MERCERIAN THEOREMS

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1. Introduction

Suppose that $k(x)$ is of bounded variation in $(-\infty, \infty)$ and that $s(x)$ is bounded and continuous in the same range. Let†

$$g(x) = \int s(x-y)\,dk(y). \tag{1·1}$$

Then it is almost trivial that

$$G \leqslant CS, \tag{1·2}$$

where $$S = \overline{\lim_{x\to\infty}} |s(x)|, \quad G = \overline{\lim_{x\to\infty}} |g(x)|, \quad C = \int |dk(x)|.$$

In particular, if $\lim_{x\to\infty} s(x) = A$ and $\int dk(x) = 1$, we have

$$g(x) - A = \int [s(x-y) - A]\,dk(y),$$

and it follows that $\lim_{x\to\infty} g(x) = A$.

Our aim in this paper is to consider the converse problem. That is, under what extra conditions can we write

$$S \leqslant F(G),$$

where the function $F(u)$ depends only on $k(x)$?

The first theorem is a straightforward extension of a known Tauberian theorem.

Theorem 1. *Suppose that $k(x)$ is of bounded variation in $(-\infty, \infty)$ and that*

$$\int e^{ixy}\,dk(y) \neq 0 \quad (-\infty < x < \infty). \tag{1·3}$$

Suppose that $s(x)$ is bounded and that for $\epsilon > 0$

$$\overline{\lim_{x\to\infty}} |s(x+\epsilon) - s(x)| = \delta(\epsilon), \quad \lim_{\epsilon\to 0} \delta(\epsilon) = 0. \tag{1·4}$$

Let

$$g(x) = \int s(x-y)\,dk(y). \tag{1·5}$$

† Integrals with unspecified limits are over $(-\infty, \infty)$.

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Then if $C_1 > 0$, *we can choose* C_2, *depending only on* C_1, $k(x)$ *and* $\delta(\epsilon)$, *so that*

$$S \leqslant C_1 + C_2 G.$$

To prove this we reduce it to the case in which $k(x)$ is absolutely continuous, and then use a form of Wiener's general Tauberian theorem. Let

$$h(x) = \pi^{-\frac{1}{2}} \int e^{-(x-y)^2} dk(y).$$

Then $h(x)$ is integrable in $(-\infty, \infty)$, and

$$\int e^{iyx} h(y)\, dy = \pi^{-\frac{1}{2}} \int e^{iyx} dk(y) \int e^{iyx} e^{-y^2} dy$$

$$= e^{-\frac{1}{4}x^2} \int e^{iyx} dk(y) \neq 0 \quad (-\infty < x < \infty).$$

Also, since $g(x)$ and $s(x)$ are bounded and continuous,

$$\int h(x-y)\, s(y)\, dy = \pi^{-\frac{1}{2}} \int s(y)\, dy \int e^{-(x-y-t)^2} dk(t) = \pi^{-\frac{1}{2}} \int s(y-t)\, dy \int e^{-(x-y)^2} dk(t)$$

$$= \pi^{-\frac{1}{2}} \int e^{-(x-y)^2} dy \int s(y-t)\, dk(t) = \pi^{-\frac{1}{2}} \int e^{-(x-y)^2} g(y)\, dy.$$

Hence $$\overline{\lim_{x \to \infty}} \left| \int h(x-y)\, s(y)\, dy \right| \leqslant G\pi^{-\frac{1}{2}} \int e^{-x^2} dx = G,$$

and the conclusion follows from a known Tauberian theorem.†

Theorem 1 is a typical *Tauberian* theorem. We use the word *Mercerian* to characterize a theorem which is formally of this type but in which the Tauberian condition (1·4) is absent. In such a theorem the essential condition on $k(x)$ is not (1·3), but the stronger condition

$$\underline{\mathrm{Bd}} \left| \int e^{iyx} dk(y) \right| > 0, \tag{1·6}$$

which is excluded in the case of absolutely continuous $k(y)$ by the Riemann-Lebesgue theorem.

We write $k(x) \epsilon V^*$ if $k(x)$ is of bounded variation and has no singular‡ component, that is, if it is the sum of an absolutely continuous function and a discontinuity function. We also write

$$K(\omega) = K(\sigma + it) = \int e^{-\omega x} dk(x) \quad (\sigma \geqslant 0)$$

when this integral converges absolutely.

† H. R. Pitt, "An extension of Wiener's general Tauberian theorem", *American Journal of Math.* 60 (1938), 532–4. We use case (*A*) of the theorem.

‡ A singular function is continuous, of bounded variation and not constant, and has zero derivative almost everywhere.

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2. General Mercerian theorems

The main results of this paper are derived from the following theorem† of N. Wiener and the author.

THEOREM 2. *Suppose that* $k(x) \epsilon V^*$ *and that*

$$\underline{\mathrm{Bd}}\, |K(it)| > 0. \tag{2·1}$$

Then

$$[K(it)]^{-1} = \int e^{-iyt}\, dp(y),$$

where $p(y)$ *is of bounded variation in* $(-\infty, \infty)$.

We can now deduce the following Mercerian theorem.

THEOREM 3. *Suppose that* $k(x) \epsilon V^*$ *and that*

$$\underline{\mathrm{Bd}}\, |K(it)| > 0.$$

Suppose that $s(y)$ *is bounded and continuous and let*

$$g(x) = \int s(x-y)\, dk(y).$$

Then

$$S \leqslant CG,$$

where the number C *depends only on* $k(x)$.

By Theorem 2, we can write

$$[K(it)]^{-1} = \int e^{-iyt}\, dp(y),$$

where $p(y)$ is of bounded variation. Hence

$$\int e^{-iyt}\, dp(y) \int e^{-ixt}\, dk(x) = 1 \quad (-\infty < t < \infty).$$

Now

$$h(x) = \int k(x-y)\, dp(y)$$

is defined and is continuous except in an enumerable set of points where it has simple discontinuities. If its values at these points are defined so that

$$h(x) = \tfrac{1}{2}[h(x+0) - h(x-0)],$$

it is of bounded variation in $(-\infty, \infty)$, and

$$\int e^{-ixt}\, dh(t) = \int e^{-iyt}\, dp(y) \int e^{-ixt}\, dk(x) = 1.$$

† N. Wiener and H. R. Pitt, "On absolutely convergent Fourier-Stieltjes transforms", *Duke Journal*, 4 (1938), 420–36, Theorem 1. When these Mercerian theorems were first proved I hoped that the class V^* could be extended to include all functions of bounded variation. However, Theorem 3 of the above paper shows that some condition on the singular component of $k(x)$ is essential, and it seems most natural to suppose that it vanishes altogether.

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It follows from the uniqueness theorem† for Fourier-Stieltjes transforms that

$$h(x) = 0 \quad (x < 1), \quad h(x) = 1 \quad (x > 1).$$

Therefore, since $g(x)$ is plainly bounded and continuous, we have

$$\int g(x-y)\,dp(y) = -\int dp(y) \int s(y)\,dk(x-y) = -\int s(y)\,dh(x-y) = s(x),$$

and hence

$$S \leqslant G \int |\,dp(y)\,| = CG.$$

We assume the continuity of $s(x)$ for simplicity, but this is not strictly necessary. To ensure the existence of the Stieltjes integrals it is sufficient to suppose that $s(x)$ is Borel-measurable.

The boundedness condition on $s(x)$ lies deeper, and one of our chief aims in this paper is to show how it can be relaxed if more is assumed about $k(x)$. We first prove the following extension of Theorem 2.

THEOREM 4. *Suppose that*

$$K(\omega) = \int e^{-\omega x}\,dk(x)$$

converges absolutely for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ *and that*

$$\underline{\mathrm{Bd}}\,|\,K(\omega)\,| > 0 \quad (\sigma_1 \leqslant \sigma \leqslant \sigma_2).$$

Then

$$[K(\omega)]^{-1} = \int e^{-\omega x}\,dp(x) \quad (\sigma_1 \leqslant \sigma \leqslant \sigma_2),$$

the integral being absolutely convergent in this range.

To prove this it is enough to apply Theorem 2 to the functions $K(\sigma_1 + it)$, $K(\sigma_2 + it)$ and use the following

LEMMA. *Suppose that* $F(\omega)$ *is continuous in* $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ *and regular in* $\sigma_1 < \sigma < \sigma_2$ *and that*

$$F(\sigma_1 + it) = \int e^{-itx}\,df_1(x), \quad F(\sigma_2 + it) = \int e^{-itx}\,df_2(x),$$

where $f_1(x), f_2(x)$ *are of bounded variation. Then we can write*

$$F(\omega) = \int e^{-\omega x}\,df(x) \quad (\sigma_1 \leqslant \sigma \leqslant \sigma_2),$$

the integral being absolutely convergent in this range.

† See, for example, S. Bochner, *Vorlesungen über Fouriersche Integrale* (Leipzig, 1932), Satz 18.

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Let $c<\sigma_1<\sigma<\sigma_2$. Then since $F(\sigma_1+it)$, $F(\sigma_2+it)$ are bounded, it follows from Cauchy's theorem that

$$\begin{aligned}
F(\omega) &= \frac{1}{2\pi}\int F(\sigma_2+it)\left[\frac{1}{\sigma_2+it-\omega}-\frac{1}{\sigma_2+it-c}\right]dt \\
&\quad -\frac{1}{2\pi}\int F(\sigma_1+it)\left[\frac{1}{\sigma_1+it-\omega}-\frac{1}{\sigma_1+it-c}\right]dt \\
&= \frac{1}{2\pi}\int\left[\frac{1}{\sigma_2+it-\omega}-\frac{1}{\sigma_2+it-c}\right]dt\int e^{-ixt}\,df_2(x) \\
&\quad -\frac{1}{2\pi}\int\left[\frac{1}{\sigma_1+it-\omega}-\frac{1}{\sigma_1+it-c}\right]dt\int e^{-ixt}\,df_1(x) \\
&= \frac{1}{2\pi}\int df_2(x)\int\left[\frac{1}{\sigma_2+it-\omega}-\frac{1}{\sigma_2+it-c}\right]e^{-ixt}\,dt \\
&\quad +\frac{1}{2\pi}\int df_1(x)\int\left[\frac{1}{\omega-\sigma_1-it}+\frac{1}{\sigma_1+it-c}\right]e^{-ixt}\,dt.
\end{aligned}$$

Now
$$\frac{1}{2\pi}\int\left[\frac{1}{\sigma_2+it-\omega}-\frac{1}{\sigma_2+it-c}\right]e^{-ixt}\,dt = e^{\sigma_2 x}[e^{-\omega x}-e^{-cx}] \quad (x<0),$$
$$= 0 \quad (x>0);$$
$$\frac{1}{2\pi}\int\left[\frac{1}{\omega-\sigma_1-it}+\frac{1}{\sigma_1+it-c}\right]e^{-ixt}\,dt = e^{-(\omega-\sigma_1)x} \quad (x>0),$$
$$= e^{(\sigma_1-c)x} \quad (x<0).$$

Therefore, if $\sigma_1<\sigma<\sigma_2$,

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{0} e^{(\sigma_1-c)x}\,df_1(x)+\int_{-\infty}^{0} e^{\sigma_2 x}[e^{-\omega x}-e^{-cx}]\,df_1(x)+\int_{0}^{\infty} e^{-(\omega-\sigma_1)x}\,df_1(x) \\
&= \int_{0}^{\infty} e^{-\omega x}\,e^{\sigma_1 x}\,df_1(x)+\int_{-\infty}^{0} e^{-\omega x}\,e^{\sigma_2 x}\,df_2(x)+C,
\end{aligned}$$

where C is a constant. These integrals plainly converge absolutely for $\sigma_1\leqslant\sigma\leqslant\sigma_2$, and the equation holds also for $\sigma=\sigma_1$, $\sigma=\sigma_2$, by continuity. This completes the proof of the lemma.

We are now in a position to prove the following extension of Theorem 3.

THEOREM 5. *Suppose that* $\sigma_2\geqslant\sigma_1\geqslant 0$, *that* $k(x)\,\epsilon V^*$ *and that*

$$K(\omega)=\int e^{-\omega y}\,dk(y)$$

converges absolutely for $0\leqslant\sigma\leqslant\sigma_2$. *Let*

$$\underline{\mathrm{Bd}}\,|K(\omega)|>0 \quad (0\leqslant\sigma\leqslant\sigma_1).$$

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Suppose that $s(x)$ is continuous and either (a) $\sigma_2 > \sigma_1 \geqslant 0$ and $s(x) = O(e^{\sigma x})$ for each fixed σ in $\sigma_1 < \sigma \leqslant \sigma_2$ or (b) $\sigma_2 = \sigma_1 = 0$ and $s(x)$ is bounded. Suppose also that

$$g(x) = \int s(x-y)\, dk(y)$$

is bounded. Then $S \leqslant CG$,
where C depends only on $k(x)$.

Case (*b*) is simply Theorem 4, so we need only consider (*a*). Plainly

$$S(\omega) = \int e^{-\omega x} s(x)\, dx$$

converges absolutely for $\sigma_1 < \sigma < \sigma_2$. Moreover, we are given that $g(x)$ is bounded for all values of x, while for negative values of x and $\sigma_1 < \sigma < \sigma_2$ we have

$$|\, g(x)\, |\, e^{-\sigma x} \leqslant \int |\, s(x-y)\, |\, e^{-\sigma(x-y)}\, e^{-\sigma y}\, |\, dk(y)\, |,$$

so that $g(x)\, e^{-\sigma x}$ is bounded. We can therefore write

$$G(\omega) = \int e^{-\omega x} g(x)\, dx \quad (\sigma_1 < \sigma < \sigma_2),$$

the integral being absolutely convergent. It follows now, by formal calculation, that

$$G(\omega) = K(\omega)\, S(\omega) \quad (\sigma_1 < \sigma < \sigma_2).$$

Now since $K(\omega)$ is uniformly continuous in $\sigma \geqslant 0$, and $|\, K(\omega)\, |$ has a positive lower bound in $0 \leqslant \sigma \leqslant \sigma_1$, the same is true in a strip $0 \leqslant \sigma \leqslant \sigma'$, where $\sigma' > \sigma_1$. There is plainly no loss of generality in supposing that $\sigma' = \sigma_2$. Therefore, by Theorem 4,

$$S(\omega) = G(\omega)\, [K(\omega)]^{-1} = G(\omega)\, P(\omega) \quad (\sigma_1 < \sigma < \sigma_2),$$

where

$$P(\omega) = \int e^{-\omega x}\, dp(x),$$

and the integral converges absolutely for $0 \leqslant \sigma \leqslant \sigma_2$. Therefore, since $g(x)$ is continuous,

$$S(\omega) = \int e^{-\omega x}\, dx \int g(x-y)\, dp(y) \quad (\sigma_1 < \sigma < \sigma_2),$$

and it follows from the uniqueness theorem for Fourier-Stieltjes transforms that

$$s(x) = \int g(x-y)\, dp(y).$$

Since $p(y)$ is of bounded variation, and depends only on $k(x)$, the conclusion follows.

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The order condition on $s(x)$ can be omitted entirely if we assume rather more about $k(x)$. The following result is a corollary of Theorem 5.

THEOREM 6.† *Suppose that $k(x)$ belongs to V^* and is constant in $(-\infty, 0)$. Let*

$$\underline{\mathrm{Bd}}\,|K(\omega)|>0 \quad (\sigma \geqslant 0).$$

Suppose that $s(x)$ is continuous in $(0,\infty)$ and that

$$g(x)=\int_0^x s(x-y)\,dk(y)$$

is bounded. Then $s(x)$ is bounded and

$$S \leqslant CG,$$

where C depends only on $k(x)$.

After Theorem 5, it is sufficient to show that $s(x) = O(e^{\sigma x})$ for some positive σ. Now

$$\lim_{\sigma\to\infty}\int_0^\infty e^{-\sigma x}\,dk(x)=k(+0)-k(0),$$

and therefore, by our hypothesis on $K(\omega)$,

$$|k(+0)-k(0)|>0.$$

We can therefore write

$$g(x)=\alpha s(x)+\int_{+0}^{\infty} s(x-y)\,dk(y) \quad (\alpha \neq 0).$$

Now suppose that $s(x)\,e^{-\sigma x}$ is not bounded in $(0,\infty)$. We can choose arbitrarily large numbers M and x_0 so that

$$|s(x_0)| \geqslant Me^{\sigma x_0}, \quad |s(y)| \leqslant Me^{\sigma y} \quad (0 \leqslant y < x_0).$$

Then

$$\begin{aligned}|g(x_0)| &\geqslant |\alpha|\,Me^{\sigma x_0} - M\int_{+0}^{x_0} e^{\sigma(x_0-y)}\,|\,dk(y)\,| \\ &= Me^{\sigma x_0}\left[\,|\alpha| - \int_{+0}^{x_0} e^{-\sigma y}\,|\,dk(y)\,|\,\right].\end{aligned}$$

But if σ is chosen so that

$$|\alpha| - \int_{+0}^{x_0} e^{-\sigma y}\,|\,dk(y)\,| > \tfrac{1}{2}|\alpha|,$$

this contradicts our assumption that $g(x)$ is bounded, since M is arbitrarily large. Therefore $s(x) = O(e^{\sigma x})$ for some σ, and the theorem is proved.

3. THEOREMS ON HAUSDORFF SUMMATION

Let μ_n be a sequence of real numbers and let

$$\mu_{\nu,n-\nu}=\sum_{p=0}^{n-\nu}(-1)^p\binom{n-\nu}{p}\mu_{\nu+p}, \tag{3·1}$$

$$t_n=\sum_{\nu=0}^{n}\binom{n}{\nu}\mu_{\nu,n-\nu}s_\nu. \tag{3·2}$$

† The special case in which $k(x)$ is absolutely continuous except at 0 has been treated by Paley and Wiener, "Fourier transforms in the complex domain", *American Math. Soc. Colloquium Publications*, vol. 19 (New York, 1934), Theorem XVII.

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Hausdorff† has shown that a necessary and sufficient condition for this transformation to be consistent is that

$$\mu_n = \int_0^1 t^n \, d\chi(t), \tag{3·3}$$

where $\chi(t)$ is of bounded variation, $\chi(+0) = \chi(0) = 1$ and $\chi(1) = 1$.

If $\chi(t)$ satisfies these conditions and if $t_n \to A$ as $n \to \infty$, we say that s_n converges (H, χ), to limit A.

In a previous paper‡ I have considered the Tauberian theorems associated with Hausdorff summation. The results of the preceding section enable us to prove some Mercerian theorems and also a Tauberian theorem (Theorem 8) of a different type.

The general Mercerian theorem is the following.

THEOREM 7. *If*

$$\chi(t) \in V^*, \quad \chi(+0) = \chi(0) = 0, \quad \chi(1) = 1, \tag{3·4}$$

a necessary and sufficient condition that a sequence s_n should converge whenever it is summable (H, χ) is that

$$\underline{\mathrm{Bd}} \left| \int_0^1 t^\omega \, d\chi(t) \right| > 0 \quad (\sigma \geqslant 0). \tag{3·5}$$

Necessity. The inverse transformation of (3·2) is easily shown to be given by

$$s_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \mu^*_{\nu, n-\nu} t_\nu,$$

where $\mu_n^* = \mu_n^{-1}$ and $\mu^*_{\nu,n-\nu} = \sum_{p=0}^{n-\nu} (-1)^p \binom{n-\nu}{p} \mu^*_{\nu+p}$.

Our hypothesis is that this transformation is consistent, so that we have

$$\mu_n^* = \int_0^1 t^n \, d\chi^*(t), \quad \int_0^1 |\, d\chi^*(t)\,| < \infty.$$

Now the function $\mu^*(\omega) = \int_0^1 t^\omega \, d\chi^*(t)$

is regular in $\sigma > 0$ and continuous in $\sigma \geqslant 0$. Therefore, if we write

$$\mu(\omega) = \int_0^1 t^\omega \, d\chi(t),$$

$\mu^*(\omega)\mu(\omega)$ is regular and bounded in $\sigma > 0$ and is equal to 1 when ω is a positive integer. It follows from Carlson's theorem§ that $\mu^*(\omega)\mu(\omega) = 1$ throughout $\sigma \geqslant 0$, and therefore (3·5) is satisfied.

† F. Hausdorff, "Summationsmethoden und Momentfolgen", *Math. Zeitschrift*, 37 (1921), 74–109.

‡ H. R. Pitt, "General Tauberian theorems", *Proc. London Math. Soc.* (2), 44 (1938), 243–88, § 4·3.

§ See, for example, L. Bieberbach, *Lehrbuch der Funktionentheorie* (Springer, 1931), p. 138.

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Sufficiency. If we assume (3·5), it follows from Theorem 4, after an exponential transformation, that

$$\mu^*(\omega) = [\mu(\omega)]^{-1} = \int_0^\infty t^\omega \, d\chi^*(t),$$

the integral being absolutely convergent for $\sigma \geqslant 0$. Moreover, $\chi^*(e^{-t})$ is of bounded variation in $(-\infty, \infty)$, and this implies that $\chi^*(+0) = \chi(0)$. We write

$$\mu^*(\omega) = \mu_1^*(\omega) + \mu_2^*(\omega),$$

$$\mu_1^*(\omega) = \int_0^1 t^\omega \, d\chi^*(t), \quad \mu_2^*(\omega) = \int_{1+0}^\infty t^\omega \, d\chi^*(t).$$

Now $\mu_1^*(\omega)$ is regular in $\sigma > 0$ and bounded and continuous in $\sigma \geqslant 0$, and the same is true of $\mu^*(\omega)$, by hypothesis. Hence $\mu_2^*(\omega)$ is regular in $\sigma > 0$ and bounded and continuous in $\sigma \geqslant 0$. But $\mu_2^*(\omega)$ is plainly regular in $\sigma < 0$ and bounded and continuous in $\sigma \leqslant 0$. It is therefore bounded and regular in the whole plane, and reduces to a constant, by Liouville's theorem. This constant must be 0, since $\mu_2^*(\omega) \to 0$ as $\sigma \to \infty$. Hence

$$[\mu(\omega)]^{-1} = \int_0^1 t^\omega \, d\chi^*(t), \quad \int_0^1 |\, d\chi^*(t)\,| < \infty.$$

If we suppose that $\chi^*(0) = 0$, then $\chi^*(1) = [\mu(0)]^{-1} = 1$, and this completes the proof.

As an immediate corollary, we have

MERCER'S THEOREM.† *If* $\alpha > 0$ *and*

$$\lim_{n\to\infty}\left[\alpha s_n + \frac{s_0 + s_1 + \ldots + s_n}{n+1}(1-\alpha)\right] = A,$$

then

$$\lim_{n\to\infty} s_n = A.$$

We put $\chi(t) = (1-\alpha)\,t (0 \leqslant t < 1)$, $\chi(1) = 1$, and the conditions of Theorem 7 are satisfied.

In the general form of Tauberian theorem for Hausdorff summation, the typical Tauberian condition is

$$s_n - s_{n-1} = O(n^{-1}).$$

It is known, however, that in certain special cases, in Euler's method, for example, it is sufficient to have

$$s_n - s_{n-1} = O(n^{-\frac{1}{2}}).$$

In the following theorem we determine general conditions under which the latter is a sufficient Tauberian condition.

THEOREM 8. *Suppose that* $\chi(t)$ *satisfies* (3·4) *and that*

$$\underline{\mathrm{Bd}}\,|\,\mu(\omega)\,| > 0 \quad (0 \leqslant \sigma \leqslant \sigma_1). \tag{3·6}$$

† J. Mercer, *Proc. London Math. Soc.* (2), 5 (1907), 206–24.

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Suppose that $s_n = O(n^{\sigma_1})$ *and that* $s_n \to A(H, \chi)$. *Suppose that*

$$s_n - s_{n-1} > -Cn^{-1} \tag{3·7}$$

or, more generally,† *that* $\sum_{n \leqslant y^1} a_n$ *belongs to* T. *Then*

$$\lim_{n\to\infty} s_n = A.$$

Let μ_n, t_n be defined by (3·2) and (3·3). Then

$$t_n = \sum_{\nu=0}^{n} \frac{n!}{\nu!(n-\nu)!} \int_0^1 y^\nu (1-y)^{n-\nu}\, d\chi(y),$$

$$\frac{t_n}{n!} = \int_0^1 \left[\sum_{\nu=0}^{n} \frac{s_\nu y^\nu (1-y)^{n-\nu}}{\nu!(n-\nu)!} \right] d\chi(y),$$

$$\sum_{n=0}^{\infty} \frac{t_n}{n!} = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{s_n y^n}{n!} \sum_{n=0}^{\infty} \frac{(1-y)^n x^n}{n!} \right] d\chi(y),$$

the series being absolutely convergent for all x, y. Hence, writing

$$T(x) = e^{-x} \sum_{n=0}^{\infty} \frac{t_n x^n}{n!}, \quad S(x) = e^{-x} \sum_{n=0}^{\infty} \frac{s_n x^n}{n!},$$

we have

$$T(x) = \int_0^1 S(xy)\, d\chi(y).$$

Now $\lim_{x\to\infty} T(x) = A$, by the consistency property of Borel summation. We can therefore deduce from Theorem 5 (after an exponential change of scale) that $\lim_{x\to\infty} S(x) = A$. The conclusion then follows from the known Tauberian theorem for Borel summation.

In conclusion, we show that if (3·6) is satisfied, and $\chi(y)$ is discontinuous at 1, then condition (3·7) is unnecessary and we have a theorem of Mercerian type.

THEOREM 9. *Suppose that* $\chi(t)$ *satisfies* (3·4) *and that*

$$\underline{\mathrm{Bd}}\, |\mu(\omega)| > 0 \quad (\sigma = 0).$$

Suppose that s_n *is bounded and* $s_n \to A(H, \chi)$. *Then*

$$\lim_{n\to\infty} s_n = A.$$

As in Theorem 8, we have

$$T(x) = \int_0^1 S(xy)\, d\chi(y), \quad \lim_{x\to\infty} S(x) = A.$$

If $0 < y < 1$, the Euler sum of order y is given by taking

$$\chi(t) = 0 \quad (0 \leqslant x < y), \quad \chi(t) = 1 \quad (y < x \leqslant 1).$$

† See H. R. Pitt, "General Tauberian theorems", *Proc. London Math. Soc.* (2), 44 (1938), 243–88, § 2. The particular form of Tauberian theorem for Borel summation which we use below is contained in Theorem 16 of the same paper.

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It is easy to show that if s_n is bounded and summable by Borel's method, then it is summable by Euler's method for any y in $0<y<1$. That is, if $0<y<1$ and $n\to\infty$,

$$E(y,n)=\sum_{\nu=0}^{n}\binom{n}{\nu}y^{\nu}(1-y)^{n-\nu}s_{\nu}\to A.$$

Now

$$t_n=\int_0^1\left[\sum_{\nu=0}^{n}\binom{n}{\nu}y^{\nu}(1-y)^{n-\nu}s_{\nu}\right]d\chi(y)$$

$$=[\chi(1)-\chi(1-0)]\,s_n+\int_{+0}^{1-0}E(y,n)\,d\chi(y),$$

$\chi(y)$ being continuous at 0. Hence, letting $n\to\infty$, we have

$$[\chi(1)-\chi(1-0)]\,s_n=A-A\int_{+0}^{1-0}d\chi(y)+o(1)$$

$$=A[\chi(1)-\chi(1-0)]+o(1),$$

since $\chi(1)=1$. Therefore $\lim_{n\to\infty}s_n=A$.

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