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On the Maximum Degree of a Random Planar Graph

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Let the random graph R_n be drawn uniformly at random from the set of all simple planar graphs on n labelled vertices. We see that with high probability the maximum degree of R_n is $\Theta(\ln n)$. We consider also the maximum size of a face and the maximum increase in the number of components on deleting a vertex. These results extend to graphs embeddable on any fixed surface.

1. Introduction

Let us start by considering planar graphs: later we shall consider graphs embeddable on any fixed surface. Let $\mathscr{P}$ denote the class of all simple planar graphs, and let $\mathscr{P}_n$ denote the set of graphs in $\mathscr{P}$ on the vertices $1,\ldots,n$. Let the random graph R_n be drawn uniformly at random from $\mathscr{P}_n$: we denote this by $R_n \in_U \mathscr{P}_n$. (Note that we are considering random graphs for which there exists an embedding in the plane, and not random embedded graphs.)

There has been much recent interest in the properties of R_n and in the related problem of estimating $|\mathscr{P}_n|$. Giménez and Noy [6] built on work in [1] and improved on a result in [9] by showing that

$$|\mathscr{P}_n| \sim g n^{-7/2} \gamma^n n!$$

where $g \approx 0.4261$ and the *growth constant* $\gamma \approx 27.2269$. They also showed, for example, that the number of components $\kappa(R_n)$ is such that $\kappa(R_n) - 1$ is asymptotically Poisson-distributed with mean $\nu \approx 0.0374$, and thus the probability that R_n is connected tends to $e^{-\nu} \approx 0.9633$; the number of edges is asymptotically normally distributed with mean κn and variance λn where $\kappa \approx 2.2133$ and $\lambda \approx 0.4303$ (improving on a result in [4]); and the number of pendant vertices (that is, vertices of degree 1) is asymptotically normally distributed with mean and variance ρn

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where $\rho = \gamma^{-1} \approx 0.04500$. It is shown in [9] that w.h.p. R_n has linearly many vertices of each fixed degree; w.h.p. it has maximum degree $\Omega(\ln n/\ln\ln n)$ (and indeed there is a vertex adjacent to at least $(1+o(1))\ln n/\ln\ln n$ pendant vertices); and w.h.p. in every embedding of R_n in the plane, the maximum face size is $\Omega(\ln n/\ln\ln n)$ and $O(\ln n)$. Here we use *w.h.p.* (with high probability) to mean 'with probability $\to 1$ as $n \to \infty$'.

Two of the major problems left open concerned the behaviour of the maximum degree and face size. Let us consider first the maximum degree $\Delta(R_n)$. As we noted above, $\Delta(R_n)$ is $\Omega(\ln n/\ln\ln n)$ w.h.p., but no decent upper bound was known. Of course, R_n has at most $3n-6$ edges. If we consider the classical random graph $G_{n,m}$ with $m \sim cn$ edges for some constant $c > 0$ then its maximum degree satisfies $\Delta \sim \ln n/\ln\ln n$ w.h.p.; see, for example, Theorem 3.7 of [2] for a much more precise result. Further, a random forest on n labelled vertices also has maximum degree $\Delta \sim \ln n/\ln\ln n$ w.h.p.; see [10]. Of course, the random planar graph R_n is very different from $G_{n,m}$ or a random forest, but still these observations led to Question 5.4 in [10], which asked whether $\Delta(R_n) \sim \ln n/\ln\ln n$ w.h.p. The answer is no.

Theorem 1.1. *There are constants $0 < c < C$ such that w.h.p. the random planar graph $R_n \in_U \mathscr{P}_n$ satisfies*

$$c\ln n \le \Delta(R_n) \le C\ln n.$$

Our proof is combinatorial, and extends to graphs embeddable on any fixed surface. For any class $\mathscr{A}$ of graphs we let $\mathscr{A}_n$ denote the set of graphs in $\mathscr{A}$ on $1,\dots,n$. Given a surface S, let $\mathscr{A}^S$ denote the class of graphs embeddable on S. See, for example, [8, 11] for definitions and discussion. The following theorem extends Theorem 1.1 above, and is our main result.

Theorem 1.2. *Let S be a fixed surface, and let $\mathscr{A} = \mathscr{A}^S$. There are constants $0 < c < C$ such that w.h.p. the random graph $R_n \in_U \mathscr{A}_n$ satisfies*

$$c\ln n \le \Delta(R_n) \le C\ln n.$$

We noted above that w.h.p. in every embedding of R_n in the plane, the maximum face size is $\Omega(\ln n/\ln\ln n)$ and $O(\ln n)$. We measure size, say, by the length of the facial walk. In fact the higher value here is the correct one.

Theorem 1.3. *Let S be a fixed surface, and let $\mathscr{A} = \mathscr{A}^S$. There are constants $0 < c < C$ such that w.h.p. the random graph $R_n \in_U \mathscr{A}_n$ satisfies the following: in every embedding in S, the maximum size of a face is at least $c\ln n$ and at most $C\ln n$.*

The upper bound parts of these last two theorems are proved in the next section, and the lower bound parts in the section after that. Finally, in Section 4, we introduce the 'component-degree' of a vertex v in a graph G, which is the increase in the number of components on deleting the edges incident with v, or equivalently the number of components of $G-v$ to which v is adjacent. We show that w.h.p. the maximum component-degree of R_n is $(1+o(1))\ln n/\ln\ln n$; and we relate this result to the fact mentioned above on the maximum number of pendant vertices adjacent to any vertex.

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In this paper we are concerned with (simple) graphs that can be embedded in a given surface, perhaps in many ways. In contrast, a *map* is a connected multigraph (with loops and multiple edges allowed) together with a given embedding in the plane. For maps, generating function methods may be used to yield far more precise results: it is shown in [3] that the maximum degree of a random rooted n-edge map is close to $\log n/\log(6/5)$ w.h.p., and indeed a much more precise result is given.

2. Upper bounds

In this section we prove the upper bound parts of Theorems 1.2 and 1.3. Throughout the section, let S be a fixed surface, let $\mathscr{A} = \mathscr{A}^S$, and let $R_n \in_U \mathscr{A}_n$. It is shown in [8] that $(|\mathscr{A}_n|/n!)^{1/n} \to \gamma$ as $n \to \infty$, where the *growth constant* γ is the same as that for the plane, and thus $\gamma = 27.2269$ [6] (as we noted earlier). Since $\mathscr{A}$ has a growth constant, we obtain the following result by Theorem 5.1 of [10].

Lemma 2.1. *There exists $\alpha > 0$ such that R_n has at least αn pendant vertices with probability $1 - e^{-\Omega(n)}$.*

We need one elementary lemma: see Theorem 4.1 below for a related sharper and more general result.

Lemma 2.2. *In R_n w.h.p. each vertex is adjacent to at most $2\ln n/\ln\ln n$ pendant vertices.*

Proof. Let k be a positive integer, and let $\mathscr{B}_n$ be the set of graphs $G \in \mathscr{A}_n$ such that vertex 1 is adjacent to at least k pendant vertices. We claim that

$$\Pr(R_n \in \mathscr{B}_n) \le 1/k!$$

which will yield the lemma, since it shows that the probability that R_n has some vertex adjacent to at least k pendant vertices is at most $n/k!$.

To prove the claim, we use a counting method which we shall use again below. Given a real number x and a positive integer r we use the notation $(x)_r$ to denote the 'falling factorial' $x(x-1)\cdots(x-r+1)$. For each $G \in \mathscr{B}_n$, consider the k least pendant vertices $u_1,\dots,u_k$ adjacent to vertex 1, remove the edges incident with these vertices u_i, and arbitrarily re-attach each vertex u_i to one vertex of G other than $u_{i+1},\dots,u_k$. Then each graph G' constructed is in $\mathscr{A}_n$, and the number of constructions is at least $|\mathscr{B}_n|(n-1)_k$.

How often can each graph $G' \in \mathscr{A}_n$ be constructed? We may guess the set of k vertices u_i and then we know the original graph G. Thus each graph G' is constructed at most $\binom{n-1}{k}$ times. Hence

$$|\mathscr{A}_n| \ge |\mathscr{B}_n|\,(n-1)_k \Big/ \binom{n-1}{k} = |\mathscr{B}_n|\,k!$$

and so

$$\Pr[R_n \in \mathscr{B}_n] = |\mathscr{B}_n|/|\mathscr{A}_n| \le 1/k!$$

as required. □

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Proof of upper bound in Theorem 1.2. Let $b = b(n) = \lceil \frac{8}{\alpha} \ln n \rceil$, where α is as in Lemma 2.1. We shall show that

$$\Delta(R_n) < b \text{ w.h.p.}, \tag{2.1}$$

which will give the required upper bound.

Let $\mathscr{B}_n$ be the set of graphs $G \in \mathscr{A}_n$ such that $\Delta(G) \geq b$, there are at least αn pendant vertices, and each vertex is adjacent to at most $2 \ln n / \ln \ln n$ pendant vertices. By Lemmas 2.1 and 2.2, it suffices to show that w.h.p. $R_n \notin \mathscr{B}_n$. The idea of the proof is similar to that of the last lemma: from the graphs in $\mathscr{B}_n$ we can build many graphs in $\mathscr{A}_n$ with little double counting, so we cannot start with many graphs in $\mathscr{B}_n$.

Here is the construction. Let $a = a(n) = \lfloor 2 \ln n \rfloor$. Let n_0 be sufficiently large that for each $n \geq n_0$ we have $a \geq 3$ and $\alpha n - 2 \ln n / \ln \ln n - a \geq \alpha n/2$. Assume that $n \geq n_0$.

Let $G \in \mathscr{B}_n$, and fix an embedding of G in S. Let v be a vertex with degree at least b. The embedding gives a clockwise order on the neighbours of v: list them in this order as $v_1, v_2, \ldots, v_d$, where $d \geq b$ is the degree of v and where v_d is the largest of the numbers $v_1, \ldots, v_d$. Choose an arbitrary ordered list of a distinct pendant vertices with none adjacent to v, say $u_1, \ldots, u_a$. Finally choose an arbitrary subset of a of the vertices v_i, which we may write as $v_{i_1}, \ldots, v_{i_a}$, where $i_1 < i_2 < \cdots < i_a$.

Now for the graph part. Delete each edge incident to v, and each edge incident to one of the chosen pendant vertices u_i. For each $i = 1, \ldots, a$, join v to u_i and join u_i to u_{i+1}(where u_{a+1} means u_1). Thus we have formed a wheel around v. For each $j = 1, \ldots, a$, join u_j to each of $v_{i_j}, v_{i_j+1}, \ldots, v_{i_{j+1}-1}$ (where i_{a+1} means i_1). This completes the construction. It is easy to see that each graph G' constructed is in $\mathscr{A}_n$.

Note that, for each $G \in \mathscr{B}_n$, we have made at least $(\alpha n - 2 \ln n / \ln \ln n)_a \geq (\alpha n/2)^a$ choices for the list of pendant vertices $u_1, \ldots, u_a$, and at least $\binom{b}{a} \geq \left(\frac{b}{a}\right)^a$ choices for the subset of the neighbours of v. Thus the number of constructions is at least

$$|\mathscr{B}_n| \left(\frac{\alpha n}{2} \cdot \frac{b}{a} \right)^a \geq |\mathscr{B}_n| (2n)^a.$$

Now consider the double counting. How many times can a given graph $G' \in \mathscr{A}_n$ be constructed? Guess the vertex v. Find the largest 'second neighbour' of v: this is v_d. This determines u_a (the unique neighbour of v adjacent to v_d). Now guess which of the two common neighbours of v and u_a is u_1 (the other is u_{a-1}). Now we know each of $u_1, u_2, \ldots, u_a$. Next guess the original neighbours of these vertices. This determines the original graph G completely. So the embedding is determined, and in particular the order $v_1, \ldots, v_d$ of the neighbours of v. But for each $j = 1, \ldots, a-1$ the vertex v_{i_j} is the earliest vertex in this list adjacent in G' to u_j, and v_{i_a} is the earliest vertex in this list which is adjacent in G' to u_a and is also after $v_{i_{a-1}}$ in the cyclic order. Hence we know $v_{i_1}, v_{i_2}, \ldots, v_{i_a}$, and all choices have been determined. Thus G' is constructed at most $n \cdot 2 \cdot n^a = 2n^{a+1}$ times.

But the number of distinct graphs constructed is at least the number of constructions divided by the maximum number of times a given graph is constructed. Hence

$$|\mathscr{A}_n| \geq |\mathscr{B}_n| (2n)^a / (2n^{a+1})$$

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and so

$$\Pr[R_n \in \mathscr{B}_n] = |\mathscr{B}_n|/|\mathscr{A}_n| \leq n/2^{a-1} = o(1).$$

This completes the proof of the claim (2.1) and thus of the upper bound in Theorem 1.2. □

Proof of upper bound in Theorem 1.3. Let c be the Euler characteristic of the surface S. Consider a graph G embedded in S with a facial walk of length j. Let H be the (simple) subgraph formed by the vertices and edges incident with this face. Then H is embeddable in S, and so by Euler's formula, $|V(H)| \geq |E(H)|/3 + c \geq j/6 + c$. Note also that the graph obtained by adding a new vertex adjacent to any subset of the vertices in $V(H)$ is embeddable in S.

By Lemma 2.1 there exists $\alpha > 0$ such that w.h.p. R_n has at least αn pendant vertices. Let $\mathscr{B}_n$ be the set of graphs $G \in \mathscr{A}_n$ with at least αn pendant vertices and such that in some embedding in S there is a facial walk with at least $2\log n + 1$ incident vertices. It suffices to show that $|\mathscr{B}_n|/|\mathscr{A}_n| = o(1)$ (since then w.h.p. the maximum length of a facial walk is at most $13\log_2 n$).

From each graph $G \in \mathscr{B}_n$ we shall construct many graphs in $\mathscr{A}_n$, as follows. We may suppose that, for each such graph G, we are given a set W of at least $2\log n + 1$ vertices such that in some embedding of G in S these vertices are incident with a common face. Pick a pendant vertex v and remove the edge incident with v, pick a subset U of $W \setminus \{v\}$, and finally add an edge between v and each vertex in U. Then each graph G' constructed is in $\mathscr{A}_n$, and for each $G \in \mathscr{B}_n$ we make at least $2^{2\log n}\alpha n = \alpha n^3$ constructions. How often can a given graph $G' \in \mathscr{A}_n$ be constructed? If we guess v and its original neighbour then we know the original graph G, so each graph G' is constructed at most n^2 times. Hence

$$|\mathscr{A}_n| \geq |\mathscr{B}_n|\, \alpha n^3/n^2,$$

so

$$|\mathscr{B}_n|/|\mathscr{A}_n| \leq 1/\alpha n = o(1),$$

as required. □

3. Lower bounds

In this section we shall prove the lower bound parts of Theorems 1.2 and 1.3, by giving more general results. As before, throughout the section, let S be a fixed surface, let $\mathscr{A} = \mathscr{A}^S$, and let $R_n \in_U \mathscr{A}_n$.

Suppose that we have a (small) graph H with h vertices including a specified root vertex r, a (large) graph G, and a set W of h vertices of G. We say that there is a *pendant copy* of H in G at W if there is an isomorphism ϕ from H to the induced subgraph of G on W, there is a unique edge between the vertices in W and the rest of G, and that edge is incident with $\phi(r)$. This is similar to the definition in [9] of H 'appearing' in G except that we have freedom here over the isomorphism ϕ. We may think of the graph H as being unlabelled.

The following result is more general than we need here but may be of independent interest. Let us say that a set $\mathscr{H}$ of graphs 'has at most exponentially many automorphisms' if there is a constant c such that each graph $H \in \mathscr{H}$ has at most $c^{|V(H)|}$ automorphisms.

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Theorem 3.1. *Let $\mathscr{H}$ be a set of connected planar graphs which have at most exponentially many automorphisms, and let $\epsilon > 0$. Then there exists $\eta > 0$ such that w.h.p. R_n has at least $n^{1-\epsilon}$ pendant copies of each graph $H \in \mathscr{H}$ with at most $\eta \ln n$ vertices.*

There is a 'converse' result indicating that Theorem 3.1 is in a sense best possible.

Proposition 3.2. *Let $\eta > 0$ and let $k = k(n) = \lfloor \eta \ln n \rfloor$. There is a constant $c > 0$ such that the following holds. Let (H_j) be a sequence of connected planar graphs, where H_j has j vertices and $\Omega(c^j)$ automorphisms: then w.h.p. R_n does not contain a pendant copy of H_k.*

We shall prove these results below, but first let us note some corollaries of Theorem 3.1. If H is a 3-connected planar graph or a subdivision of such a graph, then by a classical result of Whitney [12], H has an essentially unique embedding in the sphere; and it follows that the number of automorphisms of H is at most $4|E(H)| \leq 12|V(H)|$. Thus Theorem 3.1 gives us the following result.

Corollary 3.3. *For any $\epsilon > 0$ there exists $\eta > 0$ such that w.h.p. R_n has at least $n^{1-\epsilon}$ pendant copies of each graph H with at most $\eta \ln n$ vertices, which is a 3-connected planar graph or is a subdivision of such a graph.*

Observe that we need some condition on the planar graphs H above: by Lemma 2.2 above, w.h.p. the largest order of a pendant copy of a star is $O(\ln n / \ln \ln n)$; and we may obtain the same bound on the largest order of a pendant copy of a complete bipartite graph $K_{2,k}$ (which is 2-connected). From Corollary 3.3 we immediately obtain our next result.

Corollary 3.4. *There exists $\eta > 0$ such that w.h.p. R_n has a pendant copy of the wheel with $\lfloor \eta \ln n \rfloor$ vertices, and contains a copy of a path of length at least $\eta \ln n$ with each internal vertex of degree 2.*

Corollary 3.4 yields the lower bounds on both maximum degree and maximum face size (for any embedding). Let us note here that Corollary 3.3 also yields a result on the existence of many disjoint copies of general planar graphs.

Corollary 3.5. *For any $\epsilon > 0$ there exists $\eta > 0$ such that w.h.p. R_n has at least $n^{1-\epsilon}$ pairwise vertex-disjoint induced copies of each planar graph with at most $\eta \ln n$ vertices.*

Now we shall prove Theorem 3.1, Proposition 3.2 and Corollary 3.5.

Proof of Theorem 3.1. By Lemma 2.1 there exists $\alpha > 0$ such that, if $\mathscr{G}_n$ denotes the set of graphs $G \in \mathscr{A}_n$ with at least αn pendant vertices, then

$$\Pr[R_n \notin \mathscr{G}_n] = e^{-\Omega(n)}.$$

Let the constant $c > 0$ be such that each graph $H \in \mathscr{H}$ with h vertices has at most c^h automorphisms. Let $\mathscr{H}^j$ denote the set of graphs in $\mathscr{H}$ with at most j vertices. We may assume that

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the graphs H are unlabelled, and so $|\mathscr{H}^j| \le \tilde{c}^j$ for a suitable constant $\tilde{c}$. Let $\epsilon > 0$. Let $\eta > 0$ be sufficiently small and n_0 sufficiently large that, if we set $k = k(n) = \lfloor \eta \ln n \rfloor$, then for each $n \ge n_0$ we have $|\mathscr{H}^k| \le n^{\epsilon/5}$, $2\alpha^{-k} \le n^{\epsilon/5}$ and $2c^k \le n^{\epsilon/5}$. We shall show that the probability that R_n fails to have at least $n^{1-\epsilon}$ pendant copies of each $H \in \mathscr{H}^k$ is $O(n^{-\epsilon/5})$.

Let $n_1 \ge n_0$ be sufficiently large that for each $n \ge n_1$ the following three conditions hold: $1 \le k \le \alpha n$, $(\alpha n)_k \ge \frac{1}{2}(\alpha n)^k$, and $(n^{1-\epsilon} + k)c^k \le n^{1-4\epsilon/5}$. Assume that $n \ge n_1$. Consider any graph $H \in \mathscr{H}^k$ with $h \le k$ vertices, and fix a labelled copy of H on vertices $1, \dots, h$ rooted at vertex 1. Let $\mathscr{B}_n^H$ be the set of graphs in $\mathscr{G}_n$ with less than $n^{1-\epsilon}$ pendant copies of H. The idea of the proof is that from each $G \in \mathscr{B}_n^H$ we can construct many graphs in $\mathscr{A}_n$, as follows.

For each $G \in \mathscr{B}_n^H$, choose an arbitrary ordered list of h pendant vertices $u_1, \dots, u_h$. Delete the edge incident with u_j for each $j = 2, \dots, h$; and form a pendant copy of H rooted at vertex u_1 with vertex u_j corresponding to vertex j of H for $j = 1, \dots, h$. Each graph G' constructed is in $\mathscr{A}_n$, and the number of constructions is at least

$$|\mathscr{B}_n^H|\,(\alpha n)_h \ge |\mathscr{B}_n^H|\,\frac{1}{2}(\alpha n)^h.$$

How many times can a graph $G' \in \mathscr{A}_n$ be constructed? Guess the pendant copy of H built – there are at most $n^{1-\epsilon} + h$ possibilities. Now we know u_1 and the set $\{u_2, \dots, u_h\}$. There are at most c^h automorphisms of H, so at most c^h possible orders on $\{u_2, \dots, u_h\}$. Guess one of these orders, so now we know $u_2, \dots, u_h$, and finally guess the vertices originally adjacent to $u_2, \dots, u_h$. This identifies the original graph G. Hence G' is constructed at most $(n^{1-\epsilon} + h) \cdot c^h \cdot n^{h-1} \le n^{h-4\epsilon/5}$ times.

From the above we have

$$|\mathscr{A}_n| \ge \left|\mathscr{B}_n^H\right|\,\frac{1}{2}(\alpha n)^h / n^{h-4\epsilon/5} = \left|\mathscr{B}_n^H\right|\,\frac{1}{2}\alpha^h n^{4\epsilon/5},$$

and so

$$\left|\mathscr{B}_n^H\right| \le |\mathscr{A}_n|\,2\alpha^{-h} n^{-4\epsilon/5} \le |\mathscr{A}_n|\,n^{-3\epsilon/5}.$$

Hence

$$\left|\bigcup_{H \in \mathscr{H}^k} \mathscr{B}_n^H\right| \le |\mathscr{H}^k|\,|\mathscr{A}_n|\,n^{-3\epsilon/5} \le |\mathscr{A}_n|\,n^{-2\epsilon/5}.$$

Finally, the probability that R_n fails to have at least $n^{1-\epsilon}$ pendant copies of each $H \in \mathscr{H}^k$ is at most $n^{-2\epsilon/5} + \Pr[R_n \notin \mathscr{G}_n] = O(n^{-\epsilon/5})$. □

Proof of Proposition 3.2. Let $c > e^{1/\eta}$. Let (H_j) be a sequence of connected planar graphs, where H_j has j vertices and $\Omega(c^j)$ automorphisms. Denote the number of automorphisms of a graph G by $\mathrm{aut}(G)$. Let $\mathscr{B}_n$ be the set of graphs $G \in \mathscr{A}_n$ which have a pendant copy of H_k.

Let $G \in \mathscr{B}_n$. Pick a pendant copy of H_k with set W of k vertices, rooted at a vertex $u \in W$. Delete all the edges in this copy of H_k, and attach the $k-1$ vertices in $W \setminus \{u\}$ one after another arbitrarily by a single edge to the rest of the graph. Then each graph G' constructed is in $\mathscr{A}_n$, and there are at least $|\mathscr{B}_n|\,(n-1)_{k-1}$ constructions.

How often can a graph $G' \in \mathscr{A}_n$ be constructed? Guess the set W and vertex $u \in W$. Delete the edges incident with the vertices in $W \setminus \{u\}$, and build a copy of H_k on W rooted at u. The

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number of possibilities is at most

$$k\binom{n}{k}\, k!/\mathrm{aut}(H_k) = k(n)_k/\mathrm{aut}(H_k),$$

and so G' can be constructed at most this number of times. Hence

$$|\mathscr{A}_n| \geq \frac{|\mathscr{B}_n|\ (n-1)_{k-1}}{k(n)_k/\mathrm{aut}(H_k)} = \frac{|\mathscr{B}_n|\ \mathrm{aut}(H_k)}{kn},$$

and so

$$\frac{|\mathscr{B}_n|}{|\mathscr{A}_n|} \leq \frac{kn}{\mathrm{aut}(H_k)} = o(1)$$

as required. □

Proof of Corollary 3.5. Let $\eta > 0$ be as in Corollary 3.3, and let $k = k(n) = \lfloor(\eta/3)\ln n\rfloor$. Let H be any planar graph on $2 \leq h \leq k$ vertices. (We already know about pendant vertices.) We claim that there is a graph H^+ on at most $3h$ vertices which is a subdivision of a 3-connected planar graph and which has H as an induced subgraph. Given the claim, the corollary will follow immediately from Corollary 3.3, since $3h \leq \eta \ln n$.

To establish the claim, note that we can add a new vertex with an edge to each component of H, then triangulate this graph by adding at most $3(h+1) - 6 - h = 2h - 3$ edges, and finally form H^+ by subdividing each of these last added edges. Then H is an induced subgraph of H^+ (because of the subdivision), H^+ is a subdivision of a 3-connected graph, and H^+ has at most $h + 1 + 2h - 3 \leq 3h$ vertices. □

4. The maximum component-degree of R_n

We have noted that the maximum number of pendant vertices adjacent to any vertex in R_n is $(1 + o(1))\ln n/\ln\ln n$ w.h.p. This is an example of a more general phenomenon.

Let $\kappa(G)$ denote the number of components of a graph G. Given a vertex v in a graph G, let G' denote the graph obtained by deleting all edges incident with v: then the *component-degree* of v is $\kappa(G') - \kappa(G)$, which is the number of components of G' to which v is adjacent. Thus, if v is isolated then its component-degree is 0, if v is not isolated but is not a cut-vertex then its component-degree is 1, and if v is a cut-vertex then its component-degree equals its degree in the block tree of G. Let us denote the maximum component-degree of a vertex in G by $\Delta^\kappa(G)$.

Let G be a graph with vertex set V, let $W \subset V$ where $|W| \geq 2$, let $v \in W$ and let H be the subgraph $G[W]$ of G induced on W. We say that H is a *vertex-pendant subgraph* rooted at v on W if there are no edges between $W \setminus \{v\}$ and $V \setminus W$, there is at least one edge between v and $W \setminus \{v\}$ and at least one edge between v and $V \setminus W$, and the induced subgraph $H[W \setminus \{v\}] = G[W \setminus \{v\}]$ is connected. Given an (unlabelled) graph H_1 with a specified root vertex r, we say that there is a *pendant copy* of H_1 at v on W if there is an isomorphism between H_1 and H taking r to v. Note that for this to be possible both H_1 and $H_1 - r$ must be connected.

For a cut-vertex v its component-degree is the number of vertex-pendant subgraphs rooted at v. (If v is not a cut-vertex then there cannot be a vertex-pendant subgraph rooted at v.) Given a planar graph H of order at least 2 with a specified root vertex r and such that both H and $H - r$

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are connected, let $\Delta_H^\kappa(G)$ be the maximum over all vertices v of G of the number of pendant copies of H rooted at v. Observe that for each such H trivially we have $\Delta^\kappa(G) \geq \Delta_H^\kappa(G)$. When H is the single edge graph K_2, $\Delta_H^\kappa(G)$ is the maximum number of pendant vertices adjacent to any vertex in G.

Theorem 4.1. *The maximum component-degree $\Delta^\kappa(R_n)$ satisfies*

$$\Delta^\kappa(R_n)\ \ln\ln n/\ln n \to 1 \quad \textit{in probability as } n \to \infty;$$

and for any fixed planar graph H of order at least 2 with a specified root vertex r such that both H and $H - r$ are connected, we have

$$\Delta_H^\kappa(R_n)\ \ln\ln n/\ln n \to 1 \quad \textit{in probability as } n \to \infty.$$

Proof. **(a)** We first give a lower bound for $\Delta_H^\kappa(R_n)$, following the idea of the proof of Corollary 5.3 of [10]. Let H be as above and let $d = \Delta(H) + 1$. Consider a graph G on $V = \{1, \ldots, n\}$. Let $A = A(G)$ be the set of vertices v in G which are incident with at least d edges other than edges within vertex-pendant copies of H. Consider all the vertex-pendant subgraphs isomorphic to H such that the root is in A: suppose that there are $t = t(G)$ of them, and list them as $H_1, \ldots, H_t$ where the order is by lexicographic order on the sets W_i consisting of the vertex set of the subgraph less the root vertex. For each $i = 1, \ldots, t$ let r_i be a new vertex and form the graph H_i' by replacing the root vertex by r_i.

Let $\mathscr{H} = \mathscr{H}(G)$ be the family $(H_1', \ldots, H_t')$. Write $|\mathscr{H}|$ for $t = t(G)$. Let $W = W(G) = \cup_{i=1}^t W_i$, the set of all vertices in the pendant copies H_i less their roots. Then G is specified completely by the family $\mathscr{H}(G)$ (which determines W), the subgraph $\hat{G}$ induced by G on $V \setminus W$ (which determines A) and the function $f : \{1, \ldots, t\} \to A$ (which specifies the vertex $f(i)$ in A with which the root vertex r_i should be identified). Further, if we change the function f to another function $f' : \{1, \ldots, t\} \to A$, we form a new graph G' such that $\mathscr{H}(G') = \mathscr{H}(G)$ and $\hat{G}' = \hat{G}$. (There can be no additional vertex-pendant copies of H since no vertex in A can be in such a subgraph.)

Now let $\mathscr{H}$ and G_0 be such that $\Pr[(\mathscr{H}(R_n) = \mathscr{H}) \wedge (\hat{R}_n = G_0)] > 0$. By the discussion above, conditional on $(\mathscr{H}(R_n) = \mathscr{H}) \wedge (\hat{R}_n = G_0)$, the maximum number of vertex-pendant copies of H rooted at a vertex in A (the set A is determined by G_0) has the same distribution as the maximum load of a bin when we throw $|\mathscr{H}|$ balls uniformly at random into $|A|$ bins. By the appearances theorem, Theorem 5.1 of [10], applied to a suitable extension H^+ of the graph H, we see that w.h.p. $|\mathscr{H}(R_n)| \geq \alpha n$ for some constant $\alpha > 0$. (We may form H^+ from H by adding a disjoint copy of the complete bipartite graph $K_{2,d}$ with vertices a and b of degree d, identifying the root of H with a and rooting H^+ at b. Observe that if H^+ is a pendant subgraph in G, then H is a vertex-pendant subgraph of G which is rooted at a vertex incident with d edges that cannot be in any pendant copy of H.) But a standard result concerning balls and bins shows that if we throw between αn and n balls into at most n bins then the maximum bin load is at least $(1 + o(1)) \ln n/\ln\ln n$ w.h.p. (see, for example, Lemma 2.5 of [7] to obtain a more precise result). This gives the required lower bound on $\Delta_H^\kappa(R_n)$. The upper bound will follow from the upper bound in part (b) of the proposition.

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(b) Now let us give an upper bound for $\Delta^\kappa(R_n)$. Let $\epsilon > 0$ and let $k = k(n) = \lfloor (1+\epsilon)\ln n/\ln\ln n \rfloor$. We want to show that $\Delta^\kappa(R_n) \le k$ w.h.p. By Lemma 2.1 there is an $\alpha > 0$ such that w.h.p. R_n has at least αn pendant vertices. Let $\mathscr{B}_n$ be the set of graphs $G \in \mathscr{A}_n$ such that $\Delta^\kappa(G) > k$ and G has at least αn pendant vertices. It suffices to show that $\Pr[R_n \in \mathscr{B}_n] = o(1)$.

Let $G \in \mathscr{B}_n$, and let the vertex v in G have component-degree $> k$. Let the vertex-pendant subgraphs of G rooted at v be $H_1, \dots, H_{k'}$, where $k' \ge k$, and we have listed the components by lexicographic order of the sets W_i consisting of the vertex set $V(H_i)$ less the root vertex v. We shall consider only the first k of these components. Let X be the union of the singleton sets W_i for $i = 1, \dots, k$. Observe that if W_i is not a singleton set then it must contain a vertex of degree at least 2.

Choose an arbitrary ordered list of k distinct pendant vertices $u_1, \dots, u_k$ in $V(G) \setminus X$; and choose an arbitrary ordering $i_1, \dots, i_k$ of $1, \dots, k$. For each $j = 1, \dots, k$, form H'_j from H_{i_j} by deleting any vertices u_i in the list (at least one vertex in W_{i_j} must remain) and replacing the root vertex v by u_j. Now we start from G less the vertices in the sets $V(H'_1), \dots, V(H'_k)$, together with the k separate graphs H'_j. For each $j = 1, \dots, k$, add an edge between v and its 'image' u_j in H'_j, and add an edge between u_j and u_{j+1} (where u_{k+1} means u_1). Thus we have built a wheel around v with $u_1, \dots, u_k$ around the rim, with H'_j a vertex-pendant subgraph rooted at u_j for each j. Then each graph G' constructed is in $\mathscr{A}_n$; and the number of constructions is at least

$$|\mathscr{B}_n|\,(\alpha n - k)_k k! \ge |\mathscr{B}_n|\,\frac{1}{2}\alpha^k n^k k!$$

for n sufficiently large.

How often can a graph G' in $\mathscr{A}_n$ be constructed? Observe that G' can have at most $4n$ edges for n sufficiently large. Guess the adjacent vertices v and u_1, and guess the way around the wheel. There are at most $16n$ possibilities so far, and now we know $u_1, \dots, u_k$ and $H'_1, \dots, H'_k$. Finally guess the original neighbours in G of $u_1, \dots, u_k$: now we know G completely, and thus the entire construction including the ordering $i_1, \dots, i_k$. Hence the number of times G' can be constructed is at most $16n^{k+1}$. But now we have

$$|\mathscr{A}_n| \ge |\mathscr{B}_n|\,\frac{1}{2}\alpha^k n^k k!/(16n^{k+1}),$$

and so

$$|\mathscr{B}_n|/|\mathscr{A}_n| \le 32\alpha^{-k} n/k! = o(1). \qquad \square$$

5. Concluding remarks

We have seen that, for a fixed surface S and $R_n \in_U \mathscr{A}_n^S$ (that is, R_n sampled uniformly at random from the labelled n-vertex graphs embeddable on S), the maximum degree $\Delta(R_n)$ is $\Theta(\ln n)$ w.h.p. We discussed also the maximum size of a face and the maximum component-degree, but let us focus here on the maximum degree.

It would be interesting to know more. Is there a constant $c = c(S)$ such that $\Delta(R_n)/\ln n \to c$ in probability as $n \to \infty$? If so, what is this constant for the plane, and is it the same constant for all surfaces? Is there w.h.p. a unique vertex of maximum degree? What happens if we constrain the number of edges in R_n (as in [4, 5])? What is the behaviour of the maximum degree for random unlabelled planar graphs?

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