)$, $\Pi(p,\theta,z,w,x,\phi,v) = D(p,\theta,z,w,x,\phi,v)$ and $\tilde{\pi}(p,\theta,z,w,x,\phi,v) = (x,v)$.

4.2. *The Hamiltonian flow.* Let

$$\tilde{\omega} := dp \wedge d\theta + dz \wedge dw + dx \wedge d\phi + \tilde{\pi}^*\eta, \tag{13}$$

be the natural symplectic form on $\tilde{M}$ and let $\widetilde{\mathcal{P}}$ be the associated non-degenerate Poisson tensor. Let $\varphi, F \in C^\infty(\mathbb{T}^1)$ and $f = F'$ and define the Hamiltonian $\tilde{H} = \tilde{H}_{\varphi,F}$:

$$\tilde{H}(p,\theta,z,w,x,\phi,v) := \varphi(z)p + f(\theta)\cos 2\pi\phi. \tag{14}$$

Then $\widetilde{\omega}$, $\widetilde{\mathcal{P}}$ and $\tilde{H}$ are D invariant C^∞ tensor fields on $\tilde{M}$ and induce C^∞ tensor fields ω, $\mathcal{P}$ and H on M. Let $\tilde{m}$ denote the measure induced by the volume form $\tilde{\Omega} = \tilde{\omega}^4$; $\tilde{\Omega}$ $(\tilde{m})$ is D-invariant and induces a smooth volume form Ω (measure m) on M.

We view φ as a function on M and let $C = \{\mathrm{m} \in M : \varphi(\mathrm{m}) = 0\}$.

THEOREM 4.1. *If C is nowhere dense and $\mathbf{F} \in \mathrm{Diff}_\eta(\Sigma^2)$, then the flow $\phi_t : M^8 \to M^8$ of the Hamiltonian vector field $X_H = \mathcal{P}\,dH$ on M satisfies:*

(1) *ϕ_t is C^∞ Liouville integrable;*
(2) *ϕ_t is C^∞ non-commutatively integrable;*
(3) *$h_{\mathrm{top}}(\phi_t) = h_{\mathrm{top}}(\mathbf{F})$;*
(4) *the set C is ϕ_t invariant; if C is non-empty and $\mathbf{F}$ is Bernoulli, then there is a ϕ_t-invariant Borel probability measure ν and a full-ν-measure subset $K \subset C$ such that ϕ_t has non-zero Lyapunov exponents everywhere in K.*

Let $L_{\varphi,F}$ denote the set of invariant Liouville tori of the integrable Hamiltonian vector field $X = \mathcal{P}\,dH_{\varphi,F}$; let $S_{\varphi,F} = M - L_{\varphi,F}$ denote the complement of $L_{\varphi,F}$ (the 'singular' set). For each $r > 0$, let $K_r := \{\mathrm{m} \in M : |p(\mathrm{m})| \le r\}$.

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COROLLARY 4.2. *Given* $0 < \epsilon \leq 1$, *there exists* φ, $F \in C^\infty(\mathbb{T}^1)$ *such that for all* $r > 0$

$$m(S_{\varphi,F} \cap K_r) \geq (1-\epsilon)m(K_r). \tag{15}$$

In particular, the measure of the singular set $m(S_{\varphi,F}) = \infty$.

COROLLARY 4.3. *There exists a non-degenerately Liouville-integrable Hamiltonian vector field* X *on* M *that satisfies the conclusions (1), (3) and (4) of Theorem 4.1. In addition, the statement of Corollary 4.2 is also true with a non-degenerately Liouville-integrable Hamiltonian vector field.*

5. *Proof of Theorem 2.1*

Notation in this section is as defined in the preamble to the statement of Theorem 2.1. Let $\widetilde{\mathcal{P}}_2$ denote the Poisson tensor on Σ^2 that is induced by the symplectic form η. We view $\widetilde{\mathcal{P}}_2$ as a Poisson tensor on $\tilde{Q}$ and $\tilde{M}$.

LEMMA 5.1. *The vector fields* $\partial/\partial x$, $\partial/\partial\theta$, *the Poisson tensor* $\widetilde{\mathcal{P}}_2 = \eta^{-1}$ *and the* 1*-form* dx *are invariant under the action of* D.

Proof. The map $\mathbf{F} : \Sigma^2 \to \Sigma^2$ preserves η by hypothesis, so $\widetilde{\mathcal{P}}_2$ is invariant under D; the vector fields $\partial/\partial x$, $\partial/\partial\theta$ are clearly D-invariant. □

LEMMA 5.2. $\widetilde{\mathcal{P}}$ *is a smooth* D*-invariant Poisson tensor;* $\tilde{\alpha}$ *is a* D*-invariant smooth* 1*-form.*

Proof. $\tilde{\alpha}$ is clearly smooth; since dx is D-invariant, $\tilde{\alpha}$ is D-invariant. $\widetilde{\mathcal{P}}$ is obviously both smooth and a Poisson structure. Let $\widetilde{\mathcal{P}}_1 = \partial/\partial x \wedge \partial/\partial\theta$; by the previous lemma this is D-invariant. Then $\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}_1 + \widetilde{\mathcal{P}}_2$ is the sum of D-invariant bivectors. □

LEMMA 5.3. *Let* $\tilde{\phi}_t : \tilde{Q} \to \tilde{Q}$ *be the flow of* $X_{\tilde{\alpha}} = \widetilde{\mathcal{P}}\,\tilde{\alpha}$. *Then*

$$\tilde{\phi}_t(z,\theta,x,v) = \begin{cases} (z, \theta + \varphi(z)t, x + \dfrac{1}{2\pi\varphi(z)} \sin 2\pi(\theta+\varphi(z)t), v) & \text{if } \varphi(z) \neq 0, \\ (z, \theta, x + t\cos 2\pi\theta, v) & \text{if } \varphi(z) = 0. \end{cases} \tag{16}$$

If $\varphi^{-1}(0) \subset \tilde{Q}$ *is nowhere dense, then the set of periodic points of* $\tilde{\phi}_t$ *is open and everywhere dense.*

Proof. The vector field $X_{\tilde{\alpha}}$ is

$$X_{\tilde{\alpha}} = \begin{cases} \dot{z} = 0, & \\ \dot{x} = \cos 2\pi\theta, & \dot{\theta} = \varphi(z), \\ \dot{v}^1 = 0, & \dot{v}^2 = 0. \end{cases} \tag{17}$$

The claim is now apparent. □

LEMMA 5.4. *Let* $\phi_t : Q \to Q$ *denote the flow induced by* $\tilde{\phi}_t$ *with infinitesimal generator* $X_\alpha = \mathcal{P}\alpha$. *Let* $Q = \mathbb{T}^2[z,\theta] \times M_{\mathbf{F}}$ *and set* $q = (z,\theta,\Gamma(x,v)) \in Q$. *Let* $\hat{\mathrm{C}} = \{q \in Q : \varphi(z(q)) = 0, \cos 2\pi\theta(q) \neq 0\}$.

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(i) *If* $q \in Q - \hat{\mathrm{C}}$, *and* $T := 1/\varphi(z)$, *then*

$$\phi_T(z, \theta, \Gamma(x, v)) = (z, \theta, \Gamma(x, v)). \tag{18}$$

(ii) *If* $q \in \hat{\mathrm{C}}$ *and* $T := 1/\cos 2\pi\theta(q)$, *then*

$$\phi_T(z, \theta, \Gamma(x, v)) = (z, \theta, \Gamma(x, \mathbf{F}(v))). \tag{19}$$

Proof. (i) From the previous lemma, $\tilde{\phi}_T(z, \theta, x, v) = (z, \theta, x, v)$ for $T = 1/\varphi(z)$.

(ii) Let Y denote the vector field on Q induced by $\partial/\partial x$ on $\tilde{Q}$. By (17) from the previous lemma, $X_\alpha(q) = \cos 2\pi\theta(q) Y(q)$ for all $q \in \hat{\mathrm{C}}$. The time-1 map of Y is the map $(z, \theta, \Gamma(x, v)) \to (z, \theta, \Gamma(x, \mathbf{F}(v)))$ by construction. This verifies claim (i). □

Let us remark that if ξ, η are smooth, closed-with-respect-to-$\widetilde{\mathcal{P}}$ 1-forms such that $\xi - \eta \in \ker \widetilde{\mathcal{P}}$ then the locally Hamiltonian vector fields X_ξ and X_η are equal. In particular, $\tilde{\alpha} \equiv d\tilde{\beta} \bmod \ker \widetilde{\mathcal{P}}$ where

$$\tilde{\beta} = \varphi(z)x - \frac{1}{2\pi} \sin 2\pi\theta. \tag{20}$$

By the skew-symmetry of $\widetilde{\mathcal{P}}$, $\tilde{\beta}$ is a first integral of $X_{\tilde{\alpha}} = \widetilde{\mathcal{P}}\tilde{\alpha} = \widetilde{\mathcal{P}}\, d\tilde{\beta}$. For all points $\tilde{q} \in \tilde{Q}$ such that $\varphi(z(\tilde{q})) \neq 0$, let

$$\tilde{\chi}(\tilde{q}) := x(\tilde{q}) - \frac{\sin 2\pi\theta(\tilde{q})}{2\pi\varphi(z(\tilde{q}))} = \tilde{\beta}(\tilde{q})/\varphi(z(\tilde{q})). \tag{21}$$

It is clear that $\tilde{\chi}$ is smooth on its domain and $X_{\tilde{\alpha}}(\tilde{\chi}) = 0$ since both $\tilde{\beta}$ and z are first integrals.

LEMMA 5.5. *Let*

$$\tilde{f}_1(z, \theta, x, v) := \exp\left(-\frac{1}{\varphi(z)^2}\right) \cos 2\pi\, \tilde{\chi}(z, \theta, x, v). \tag{22}$$

Then $f_1 \in C^\infty(\tilde{Q})^D$ *and* f_1 *is a first integral of* $X_{\tilde{\alpha}}$.

Proof. $\tilde{f}_1$ is clearly smooth and D-invariant. Since $\tilde{f}_1$ is a function of first integrals of $X_{\tilde{\alpha}}$ it is a first integral. □

Let $g \in C^\infty(M_{\mathbf{F}})$, let $\tilde{g} = \Pi_{\mathbf{F}}^* g \in C^\infty(\tilde{M}_{\mathbf{F}})$ denote the Γ-invariant function induced by g and let $\tilde{G} = \tilde{\pi}^* \tilde{g} \in C^\infty(\tilde{Q})$ denote the D-invariant smooth function induced by $\tilde{g}$. Let

$$\tilde{f}_g(z, \theta, x, v) := \exp\left(-\frac{1}{\varphi(z)^2}\right) \tilde{G}(z, \theta, \tilde{\chi}(z, \theta, x, v), v). \tag{23}$$

LEMMA 5.6. *For each* $g \in C^\infty(M_{\mathbf{F}})$, $\tilde{f}_g \in C^\infty(\tilde{Q})^D$ *and* $\tilde{f}_g$ *is a first integral of* $X_{\tilde{\alpha}}$.

Proof. $\tilde{f}_g$ is obviously D-invariant; it is smooth because $\tilde{G}$ is a smooth D-invariant function and all derivatives of $\tilde{G}$ are therefore bounded. Equation (17) reveals that $\tilde{f}_g$ is a first integral of $X_{\tilde{\alpha}}$. □

Let $V = \ker dx \subset TM_{\mathbf{F}}$ denote the vertical subbundle that is annihilated by the differential of the submersion $q \in M_{\mathbf{F}} \to x(q) \in \mathbb{T}^1$. For each $g \in C^\infty(M_{\mathbf{F}})$, let $V_g^0 := \{q \in M_{\mathbf{F}} : dg_q \mid V_q = 0\}$ denote the set of points where dg annihilates V. It is clear that there exists $g \in C^\infty(M_{\mathbf{F}})$ such that the set V_g^0 is nowhere dense in Q; let NV denote the set of all such functions. Let $g \in NV$ be one such function and let $\tilde{f}_g : \tilde{Q}^5 \to \mathbb{R}$ be the function constructed in Lemma 5.6.

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LEMMA 5.7. *Let* $\tilde{F} = (\tilde{f_1}, \tilde{f_g}, z) : \tilde{Q}^5 \to \mathbb{R}^3$ *and let* **dvol** *denote the canonical volume form on* $\mathbb{R}^3$. *For all* $q \in \tilde{Q}$ *such that* $\varphi(z(q)) \neq 0$ *we have*

$$\tilde{F}^*\mathbf{dvol} = -\exp\left(-\frac{2}{\varphi(z)^2}\right) 2\pi \sin\tilde{\chi}\, d\tilde{\chi} \wedge (\tilde{g}_{v^1}\, dv^1 + \tilde{g}_{v^2}\, dv^2) \wedge dz. \tag{24}$$

In particular, the set $\{q \in \tilde{Q} : \tilde{F}^*\mathbf{dvol} = 0\}$ *is nowhere dense if* $\varphi^{-1}(0)$ *is nowhere dense.*

Proof. By hypothesis, $\tilde{g}(z, \theta, x, v) = g(x, v)$. The proof is now a calculation. □

THEOREM 5.8. (Theorem 2.1(1)) *The flow* $\phi_t : Q \to Q$ *is Liouville integrable.*

Proof. The rank of $\mathcal{P}$ is everywhere 4 and the Casimirs of $\mathcal{P}$ are generated by z, so we require two additional Poisson commuting first integrals of ϕ_t that are functionally independent on an open, dense subset of Q. Let $\tilde{f_1}$ and $\tilde{f_g}$ be the independent first integrals constructed in Lemmas 5.5 and 5.6. By Lemma 5.7, $(\tilde{f_1}, \tilde{f_g}, z)$ are independent on an open dense subset of Q. Clearly, $\widetilde{\mathcal{P}}(d\tilde{f_1}, d\tilde{f_g}) \equiv 0$. By Lemmas 5.5 and 5.6 these functions are D-invariant and smooth, so we conclude that ϕ_t is Liouville integrable on Q with the induced smooth first integrals. □

THEOREM 5.9. (Theorem 2.1(2)) *The flow* $\phi_t : Q \to Q$ *is non-commutatively integrable.*

Proof. It is clear from Lemma 5.7 that there exist $g_2, g_3 \in NV$ that are functionally independent on an open, dense subset of $M_{\mathbf{F}}$. It is apparent that $\tilde{f_1}, \tilde{f_{g_2}}, \tilde{f_{g_3}}$ are functionally independent on an open, dense subset of $\tilde{Q}$ and are first integrals of $\tilde{\phi_t}$. □

THEOREM 5.10. (Theorem 2.1(3)) *The set of periodic orbits of* ϕ_t *is an open, dense subset of* Q.

Proof. See Lemma 5.4. □

THEOREM 5.11. (Theorem 2.1(4)) *Let* $\mathbf{F} \in \mathrm{Diff}_\eta(\Sigma)$ *and* $C := \varphi^{-1}(0) \subset Q$. *Then* $h_m(\phi_1) = (2m(C)/\pi)h_{m_\eta}(\mathbf{F})$.

Proof. We assume that $1 > m(C) > 0$, otherwise the result is trivial. We present the proof in steps.

(1) The 2-form $\omega = d\theta \wedge dx + \eta$ restricted to each $\mathcal{P}$-symplectic leaf $\{z = \text{constant}\}$ is the symplectic form induced by $\mathcal{P}$, so $\omega \wedge \omega$ is ϕ_t-invariant. Since dz is ϕ_t-invariant, ϕ_t preserves the volume form $dz \wedge d\theta \wedge dx \wedge \eta$. Hence, ϕ_t is an automorphism of the measure space $(Q, \mathcal{B}, m)$, where $\mathcal{B}$ are the Borel sets in Q.

(2) Recall that $C := \{q \in Q : \varphi(z(q)) = 0\}$. For $B \in \{C, Q - C\}$ let $m_B(A) := (1/m(B))m(B \cap A)$ for any measurable set $A \subset Q$. m_B is a Borel probability measure and since both C and $Q - C$ are ϕ_t-invariant, m_B is a ϕ_t-invariant Borel probability measure and $m = m(C)m_C + m(Q - C)m_{Q-C}$. By Lemma 5.4, $h_{m_{Q-C}}(\varphi_1) = 0$ and so by the linearity of h_m (see [**11**]):

$$h_m(\varphi_1) = m(C)h_{m_C}(\varphi_1). \tag{25}$$

(3) Let $\mathbf{p} : Q \to \mathbb{T}^2$ denote the projection map $\mathbf{p}(q) = (z(q), \theta(q))$. We will denote $\mathbf{p}|C$ by the same map. For each $\Theta \in \mathbb{T}^2$, denote by m_Θ the measure induced

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on $\mathbf{p}^{-1}(\Theta) = \{\Theta\} \times M_{\mathbf{F}}$ by the form $\pi^*(dx \wedge \eta)$. It is apparent that

$$m_C = \int_{\Theta \in C_z \times \mathbb{T}^1} m_\Theta \, dz \, d\theta, \tag{26}$$

where $C_z = z(C)$.

(4) Let Y denote the vector field on Q induced by $\partial/\partial x$ and let $\phi_t^{\mathbf{F}}$ denote the flow of Y: it is clear that $\phi_1^{\mathbf{F}}(z, \theta, \Gamma(x, v)) = (z, \theta, \Gamma(x, \mathbf{F}(v)))$. From Lemma 5.4, for any $\Theta = (z, \theta) \in \mathbf{p}(\hat{\mathrm{C}})$, we have that $\mathbf{p}^{-1}(\Theta)$ is ϕ_t-invariant and that $\phi_{1/\cos 2\pi\theta}|\mathbf{p}^{-1}(\Theta) = \phi_1^{\mathbf{F}}|\mathbf{p}^{-1}(\Theta)$. Therefore,

$$h_{m_\Theta}(\phi_1) = |\cos 2\pi\theta| h_{m_\Theta}(\phi_1^{\mathbf{F}}) = |\cos 2\pi\theta| h_{m_\eta}(\mathbf{F}). \tag{27}$$

Equations (26) and (27) along with the ergodic decomposition theorem [**17**] imply that

$$\begin{aligned} h_{m_C}(\phi_1) &= \frac{1}{m(C)} \int_{\Theta \in C_z \times \mathbb{T}^1} h_{m_\Theta}(\phi_1) \, d\theta \, dz \\ &= h_{m_\eta}(\mathbf{F}) \int_{\theta \in \mathbb{T}^1} |\cos 2\pi\theta| \, d\theta, \\ &= \frac{2}{\pi} h_{m_\eta}(\mathbf{F}). \end{aligned} \tag{28}$$

Equations (25) and (28) together imply that

$$h_m(\phi_1) = \frac{2m(C)}{\pi} h_{m_\eta}(\mathbf{F}). \qquad \square$$

THEOREM 5.12. (Theorem 2.1(5)) *Assume that* $\mathbf{F} \in \mathrm{Diff}_\eta(\Sigma)$ *is a Bernoulli diffeomorphism. Then on the set* $\hat{\mathrm{C}} \subset Q$, *the Lyapunov exponents of* ϕ_t *with respect to the measure* m *are nowhere vanishing.*

Proof. This is clear in light of the proof of the previous theorem. $\square$

Proof of Corollary 2.2. The coordinate transformation $(z, \theta, x, v) \to (z, \theta, \tilde{\chi}, v)$ is a Poisson map on $\tilde{Q}$. The equations of motion of the Hamiltonian vector field $X_{\tilde{f}_g}$ are

$$X_{\tilde{f}_g} = \begin{cases} \dot{z} = 0, \\ \dot{\tilde{\chi}} = 0, & \dot{\theta} = \tilde{G}_x, \\ \dot{v}^1 = -\tilde{G}_{v^2}, & \dot{v}^2 = \tilde{G}_{v^1}, \end{cases} \tag{29}$$

and after a second Poisson transformation

$$X_{\tilde{f}_g} = \begin{cases} \dot{z} = 0, \\ \dot{\tilde{\chi}} = 0, & \dot{\theta} = \tilde{G}_x, \\ \dot{J} = 0, & \dot{\gamma} = \tilde{G}_J, \end{cases} \tag{30}$$

where $\tilde{G} = \tilde{G}(z, \tilde{\chi}, J)$. It is clear that a $\tilde{G}$ exists such that the vector field $X_{\tilde{f}_g}$ is non-degenerate. $\square$

Proof of Corollary 2.4. (1) Let $0 < \epsilon \leq 1$ be given. Then there exists a Cantor set $C_z \subset \mathbb{T}^1$ such that $m_{dz}(C_z) \geq 1 - \epsilon$ and a smooth function $\varphi \in C^\infty(\mathbb{T}^1)$ such that $\varphi|C_z \equiv 0$ and $\varphi(z) > 0$ for $z \notin C_z$. From Lemma 5.4, the flow of X_{α_φ} has a set of invariant tori of m-measure $1 - m_{dz}(C_z) \leq \epsilon$. (2) and (3) follow from (1) and Theorem 2.1(4). $\square$

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5.1. *Proof of Corollary 2.6.* We first observe the following.

LEMMA 5.13. *Let* $\hat{\mathrm{Q}} = (0,1)[z] \times \mathbb{T}^1[\theta] \times M_{\mathbf{F}}$. *Then there exists a* C^∞ *embedding* $j : \hat{\mathrm{Q}} \to \mathbb{R}^5$.

Proof. By [**9**], there exists an embedding of the smooth, orientable 3-manifold $M_{\mathbf{F}}$ into $\mathbb{R}^5$ with a trivial normal bundle. That is, the trivial bundle $\mathbb{R}^2 \times M_{\mathbf{F}}$ smoothly embeds into $\mathbb{R}^5$. Hence $\hat{\mathrm{Q}}$ smoothly embeds into $\mathbb{R}^5$. □

Proof of Corollary 2.6. Let us abuse notation and consider the vector field X as a vector field on $\hat{\mathrm{Q}}$; clearly everything that has been proven until now about the vector field X on Q is equally true for X on $\hat{\mathrm{Q}}$. Let $\gamma(z)$ be a non-negative smooth function that vanishes to all orders at $z = 0, 1$, is strictly positive elsewhere and has unit mass $\int_0^1 \gamma(z)\,dz = 1$. Let $\mathcal{R} := \gamma\mathcal{P}$, $\xi := \gamma\alpha$, $Y = \mathcal{R}\xi = \gamma^2 X$ and $\Omega = \gamma(z)\,dz \wedge d\theta \wedge dx \wedge dv^1 \wedge dv^2$. The 1-form ξ is closed-with-respect to $\mathcal{R}$, so Y is locally Hamiltonian. The volume form Ω is clearly X-invariant and induces a smooth probability measure on $\hat{\mathrm{Q}}$. Because Y is a non-zero time change of X, everything that has been proven for X is true of Y. The embedding $j : \hat{\mathrm{Q}} \to \mathbb{R}^5$ pushes the Poisson structure $\mathcal{R}$, the 1-form ξ, Y and Ω forward onto $j(\hat{\mathrm{Q}})$. Because each of these smooth objects vanishes to all orders on the boundary of $j(\hat{\mathrm{Q}})$, they can be smoothly extended by zero to all of $\mathbb{R}^5$.

Consider the action of the homotheties $h_{a,b} : x \in \mathbb{R}^5 \to a(x - b)$ for some $0 \neq a \in \mathbb{R}$ and $b \in \mathbb{R}^5$. These diffeomorphisms act on $j(\hat{\mathrm{Q}})$; let $\hat{\mathrm{Q}}_{a,b} = h_{a,b} \circ j(\hat{\mathrm{Q}})$ and let $\mathcal{R}_{a,b} := (h_{a,b} \circ j)_* \mathcal{R}$, $\xi_{a,b} := a^2 (h_{a,b} \circ j)_* \xi$, $Y_{a,b} := \mathcal{R}_{a,b}\xi_{a,b}$ and $\Omega_{a,b} := a^{10}(h_{a,b} \circ j)_*\Omega$ denote the rescaled push forwards, which are smoothly extended by zero on the complement of $\hat{\mathrm{Q}}_{a,b}$.

It is clear that given any open subset $U \subset \mathbb{R}^5$, there exists a countable sequence of homotheties $\{h_{a_n,b_n}\}_{n=1}^\infty$ such that $V := \bigcup_{n=1}^\infty \hat{\mathrm{Q}}_{a_n,b_n}$ is dense in U and the sets $\hat{\mathrm{Q}}_{a_n,b_n}$ are pairwise disjoint. In addition, the tensors

$$\mathcal{S} := \sum_{n=1}^\infty \mathcal{R}_{a_n,b_n}, \quad \eta := \sum_{n=1}^\infty \xi_{a_n,b_n},$$
$$Z := \sum_{n=1}^\infty Y_{a_n,b_n}, \quad \Xi := \sum_{n=1}^\infty \Omega_{a_n,b_n}$$

are smooth and have support $\overline{U}$ (we needed to rescale the pushed-forward forms to ensure smoothness). They also satisfy the property that $\mathcal{S}$ is a Poisson structure on $\mathbb{R}^5$, η is closed with respect to $\mathcal{S}$, $Z = \mathcal{S}\eta$ and Ξ induces a measure that is positive on U. In particular, if U is an open cube in $\mathbb{R}^5$ then the vector field Z satisfies the conclusions of the corollary.

By the inclusion of $\mathbb{R}^5 \to \mathbb{R}^k = \mathbb{R}^5 \times \mathbb{R}^{k-5}$ for all $k \geq 5$, it is clear that the statement of the corollary is also true for all open subsets of $\mathbb{R}^k$, $k \geq 5$.

Now let M^k be a manifold of dimension $k \geq 5$ and let C_n be a countable family of pairwise disjoint subsets of M such that $\bigcup_{n=1}^\infty C_n$ is dense in M and each C_n is diffeomorphic to an open cube $U_{r_n} = (-r_n, r_n)^k$ under a coordinate mapping $\mathbf{x}_n : C_n \to U_{r_n}$. Repeating the arguments of the previous two paragraphs with the Poisson structure/1-form/vector field obtained there shows that there exists a smooth Poisson

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structure $\mathcal{Q}$ and a smooth, closed-with-respect-to $\mathcal{Q}$ 1-form ϑ on M such that the locally Hamiltonian vector field $V = \mathcal{Q}\vartheta$ satisfies conclusions (1)–(3) of the corollary. The measure on M obtained from the arguments in the previous two paragraphs is also clearly positive, so conclusion (4) is satisfied with respect to this measure on M; hence, conclusion (4) is satisfied with respect to any positive, invariant measure. □

Remark. This corollary is not very surprising in some sense. The Poisson structure is a much more flexible object than a symplectic structure. In addition, we do not constrain the vector field V or the Poisson structure $\mathcal{Q}$ to have any global properties. It is a much more difficult problem to determine, given a non-trivial Poisson structure on a manifold M^k, $k \geq 4$, whose rank is generically four or more, if there exists a vector field V with the properties described in the corollary that is also (locally) Hamiltonian with respect to the *given* Poisson structure. In the next sections we pursue this question further for some cases. Meanwhile, let us note the following.

COROLLARY 5.14. *Let (N^{2n}, ω) be a smooth, oriented manifold and $\mathbf{F} \in \mathrm{Diff}_\omega(N^n)$ be a volume-preserving diffeomorphism and let $\mathcal{P}_2$ be an $\mathbf{F}$-invariant Poisson structure. Let $Q^{2n+3} = \mathbb{T}^2[z, \theta] \times M_{\mathbf{F}}$ where $M_{\mathbf{F}}$ is $\mathbf{F}$'s mapping cylinder. Let*

$$\mathcal{P} := \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \theta} + \mathcal{P}_2. \tag{31}$$

Then there exists a $\mathcal{P}$-locally Hamiltonian, volume-preserving vector field X on Q^{2n+3} that satisfies conclusions (1)–(5) of Theorem 2.1.

Remark. Theorem 2.1 and Corollary 5.14 are remarkable because the integrability of the vector field X is really independent of the mapping $\mathbf{F} : N \to N$ and the invariant Poisson structure $\mathcal{P}_2$ (indeed, it might very well be zero). The explanation for this is straightforward: the vector field X in (17) is already locally Hamiltonian with respect to the Poisson structure $\mathcal{P}_1 = \partial/\partial x \wedge \partial/\partial\theta$. Indeed $X = \mathcal{P}_1 \alpha$. Because the first integral $\tilde{f}_1$ is generically proper when restricted leaf-wise to the symplectic leaves of $\widetilde{\mathcal{P}}_1$ on $\tilde{Q}$, this means that the intersection of a leaf with a generic level set of $\tilde{f}_1$ is a union of circles. Hence, the trajectories of X are generically circles. When the leafwise restriction of $\tilde{f}_1$ ceases to be proper, however, the trajectories of $\tilde{X}$ are either copies of $\mathbb{R}$ or points and so the trajectories of X have the potential to exhibit wild behaviour—as they do when the diffeomorphism $\mathbf{F}$ exhibits complicated behaviour.

6. *Proof of Theorem 3.1*

The notation of this section follows the notation in the preamble to the statement of Theorem 3.1.

LEMMA 6.1. *The vector field $X_{\tilde{H}} = \widetilde{\mathcal{P}}\, d\tilde{H}$ is*

$$X_{\tilde{H}} = \begin{cases} \dot{z} = 0, & \\ \dot{x} = p_x, & \dot{p}_x = -\varphi(z) p_w, \\ \dot{w} = p_w, & \dot{p}_w = \varphi(z) p_x, \\ \dot{v}^1 = 0, & \dot{v}^2 = 0, \end{cases} \tag{32}$$

on $\tilde{M}$.

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LEMMA 6.2. *Let*

$$\tilde{\chi} := \frac{p_w}{\varphi(z)} + w, \quad \tilde{\Xi} := \frac{p_x}{\varphi(z)} - x, \tag{33}$$

for all $(z, p, u, v) \in \tilde{M}$ *such that* $\varphi(z) \neq 0$. *Then* $\tilde{\chi}$ *and* $\tilde{\Xi}$ *are smooth functions on their domains, are mod* $\mathbb{Z}$ *invariant under the action of* D *and are first integrals of* $X_{\tilde{H}}$.

Proof. Let $\tilde{\alpha} = p_x + w\varphi(z)$ and $\tilde{\beta} = p_w - x\varphi(z)$. By Lemma 6.1, $\dot{\tilde{\alpha}} = \dot{\tilde{\beta}} = 0$. Then $\tilde{\chi} = \tilde{\alpha}/\varphi(z)$ ($\tilde{\Xi} = \tilde{\beta}/\varphi(z)$) is a function of the first integrals of $X_{\tilde{H}}$, so it is a first integral, too. The other claims are clear. □

LEMMA 6.3. *Let*

$$\tilde{f_3}(z, p, u, v) := \exp\left(-\frac{1}{\varphi(z)^2}\right) \sin 2\pi\tilde{\chi}(z, p, u, v), \tag{34}$$

$$\tilde{f_5}(z, p, u, v) := \exp\left(-\frac{1}{\varphi(z)^2}\right) \sin 2\pi\tilde{\Xi}(z, p, u, v), \tag{35}$$

for all $(z, p, u, v) \in \tilde{M}$ *such that* $\varphi(z) \neq 0$ *and zero otherwise. Then* $\tilde{f_3}, \tilde{f_5} \in C^\infty(\tilde{M})^D$ *are first integrals of* $X_{\tilde{H}}$.

LEMMA 6.4. *If* $g \in C^\infty(M_{\mathbf{F}})$ *and* $\tilde{G} = (\Pi \circ \pi)^* g \in C^\infty(\tilde{M})^D$ *is the induced* D*-invariant function on* $\tilde{M}$, *then*

$$\tilde{f_g}(z, p, u, v) := \exp\left(-\frac{1}{\varphi(z)^2}\right) \tilde{G}(z, p, \tilde{\chi}(z, p, u, v), v) \tag{36}$$

satisfies $\tilde{f_g} \in C^\infty(\tilde{M})^D$ *and* $\tilde{f_g}$ *is a first integral of* $X_{\tilde{H}}$.

Proof. Apply Lemma 6.2. □

LEMMA 6.5. *Let* $\tilde{F} = (\tilde{f_1} = z, \tilde{f_2} = \tilde{H}, \tilde{f_3}, \tilde{f_4}) : \tilde{M} \to \mathbb{R}^4$ *where* $\tilde{f_4} = \tilde{f_g}$ *for some* $g \in C^\infty(M_{\mathbf{F}})$. *If* **dvol** *denotes the canonical volume form on* $\mathbb{R}^4$, *then for all* $(z, p, u, v) \in \tilde{M}$ *such that* $\varphi(z) \neq 0$:

$$\tilde{F}^*\mathbf{dvol} = \exp\left(-\frac{1}{\varphi(z)^2}\right) [2\pi \cos 2\pi\tilde{\chi} + \tilde{G}_x]\, d\tilde{H} \wedge dz \wedge d\tilde{\chi} \wedge [\tilde{G}_{v^1}\, dv^1 + \tilde{G}_{v^2}\, dv^2]. \tag{37}$$

In particular, there exists $g \in C^\infty(\tilde{M})^D$ *such that* $\operatorname{rank} d\tilde{F}_q = 4$ *for an open, dense subset of* $q \in \tilde{M}$.

Proof. The formula is a calculation. The second claim follows by the same arguments as in the proof of Lemma 5.7. □

LEMMA 6.6.

$$X_{\tilde{f_i}}(\tilde{f_j}) \equiv 0 \tag{38}$$

for all $i, j = 2, 3, 4$.

Proof. Only $X_{\tilde{f_3}}(\tilde{f_4})$ needs to be computed. However, $\tilde{f_3} = \tilde{f_3}(z, \tilde{\chi}, v)$ and $\tilde{f_4} = \tilde{f_4}(z, \tilde{\chi})$, so these functions Poisson commute. □

THEOREM 6.7. (Theorem 3.1(1)) *The flow* $\phi_t : M \to M$ *of the vector field* X_H *is Liouville integrable.*

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Proof. It only remains to prove that the connected components of the regular levels of the first-integral map $F : M \to \mathbb{R}^4$ are compact; but H is a proper map, therefore so is F. □

THEOREM 6.8. (Theorem 3.1(2)) *The flow $\phi_t : M \to M$ of the vector field X_H is non-commutatively integrable.*

Proof. The functions f_i, $i = 1, \ldots, 5$, are all first integrals of X_H and X_{f_3} commutes with X_H. Since the common, regular levels of these first integrals are compact, the vector field X_H is non-commutatively integrable with two-dimensional invariant tori. In fact, it is clear that there is a further, sixth, independent first integral of X_H obtained in a manner analogous to f_4. □

LEMMA 6.9. (Theorem 3.1(3)) *The flow*

$$\tilde{\phi}_t(z, p, u, v) = \begin{cases} (z, e^{it\varphi(z)}p, u - i\varphi(z)^{-1}(e^{it\varphi(z)} - 1)p, v), & \text{if } \varphi(z) \neq 0, \\ (z, p, u + tp, v), & \text{if } \varphi(z) = 0. \end{cases} \tag{39}$$

If $\{z \in \mathbb{T}^1 : \varphi(z) = 0\}$ is nowhere dense, then the periodic points of $\tilde{\phi}_t$ (ϕ_t) are open and dense.

LEMMA 6.10. *Let $\phi_t : M \to M$ be the flow with infinitesimal generator $X_H = \mathcal{P}\, dH$ that is induced by $\tilde{\phi}_t$. Let $M = \mathbb{T}^2[z, w] \times \mathbb{C}[p] \times M_{\mathbf{F}}$ and let $q = (z, w, p, \Gamma(x, v)) \in M$. Let $\hat{\mathrm{C}} = \{q \in M : \varphi(z(q)) = 0, p_x \neq 0\}$.*

(i) *If $q \in Q - \hat{\mathrm{C}}$ and $T := 1/\varphi(z)$, then*

$$\phi_T(z, w, p, \Gamma(x, v)) = (z, w, p, \Gamma(x, v)). \tag{40}$$

(ii) *If $q \in \hat{\mathrm{C}}$ and $T := 1/p_x$, then*

$$\phi_T(z, w, p, \Gamma(x, v)) = \left(z, w + \frac{p_w}{p_x}, p, \Gamma(x, \mathbf{F}(v))\right). \tag{41}$$

The remaining claims in Theorem 3.1 and Corollaries 3.2, 3.3 and 3.4 are now proven in precisely the same way as in the previous section.

7. *Proof of Theorem 4.1*

The proofs of these statements are virtually identical to the previous sections and are thus omitted.

8. *Conclusions*

In the Bolsinov–Taĭmanov example of an integrable geodesic flow on T^*M_A, where M_A is the suspension manifold of the hyperbolic toral automorphism $A : \mathbb{T}^2 \to \mathbb{T}^2$, the Liouville tori have full measure and the positive topological entropy arises because the geodesic flow along the annihilator of the vertical tangent bundle $VM_A^{\perp}$ is a time-change of the lift of the suspension vector field to T^*M_A. On the covering $T^*\tilde{M}_A = \mathbb{R}[x] \times \mathbb{T}^2[y] \times \mathbb{R}[p_x] \times \mathbb{R}^2[p_y]$, the geodesic vector field can be written as

$$X_H = \begin{cases} \dot{x} = p_x, & \dot{p}_x = \langle Be^{-xB}p_y, e^{-xB}p_y \rangle, \\ \dot{y} = e^{-xB'}e^{-xB}p_y, & \dot{p_y} = 0, \end{cases} \tag{42}$$

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where $e^B = A'$ and $2H = p_x^2 + \langle e^{-xB} p_y, e^{-xB} p_y \rangle$. On this covering space, it is apparent that in the $x - p_x$ coordinate plane the level set $S_c = \{H = \frac{1}{2}, p_y = c\}$ is compact provided that $c|E_i \neq 0$ for each of the eigenspaces E_i of A. The motion of the geodesic flow in the $x - p_x$ plane is therefore periodic for these values of the first integrals, but as $c_i \to 0$ for either $i = 1$ or $i = 2$, $\operatorname{diam} S_c \to \infty$. When $c = 0$, $X_H = p_x(\partial/\partial x)$; on T^*M_A, the Poincaré return map of X_H has an invariant subset on which it is conjugate to $A : \mathbb{T}^2 \to \mathbb{T}^2$. This generates the positive topological entropy, while it is clear that for any smooth measure, the measure of the singular set is zero and so the measure-theoretic entropy is zero with respect to any invariant smooth probability measure. The present paper utilizes this basic mechanism to construct dynamical systems which are both integrable and have very large-in-measure singular sets with similar behaviour. Unfortunately, it does not appear possible to construct integrable geodesic flows on T^*M_A with a naïve application of the ideas in the present paper.

This paper, therefore, raises some interesting questions: is it possible to replicate the results of Theorem 3.1 on a compact symplectic manifold (or, more modestly, with a proper Hamiltonian)? In the proof of Theorem 4.1, the non-compactness of the isoenergy surfaces seems essential. It would also be very interesting to know if the mixture of completely integrable and completely non-integrable behaviour that is exhibited here also can be exhibited by geodesic flows.

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