J. Plasma Physics (2001), *vol.* 65, *part* 1, *pp.* 73–77. Printed in the United Kingdom

© 2001 Cambridge University Press

Theory of nonlinear plasma surface waves

O. M. GRADOV[1] and L. STENFLO[2]

[1] N. S. Kurnakov Institute, Russian Academy of Sciences, 117907 Moscow, Russia
[2] Department of Plasma Physics, Umeå University, S-90187 Umeå, Sweden

(Received 30 March 2000)

Abstract. By means of the cold-electron fluid equations, we deduce a new strongly nonlinear surface solitary-wave solution for a semi-infinite plasma.

Many previous authors (e.g. Yu and Zhelyazkov 1978; Stenflo et al. 1983) have shown that finite-amplitude surface waves can propagate in the form of solitary waves on the interface between a cold-electron plasma and vacuum. In the limit of electrostatic waves on a semi-infinite plasma ($z > 0$) bounded by vacuum ($z < 0$), one can for example deduce simple expressions for the electrostatic potential ϕ, namely $\phi_{z>0} = \phi_0 \exp(ik_x x - |k_x| z - i\omega t)$ and $\phi_{z<0} = \phi_0 \exp(ik_x x + |k_x| z - i\omega t)$, with frequency $\omega = \omega_{p0}/\sqrt{2}$ and with amplitude (Gradov and Stenflo 1982)

$$\phi_0 = \phi_0(x_0)\,\mathrm{sech}\,[\beta\phi_0(x_0)\,(x - x_0)], \tag{1}$$

where $\beta = 4qk_x^3/m\omega^2$, q/m is the electron charge-to-mass ratio, and ω_{p0} is the electron plasma frequency. Depending on the specific boundary conditions, it is possible to find a variety of other solutions (see e.g. Brodin and Lundberg 1994; Azarenkov et al. 1995; Bychkov and Stenflo 1998; Lee and Lim 1998; Grimalskii and Rapoport 1998), which are also partly covered in recent review papers (Vladimirov et al. 1994; Stenflo 1996; Azarenkov and Ostrikov 1999). Similar solutions can of course appear also in quite different contexts (see e.g. Efremidis et al. 2000).

In this paper, we shall focus our attention on some more exotic solutions that do not appear in the almost-linear limit. One such solution was previously found for a uniform plasma cylinder with radius R, where it turned out that for sufficiently large amplitudes ϕ_m, where ϕ_m is the electrostatic potential maximum at the cylinder surface, the frequency could be estimated as (Stenflo and Yu 1989)

$$\omega \approx 2^{3/4}(q\phi_m/mR^2)^{1/2}. \tag{2}$$

The term 'oscillons' was introduced to represent these strongly nonlinear electrostatic oscillations at the plasma boundary. Similar oscillating patterns were investigated further in many other papers (see e.g. Chakraborty 1991; Stenflo and Yu 1998). One peculiar exact solution with particular initial values (Stenflo 1996) for a plasma cylinder surrounded by a dielectric of constant permittivity ε_d is a strongly nonlinear wave with density $n_e(t) = n_e(0)/(1 + \sin\omega t)$, where

$$\omega = \frac{\omega_{p0}}{(1 + \frac{1}{2}\varepsilon_d)^{1/2}}, \tag{3}$$

https://doi.org/10.1017/S0022377801008996 Published online by Cambridge University Press

which obviously differs in a remarkable way from the well-known linear solution $\omega = \omega_{p0}/(1+\varepsilon_d)^{1/2}$.

New one-dimensional nonlinear solutions for cold bounded plasmas have recently been found (Stenflo and Gradov 1998; Gradov 1999). These solutions do not exist in the linear limit. Below, we shall extend our findings to the two-dimensional case, and show that solitary-wave solutions then can be deduced.

Let us consider a cold-electron plasma with a background of stationary ions with density $n_{0i}(z)$, where $n_{0i}(z>0)$ is constant $(=N_{0i})$, and $n_{0i}(z<0)$ is zero. Limiting ourselves to electrostatic perturbations in the x and z directions (i.e. $\partial_y \equiv 0$), we write the electron density n_e as $N_0 + n(x,z,t)$, where $N_0 = N_{0i} = N_{0e}$ for $z>0$, and zero for $z<0$. We then use the electron continuity equation

$$\partial_t n + \nabla\cdot(N_0\mathbf{v} + n\mathbf{v}) = 0, \tag{4}$$

the electron momentum equation

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{q}{m}\nabla\phi, \tag{5}$$

and Poisson's equation

$$\nabla^2\phi = -\frac{q}{\varepsilon_0}n, \tag{6}$$

where $\mathbf{v}$ is the electron fluid velocity and ϕ is the electric field potential. Considering a plasma in the absence of constant drifts ($\mathbf{v}_0 = 0$) we then write $\mathbf{v}$ as $\Sigma_{l=-\infty}^{\infty}\mathbf{v}_l(x,z,\omega)\exp(-il\omega t)$, where $\mathbf{v}_{\pm 1}$ represent the linear terms. Similar expressions are used for n and ϕ. Equation (4) can thus be split into the three equations

$$-i\omega n_1 + \nabla\cdot(N_0\mathbf{v}_1 + n_0\mathbf{v}_1 + n_1^*\mathbf{v}_2 + n_2\mathbf{v}_1^*) = 0, \tag{7a}$$

$$-2i\omega n_2 + \nabla\cdot(N_0\mathbf{v}_2 + n_0\mathbf{v}_2 + n_1\mathbf{v}_1) = 0, \tag{7b}$$

and

$$\nabla\cdot(n_1\mathbf{v}_1^* + n_1^*\mathbf{v}_1) = 0, \tag{7c}$$

whereas (2) and (3) yield

$$-i\omega\mathbf{v}_1 + \mathbf{v}_2\cdot\nabla\mathbf{v}_1^* + \mathbf{v}_1^*\cdot\nabla\mathbf{v}_2 = -\frac{q}{m}\nabla\phi_1, \tag{8a}$$

$$-2i\omega\mathbf{v}_2 + \mathbf{v}_1\cdot\nabla\mathbf{v}_1 = \frac{-q}{m}\nabla\phi_2, \tag{8b}$$

$$\mathbf{v}_1\cdot\nabla\mathbf{v}_1^* + \mathbf{v}_1^*\cdot\nabla\mathbf{v}_1 = -\frac{q}{m}\nabla\phi_0, \tag{8c}$$

and

$$\nabla^2\phi_{0,1,2} = -\frac{q}{\varepsilon_0}\nu_{0,1,2} \tag{9}$$

We have here neglected terms with $l \geqslant 3$. This is a reasonable approximation, because the neglected contributions then are proportional to the small factor $1/l^2$. Furthermore, we note that $\mathbf{v}_1$ can be written as a gradient, and introduce the velocity potential φ from $\mathbf{v}_1 \equiv \nabla\varphi$.

https://doi.org/10.1017/S0022377801008996 Published online by Cambridge University Press

Using (7)–(9) to first express $\mathbf{v}_2$ in terms of $\mathbf{v}_1$, and then to eliminate $\phi_{0,1,2}$, $n_{0,1,2}$, and $\mathbf{v}_2$, we readily obtain an equation for φ

$$\boldsymbol{\nabla}\cdot\left\{\left[\varepsilon(\omega)-\frac{\nabla^2|\boldsymbol{\nabla}\varphi|^2}{\omega^2}-\frac{|\nabla^2\varphi|^2}{4\omega^2\varepsilon(2\omega)}\right]\boldsymbol{\nabla}\varphi-\frac{\nabla^2(\boldsymbol{\nabla}\varphi)^2}{2\omega^2}\boldsymbol{\nabla}\varphi^*\right.$$
$$-\frac{1}{4\omega^2\varepsilon(2\omega)}(\nabla^2\varphi^*)\,\boldsymbol{\nabla}(\boldsymbol{\nabla}\varphi)^2+\boldsymbol{\nabla}\left[\frac{|\boldsymbol{\nabla}\varphi|^2\nabla^2\varphi+(\boldsymbol{\nabla}\varphi^*)\cdot\boldsymbol{\nabla}(\boldsymbol{\nabla}\varphi)^2}{4\omega^2\varepsilon(2\omega)}\right]$$
$$\left.+\boldsymbol{\nabla}\cdot\left[\frac{(\nabla^2\varphi)\,\boldsymbol{\nabla}\varphi+\boldsymbol{\nabla}(\boldsymbol{\nabla}\varphi)^2}{2\omega^2\varepsilon(2\omega)}\right]\boldsymbol{\nabla}\varphi^*\right\}=0, \tag{10}$$

where

$$\varepsilon(\omega)=1-\frac{\omega_p^2}{\omega^2},\qquad \varepsilon(2\omega)=1-\frac{\omega_p^2}{4\omega^2},\qquad \omega_p^2=\frac{N_0\,q^2}{\varepsilon_0\,m}.$$

A corresponding derivation of an equation containing only ϕ_1 has been presented previously (Stenflo and Gradov 1986; Stenflo 1996). In the present paper, it is more convenient to use the potential φ as our key function, however. Equation (10) is actually valid for arbitrary $N_0(z)$. Below, however, it will only be used for the case where ω_p^2 is constant ($=\omega_{p0}^2=N_{0e}\,q^2/\varepsilon_0\,m$) for $z>0$, and zero for $z<0$.

Equation (10) is rather complicated. In order to simplify our further analysis, we shall therefore assume that ∂_x is so small that we only have to keep the linear contributions in the terms containing ∂_x^2 derivatives. In the purely z-dependent terms, we shall not make approximations, however. This means that our analysis below is exact in the one-dimensional case ($\partial_x\equiv 0$) and that the two-dimensional case is correct as long as ∂_x^2 is small, but that the term containing ∂_x^2 should be multiplied by a factor of order unity if ∂_x^2 is not small. Equation (10) consequently reduces to

$$\partial_z\left\{(\partial_z\varphi)\left[\varepsilon(\omega)-\frac{3}{8\omega^2}\partial_z\left(\frac{\varepsilon(\omega)}{\varepsilon(2\omega)}\partial_z(\partial_z\varphi)^2\right)\right]\right\}+\varepsilon(\omega)\,\partial_x^2\varphi=0. \tag{11}$$

When a solution of our comparatively very simple model equation (11) has been found, one has to apply the corresponding boundary conditions. As (11) contains fifth-order derivatives, we shall require the continuity of five quantities A_j ($j=1,\ldots,5$), i.e.

$$(A_j)_{z=-0}=(A_j)_{z=+0}. \tag{12}$$

The first boundary condition is due to the continuity of ϕ_1. Using (7)–(9), we thus require that

$$A_1\equiv\varphi-\frac{3\varphi_z^2\varphi_{zz}}{2(4\omega^2-\omega_p^2)} \tag{13a}$$

is continuous at $z=0$. Here we have introduced the simpler notation $\varphi_z\equiv\partial_z\varphi$ and $\varphi_{zz}\equiv\partial_z^2\varphi$, and we recall that $(\omega_p^2)_{z=+0}=\omega_{p0}^2$ and $(\omega_p^2)_{z=-0}=0$. We note that in the linear limit, (13a) corresponds to the continuity of φ.

The second boundary condition is simply obtained by integrating (11) over the surface region $z\approx 0$. As a result, we require that

$$A_2\equiv\varepsilon(\omega)\left[1-\frac{3\partial_z^2\varphi_z^2}{2(4\omega^2-\omega_p^2)}\right]\varphi_z \tag{13b}$$

https://doi.org/10.1017/S0022377801008996 Published online by Cambridge University Press

is continuous at $z = 0$. It is obvious that in the linear limit, (13b) corresponds to the continuity of $\varepsilon(\omega)\,\partial_z\varphi$.

The remaining three quantities $A_{3,4,5}$ are obtained from the continuity of ϕ_2 and from (4a) and (4b) respectively. They are

$$A_3 \equiv \frac{4+2r}{4-r}\varphi_z^2, \tag{13c}$$

where $r = \omega_{p0}^2/\omega^2$,

$$A_4 \equiv \left[\varepsilon(\omega) - \frac{(5-2r)\,(\partial_z^2\varphi_z^2) + 12\varphi_{zz}^2}{2\omega^2(4-r)} - \frac{9\varphi_{zz}\,\partial_z^2(\varphi_z\,\partial_z\varphi_z^2)}{2(4\omega^2-\omega_p^2)^2}\right]\varphi_z, \tag{13d}$$

and

$$A_5 \equiv \left[\partial_x^2\varphi + \frac{3\varphi_{zz}\,\partial_z^2\varphi_z^2}{2(4\omega^2-\omega_p^2)} + \frac{3\partial_z^2(\varphi_z\,\partial_z\varphi_z^2)}{2(4\omega^2-\omega_p^2)}\right]\varphi_z. \tag{13e}$$

We have thus finally deduced our model equation (11) together with its five necessary boundary conditions.

The limit where the nonlinear terms only slightly modify the linear surface-wave solution ($\omega \approx \omega_{p0}/\sqrt{2}$) has been investigated in many previous works, as noted in our introduction. Below, we shall therefore focus our attention on a quite different case. We consequently first rewrite (11) in the form

$$\varphi_z^2 = \int_0^z dz' \int_0^{z'} dz''\,\nu(z'') + \tfrac{1}{3}(4\omega^2-\omega_p^2)\,z^2 - F(x)\,z + v_0^2, \tag{14a}$$

where

$$\nu(x,z) = \frac{2(4\omega^2-\omega_p^2)}{3\varphi_z(x,z)}\int_0^z dz'\,\partial_x^2\varphi(x,z). \tag{14b}$$

Inserting (14) in the boundary conditions (12) and (13), assuming that the ∂_x^2 terms are small, and introducing the variable $\xi \equiv (\varphi_{zz}/\omega_{p0})_{z=-0}$, we then obtain, after straightforward but lengthy algebra, the equation

$$\xi - \frac{3}{8k^2}\left(1-\frac{1}{\alpha_1}\right)\partial_x^2\xi - \frac{3}{4}\alpha_1^2\alpha_2\,\xi^3 = 0, \tag{15}$$

where

$$\alpha_1 = \left(\frac{4+2r}{4-r}\right)^{1/2}, \qquad \alpha_2 = -1 + 3(1-\tfrac{1}{4}r)^2,$$

and

$$k^2 = \frac{\omega_{p0}^2}{(\varphi_z^2)_{x=z=-0}}.$$

A solution of (15) is

$$\xi = \xi_0\,\mathrm{sech}\,[\kappa(x-x_0)],$$

where

$$\xi_0^2 = \frac{8}{3\alpha_1^2\alpha_2} \qquad \kappa^2 = \frac{8k^2}{3(1-1/\alpha_1)}.$$

The frequency is thus a function of the amplitude ξ_0, although quite different from (2). The solitary-wave function is again of the sech type, but has otherwise no similarities to (1). We have thus added still another creature to our zoo of

https://doi.org/10.1017/S0022377801008996 Published online by Cambridge University Press

strange special surface-wave solutions. The theory of nonlinear plasma surface waves, which is an emerging subfield of nonlinear plasma science (cf. Shukla 1999; Tendler 2000) obviously deserves much more future attention.

Acknowledgements

This work was supported by the RFBR, Grant 98-02-16840, as well as by the Royal Swedish Academy of Sciences.

References

Azarenkov, N. A., Ostrikov, K. N. and Kuzmenko, A. V. 1995 *Phys. Scripta* **52**, 668.

Azarenkov, N. A. and Ostrikov, K. N. 1999 *Phys. Rep.* **308**, 333.

Brodin, G. and Lundberg, J. 1994 *Phys. Plasmas* **1**, 96.

Bychkov, V. V. and Stenflo, L. 1998 *J. Plasma Phys.* **60**, 833.

Chakraborty, B. B. 1991 *Phys. Fluids* **B3**, 857.

Efremidis, N., Hizanidis, K., Malomed, B. A., Nistazakis, H. E. and Frantzeskakis, D. J. 2000 *Physica Scripta* **T84**, 18.

Gradov, O. M. 1999 *Physica Scripta* **T82**, 63.

Gradov, O. M. and Stenflo, L. 1982 *Phys. Fluids* **25**, 983.

Grimalskii, V. V. and Rapoport, Yu. G. 1998 *Plasma Phys. Rep.* **24**, 980.

Lee, H. J. and Lim, Y. K. 1998 *Plasma Phys. Contr. Fusion* **40**, 1619.

Shukla, P. K. 1999 *Physica Scripta* **T82**.

Stenflo, L. 1996 *Physica Scripta* **T63**, 59.

Stenflo, L. and Gradov, O. M. 1986 *IEEE Trans. Plasma Sci.* **PS-14**, 554.

Stenflo, L. and Gradov, O. M. 1998 *Phys. Rev.* **E58**, 8044.

Stenflo, L. and Yu, M. Y. 1989 *Phys. Fluids* **B1**, 1543.

Stenflo, L. and Yu, M. Y. 1998 *Phys. Plasmas* **5**, 3122.

Stenflo, L., Yu, M. Y. and Zhelyazkov, I. 1983 *Beitr. Plasmaphys.* **23**, 621.

Tendler, M. 2000 *J. Plasma Phys.* **63**, 201.

Vladimirov, S. V., Yu, M. Y. and Tsytovich, V. N. 1994 *Phys. Rep.* **241**, 1.

Yu, M. Y. and Zhelyazkov, I. 1978 *J. Plasma Phys.* **20**, 183.

https://doi.org/10.1017/S0022377801008996 Published online by Cambridge University Press