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POPULATION DYNAMICS DRIVEN BY TRUNCATED STABLE PROCESSES WITH MARKOVIAN SWITCHING

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Abstract

We focus on the population dynamics driven by two classes of truncated α-stable processes with Markovian switching. Almost necessary and sufficient conditions for the ergodicity of the proposed models are provided. Also, these results illustrate the impact on ergodicity and extinct conditions as the parameter α tends to 2.

Keywords: Ergodicity; α-stable processes; Markovian switching; stationary distribution; extinction

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1. Introduction

In the famous Lotka–Volterra model, the number in the population X_t is given by $\mathrm{d}X_t = X_t(a - bX_t)\,\mathrm{d}t$, where a is population growth rate and b represents the intra-species coefficient. However, species growth suffers from all kinds of noise. To characterize the white noise impact on the growth rate, many researchers replace the term $a\,\mathrm{d}t$ by $a + \sigma\,\mathrm{d}B_t$ and consider the stochastic Lotka–Volterra model $\mathrm{d}X_t = X_t[(a - bX_t)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t]$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion. On the other hand, to model the joint impact of color noise together with white noise on the growth rate a and the intra-species intensity, Luo and Mao [11,12] first established the stochastic Loka–Volterra model with Markovian switching. This stochastic model is given by $\mathrm{d}X_t = X_t[(a_{r_t} - b_{r_t}X_t)\,\mathrm{d}t + \sigma_{r_t}\,\mathrm{d}B_t]$, where $(r_t)_{t\geq 0}$ is a Markov chain with a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$.

Moreover, population changes are highly affected by natural disasters. Earthquakes (including tsunamis) killed more people globally than all other types of disaster put together, claiming nearly 750,000 lives between 1994 and 2013. Drought affected more than one billion people between 1994 and 2013, or 25% of the global total. Species always suffer from catastrophic events, which can cause a sudden decline of various species. Empirical studies have

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demonstrated that the distribution of the number of casualties (wars, terrorist attacks, tornados, earthquakes, floods, and landslides) follows power-tailed distributions [4]–[10]. For example, [15] showed that casualty levels of wars follow a power-law distribution according to the number of 'quarrel-dead' between 1820 and 1945 in the world. Reference [9] modelled casualties throughout the wars from 1816 to 1980 by an approximate power-law distribution. Note that spiky population changes follow power-law or heavy-tailed distributions with infinite variance. By the generalized central limit theorem (see [14]), the sum of independent random variables with infinite variance converges weakly to a stable distribution. Hence, to model population changes, the stable process $(Z_t^{r_t})_{t\geq 0}$ is more appropriate than the Brownian motion $(B_t)_{t\geq 0}$. Consequently, we consider the stochastic differential equation (SDE)

$$\mathrm{d}X_t = X_t[(a_{r_t} - b_{r_t}X_t)\,\mathrm{d}t + \sigma_{r_t}\,\mathrm{d}Z_t^{r_t}], \tag{1}$$

where $(Z_t^{r_t})_{t\geq 0}$ is a stable process with Markovian switching. Namely, given $r_t \equiv i$, $(Z_t^{r_t})_{t\geq 0}$ is a stable process with Lévy jump measure $\nu(\mathrm{d}z, i)$. Throughout this paper, let

$$C_\alpha = \frac{\alpha 2^{\alpha-1}\Gamma\left(\frac{\alpha+1}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)},$$

where $\Gamma(\cdot)$ is the Gamma function defined by

$$\Gamma(s) = \int_0^{+\infty} t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t, \qquad s \in (0, +\infty).$$

Given $r_t \equiv i$, if the switching Lévy measures $\nu(\mathrm{d}z, i)$ are independent of i, and the Lévy measure takes $\frac{C_\alpha}{|z|^{1+\alpha}}\,\mathrm{d}z$, then $(Z_t^i)_{t\geq 0}$ reduces to the symmetric stable process.

The SDE (1) is termed a stochastic population driven by a stable process with Markovian switching. Although a stable process is a special case of a Lévy process, there is a paucity of literature related to the SDE (1). The Lotka–Voltera model driven by Lévy noise was first established by Bao et al. [3]. However, the Lévy jump measure was assumed to be a finite measure in this paper. It is well recognized that a stable process with σ-finite Lévy jump measure is not a finite measure. Hence, their results are not suitable for the SDE (1). Consider the simplified form of the SDE (1) without Markovian switching. Given $r_t \equiv i$, $i \in \mathbb{S}$, the SDE (1) reduces to

$$\mathrm{d}X_t^i = X_t^i\left[(a_i - b_iX_t^i)\,\mathrm{d}t + \sigma_i\,\mathrm{d}Z_t^i\right]. \tag{2}$$

To the best of our knowledge, there are limited papers considering the model (2). Tong, Ma, and Zhang [18] considered the ergodicity and extinction of the model (1) with pure positive jumps. However, the negative jump case was not considered in their study. We have emphasized that population declines are highly affected by natural disasters globally. Hence, there is a social, ethical, and economic meaning to considering the impact of negative jumps on population dynamics. The main purpose of this paper is to find the impact of negative jumps on population dynamics, and the remainder of the paper will provide a solid answer.

To proceed, we need the following basic assumptions.

Assumption 1. *The coefficients a_i, b_i, and σ_i are all positive constants for $i \in \mathbb{S}$.*

Assumption 2. *The Markov chain $(r_t)_{t\geq 0}$ is irreducible and non-periodic with finite state space. If $N \geq 2$, then the stable processes $\{Z_t^1\}_{t\geq 0}, \ldots, \{Z_t^N\}_{t\geq 0}$ are independent.*

Assumption 3. *For some $i, j \in \mathbb{S}$, $i \neq j$, $q_{ij} > 0$, $\alpha \in (1, 2)$.*

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Assumption 4. *Given $r_t \equiv i$, the jump process $\Delta Z_t^i =: Z_t^i - Z_{t^-}^i$ corresponding to $(Z_t^i)_{t\geq 0}$ is bounded by $\frac{1}{\sigma_i}$. That is, the Lévy measure corresponding to the process $(Z_t^i)_{t\geq 0}$ has*

$$\nu(\mathrm{d}z, i) = \begin{cases} \dfrac{C_\alpha I_{\left\{|z|<\frac{1}{\sigma_i}\right\}}}{|z|^{1+\alpha}}\,\mathrm{d}z, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Assumption 5. *Given $r_t \equiv i$, the jump process $\Delta Z_t^i =: Z_t^i - Z_{t^-}^i$ corresponding to $(Z_t^i)_{t\geq 0}$ is bounded from below by $-\frac{1}{\sigma_i}$. That is, the Lévy measure corresponding to the process $(Z_t^i)_{t\geq 0}$ has*

$$\nu(\mathrm{d}z, i) = \begin{cases} \dfrac{C_\alpha I_{\left\{z>-\frac{1}{\sigma_i}\right\}}}{|z|^{1+\alpha}}\,\mathrm{d}z, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Remark 1. Assumptions 4 and 5 are natural conditions and rather weak. The jump process $(\Delta Z_t)_{t\geq 0}$ corresponds to the change of population size in an instant time. Given $r_t = i$, Assumption 4 represents that the state change is bounded from below. Roughly speaking, a decreasing population number is bounded from below. From the perspective of biology, this condition obviously holds. Contrarily, $z = \frac{1}{\sigma_i}$ represents that the immigration size doubles the original size at a certain moment; this case will be discussed at the end of the paper.

Finally, the following notation is frequently used:

- $A \gg 0$: each element of A is positive.
- $\mathbb{R}^+$, $(0, +\infty)$: all positive real numbers.
- $X_{t^-} := \lim_{s\to 0^-} X_{t+s}$.
- The Poisson random measure is defined as $N^i(t, B) := \sum_{0\leq s\leq t} I_B(\Delta Z_s^i)$, for any $B \in \mathbb{B}(\mathbb{R} - \{0\})$, where $\Delta Z_s^i = Z_s^i - Z_{s^-}^i$ represents the jump at time s.
- For any $\varepsilon > 0$, let $B_\varepsilon(y) = \{x \in \mathbb{R}^+ : |x - y| < \varepsilon\}$ be the ball of radius ε and center y, and $\overline{B}_\varepsilon(y) = \{x \in \mathbb{R}^+ : |x - y| \leq \varepsilon\}$ be the ball of radius ε and center y with its boundary.
- Let $\widetilde{N}^i(\mathrm{d}t, \mathrm{d}z) = N^i(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z, i)\,\mathrm{d}t$ be the compensated compounded Poisson measure. By the famous Lévy–Itô decomposition,

$$Z_t^i = \int_0^t \int_{|z|\leq 1} z\widetilde{N}^i(\mathrm{d}t, \mathrm{d}z) + \int_0^t \int_{|z|>1} zN^i(\mathrm{d}t, \mathrm{d}z).$$

2. Preliminaries

Let $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P})$ be a complete probability space with filtration $\{\mathbb{F}_t\}_{t\geq 0}$ satisfying the common conditions (i.e. it is right continuous and increasing while $\mathbb{F}_0$ contains all $\mathbb{P}$-null

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sets). Let $(r_t)_{t\geq 0}$ be a continuous-time Markov chain on the state space $\mathbb{S}=\{1, 2, \ldots, N\}$, with $N<\infty$, satisfying

$$\mathbb{P}\{r_{t+\Delta}=j \mid r_t=i\}=\begin{cases} q_{ij}\Delta+o(\Delta), & \text{if } i\neq j,\\ 1+q_{ii}+o(\Delta), & \text{if } i=j,\end{cases}$$

where $\Delta>0$. The Q-matrix $Q=(q_{ij})_{N\times N}$ is assumed to be irreducible and conservative. Here, $q_{ij}\geq 0$ is the transition rate from i to j if $i\neq j$, and $q_{ii}=-\sum_{i\neq j} q_{ij}$. Since the Markov chain (r_t) is irreducible and conservative, it has a unique stationary distribution $\pi=(\pi_1, \pi_2, \ldots, \pi_N)$. The stationary distribution π is the solution of the following equation:

$$\pi Q=0 \quad \text{and} \quad \sum_{i=1}^{N}\pi_i=1.$$

Next, we consider the solution process of the SDE (1). Although (X_t) is not a Markov process, the process (X_t, r_t) is a Markov process. The infinitesimal generator $\mathscr{A}$ of $(X_t, r_t)_{t\geq 0}$ corresponding to the the equation (1) is given by

$$\begin{aligned}\mathscr{A}V(x, i)&=x(a_i-b_ix)V_x(x, i)+QV(x, \cdot)(i)\\ &\quad+\int_{\mathbb{R}\setminus\{0\}}\big(V(x+\sigma_i xz, i)-V(x)-V_x(x, i)\sigma_i xz1_{\{|z|\leq 1\}}\big)\nu(\mathrm{d}z, i),\end{aligned}\tag{3}$$

where $V_x(x, i)=\frac{\partial}{\partial x}V(x, i)$ and $QV(x, \cdot)(i)=\sum_{j\neq i} q_{ij}\,(V(x, j)-V(x, i))$. Here, $V(\cdot, \cdot)\in C^2(\mathbb{R}\times\mathbb{S})$ means, for all $i\in\mathbb{S}$, $V(x, \cdot)\in C^2(\mathbb{R})$.

For any Borel set A of $\mathbb{R}^+$, $x\in\mathbb{R}^+$, $i, j\in\mathbb{S}$, set

$$\mathbb{P}(t, (x, i), (A, j))=\mathbb{P}(X_t\in A, r_t=j \mid X_0=x, r_0=i).$$

Lemma 1. *Under Assumptions 1–4, for any given initial data $(x_0, r_0)\in\mathbb{R}^+\times\mathbb{S}$, the SDE (1) has a unique positive solution (X_t, r_t):*

$$\mathbb{P}\{X_t\in\mathbb{R}^+ \text{ for all } t\geq 0\}=1.$$

Proof. The proof of the existence and uniqueness of the solution is almost the same as those of [20, Theorem 2.1] and [19, Lemma 3.1]. The sketch proof is given as follows. Rewrite the Markov chain as $r_t=r_0+\sum_{n=1}^{\infty}M_nI(\tau_n\leq t)$, where τ_n, M_n have the following conditional distributions. Given $r(\tau_k)=i$, $\tau_{k+1}-\tau_k$ follows the exponential distribution with parameters q_{kk}, the jump $M_k=r(\tau_{k+1})-r(\tau_k)$ is independent of the past, and $r_{\tau_{k+1}}$ hits the state j with probability $-\frac{q_{kj}}{q_{kk}}$. According to [20, Theorem 2.1], there exists a unique solution to the equation

$$\mathrm{d}Y_t^i=Y_t^i[(a_i-b_iX_t)\,\mathrm{d}t+\sigma_i\,\mathrm{d}Z_t^i], \qquad Y_0^i=y.$$

For each i, denote the unique solution by $\{Y_t^{y,i}\}_{t\geq 0}$. With a little abuse of notation, define, on $t\in[0, \tau_1)$, $X_t=Y_t^{X_0, r_0}$. Iterating for n times, for $\tau_{n-1}\leq t<\tau_n$, we define $X_t=Y_t^{X_{\tau_{n-1}}, r_{\tau_{n-1}}}$. It is not hard to see that, for any $t\in\mathbb{R}^+$,

$$\begin{aligned}X_t&=X_0+\sum_{k=1}^{\infty}Y_t^{Y_{\tau_{n-1}}, r_{\tau_{n-1}}}I(\tau_{k-1}\leq t<\tau_k)\\ &=X_0+\int_0^t X_s(a_{r_s}-b_{r_s}X_s)\,\mathrm{d}s+\int_0^t\sigma_{r_s}X_s\,\mathrm{d}Z_s^{r_s}.\end{aligned}\tag{4}$$

Then, the construction solution $\{X_t\}_{t\geq 0}$ is the unique solution to the SDE (1).

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To prove the non-negativity of the solution, let t_0 be any jump moment of the stable process (Z_t). First, we prove that no jump causes the state of X_t to be negative. Rewriting (1), we have, for any $r_{t_0^-} = j$, $X_{t_0} = X_{t_0^-} + \sigma_j X_{t_0^-} \triangle Z^j_{t_0}$, where $\triangle Z^j_{t_0} = Z^j_{t_0} - Z^j_{t_0^-}$ represents a jump of the stable process (Z^j_t) at time t_0^-. Using Assumption 4, we have $\triangle Z^j_{t_0} > -\frac{1}{\sigma_j}$. If $X_{t_0^-} > 0$, then, almost surely (a.s.), $\sigma_j X_{t_0^-} \triangle Z^j_{t_0} > -X_{t_0^-}$. It follows from this that $X_{t_0} > 0$ a.s.

Next, we prove that if there is no jump for the stable noise between $[t, t + \Delta t)$, then the continuous motion corresponding to the drift term cannot cause the path of $X_{t+\triangle t}$ to be negative if $X_t > 0$. To prove the assertion, note that the continuous motion satisfies the SDE

$$\mathrm{d}X_t = X_t(a_{r_t} - b_{r_t} X_t)\,\mathrm{d}t, \qquad X_0 = x > 0. \tag{5}$$

According to the construction of the solution (4), to prove the non-negativity of the solution to the SDE (5), we only need to prove that, for each $i \in \mathbb{S}$, the solution to the SDE

$$\mathrm{d}X^i_t = X^i_t(a_i - b_i X^i_t)\,\mathrm{d}t, \qquad X^i_0 = x > 0 \tag{6}$$

is non-negative. Observe that (6) is the famous logistic differential equation. Let $Y_t = \frac{1}{X^i_t}$. Then, (6) can be rewritten as $\mathrm{d}Y_t = (b_i - a_i Y_t)\,\mathrm{d}t$.

For any $a_i > 0$, $b_i > 0$, $X_0 = x > 0$, $t \in \mathbb{R}^+$, $Y_t = \frac{b_i}{a_i} + \left(\frac{1}{x} - \frac{b_i}{a_i}\right)\mathrm{e}^{-a_i t} = \frac{b_i}{a_i}(1 - \mathrm{e}^{-a_i t}) + \frac{1}{x}\mathrm{e}^{-a_i t} > 0$. Thus, $X^i_t = \frac{1}{Y_t} > 0$ for all $t \in \mathbb{R}^+$. This, together with (4), implies that the solution $\{X_t\}$ to the SDE (5) is non-negative. This completes the proof. □

Remark 2. From the above proof, the global positive solution of Lemma 1 still holds if Assumption 4 in Lemma 1 is replaced by Assumption 5.

Finally, we provide a modified definition of ergodicity.

Definition 1. If, for any initial value $x \in \mathbb{R}^+$, $i \in \mathbb{S}$, there exists an invariant distribution $\pi(A, j)$ such that the transition probability $P(t, (x, i), (\cdot, \cdot))$ converges weakly to the invariant distribution $\pi(\cdot, \cdot)$, then the SDE (1) is said to be ergodic.

Remark 3. It is noteworthy that our ergodicity definition of the SDE (1) is different from that of general stochastic processes. Strictly speaking, the state space is restricted to $\mathbb{R}^+ \times \mathbb{S}$. If this initial condition is removed, the solution process $(X_t)_{t\geq 0}$ corresponding to SDE (1) has at least two invariant distributions. Hence, the solution process $(X_t)_{t\geq 0}$ is not ergodic without the initial condition. Moreover, it is more reasonable that 'ergodic' in Definition 1 is replaced by 'locally ergodic' or 'constrained ergodic'. Finally, the permanence of the process (X_t) is not considered directly because the ergodicity of (X_t, r_t) implies the permanence of (X_t).

3. Ergodicity

In this section we will consider the ergodicity of the population dynamics model (1). Our methods depend heavily on a modification of Khasminskii's results.

Lemma 2. *Suppose there exists a bounded open domain $U \subset \mathbb{R}^+$, with closure $\bar{U} \subset \mathbb{R}^+$, satisfying the following conditions:*

(i) $\inf_{(x,i)\in U\times\mathbb{S}} \sigma_i^2 x^2 > 0$;

(ii) If $x_0 \in \mathbb{R}^+ - U$, then $\sup_{(x_0,r_0)\in(G-U)\times\mathbb{S}} \mathbb{E}\tau_U < \infty$ holds for every compact subset G of $\mathbb{R}^+$ such that $U \subset G$, where $\tau_U = \inf\{t \geq 0, X^{x_0,r_0}_t \in U\}$.

Then the SDE model (1) has a unique stationary distribution.

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Proof. The proof is similar to that of [17, Theorem 4.1] and [1, Theorem 5.1]. From condition (ii), the process (X_t, r_t) is positive recurrent with respect to the domain $U \times \mathbb{S}$, and recurrent in the domain $\mathbb{R}^+ \times \mathbb{S}$. By the maximum principle associated with the operator in (3) (see Appendix A), we get the existence of a stationary distribution. The proof of uniqueness is presented in Appendix A. □

Remark 4. Condition (i) is essentially to ensure that the generator of the SDE (1) is a uniformly elliptic operator in domain U, which ensures that the process (X_t) will exit the domain U and hit $\mathbb{R}^+ - U$ in finite time. By condition (ii), if the process (X_t) is in $\mathbb{R}^+ - U$ then it will return to U in a finite time.

Theorem 1. *For $\alpha \in (1, 2)$, let Assumptions 1–4 hold. If*

$$\sum_{i=1}^{N} \pi_i \left(a_i - \sum_{n=1}^{+\infty} \frac{C_\alpha \sigma_i^\alpha}{n(2n-\alpha)} \right) > 0,$$

then the SDE (1) is ergodic and has a unique stationary distribution.

Proof. We will prove this theorem according to Lemma 2. We will verify that the conditions (i) and (ii) hold in our setting. Let N_1 be a sufficiently large number and set $U = \left\{ x \in \mathbb{R}^+ : \frac{1}{N_1} < x < N_1 \right\}$. Since $\sigma_i > 0$, $i \in \mathbb{S}$, $\inf_{x \in U} \sigma_i^2 x^2 = \frac{\sigma_{\min}^2}{N^2} > 0$, where $\sigma_{\min}^2 = \min\left\{ \sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2 \right\}$. This verifies condition (i) in Lemma 2.

Next, we will verify condition (ii) in Lemma 2. For simplicity, the column vector $\mu = (\mu_1, \mu_2, \ldots, \mu_N)^\top \in \mathbb{R}^N$ is defined as

$$\mu_i = \sum_{n=1}^{+\infty} \frac{C_\alpha \sigma_i^\alpha}{n(2n-\alpha)} - a_i.$$

Also, let

$$\beta = -\pi \mu = \sum_{i=1}^{N} \pi_i \left(a_i - \sum_{n=1}^{+\infty} \frac{C_\alpha \sigma_i^\alpha}{n(2n-\alpha)} \right).$$

Note that $\beta > 0$. Let $\mathbf{1} = (1, \ldots, 1)^\top$; then $\pi(\mu + \beta \mathbf{1}) = -\beta + \beta = 0$. From [16, Lemma A.12], this implies that the linear equation $Qc = \mu + \beta \mathbf{1}$ has a solution $c = (c_1, c_2, \ldots, c_N)^\top \in \mathbb{R}^N$. Thus, we have

$$\beta = \sum_{j=1}^{N} q_{ij} c_j - \mu_i, \qquad i \in \mathbb{S}.$$

For each $i \in \mathbb{S}$, consider the Lyapunov function $V : \mathbb{R}^+ \times \mathbb{S} \to \mathbb{R}$, $V(x, i) = (1 - \gamma c_i)(x^\gamma + x^{-\gamma})$, where $\gamma \in (0, \alpha - 1)$ is sufficiently small that $1 - \gamma c_i > 0$ for each $i \in \mathbb{S}$. Hence, it follows that

$$\begin{aligned} \mathscr{A}V(x, i) = {} & (a_i - b_i x)(1 - \gamma c_i)\gamma(x^\gamma - x^{-\gamma}) - \sum_{j \neq i} q_{ij} \gamma (c_j - c_i)(x^\gamma + x^{-\gamma}) \\ & + (1 - \gamma c_i) \int_{|z| \le \frac{1}{\sigma_i}} \Big((x + \sigma_i x z)^\gamma + (x + \sigma_i x z)^{-\gamma} - x^\gamma - x^{-\gamma} \\ & \qquad - \gamma \sigma_i x z I_{\{|z| \le K_i\}} \left(x^{\gamma - 1} - x^{-\gamma - 1} \right) \Big) \frac{C_\alpha}{|z|^{1+\alpha}} \, \mathrm{d}z, \end{aligned}$$

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where $K_i = \begin{cases} \frac{1}{\sigma_i}, \sigma_i \geq 1, \\ 1, \ \sigma_i < 1. \end{cases}$

By the variable substitution $y = \sigma_i z$, set

$$\mathscr{A}V(x, i) := A_1(x, i) + A_2(x, i) + A_3(x, i), \tag{7}$$

where

$$A_1(x, i) = (a_i - b_i x)(1 - \gamma c_i)\gamma(x^\gamma - x^{-\gamma}) - \sum_{j \neq i} q_{ij}\gamma(c_j - c_i)(x^\gamma + x^{-\gamma}),$$

$$A_2(x, i) = (1 - \gamma c_i)x^\gamma \sigma_i^\alpha \int_{|y| \leq 1} \left((1 + y)^\gamma - 1 - \gamma y I_{\{|y| \leq \sigma_i K_i\}}\right) \frac{C_\alpha}{|y|^{1+\alpha}} \, \mathrm{d}y,$$

$$A_3(x, i) = (1 - \gamma c_i)x^{-\gamma} \sigma_i^\alpha \int_{|y| \leq 1} \left((1 + y)^{-\gamma} - 1 + \gamma y I_{\{|y| \leq \sigma_i K_i\}}\right) \frac{C_\alpha}{|y|^{1+\alpha}} \, \mathrm{d}y.$$

First, we calculate the term $A_1(x, i)$:

$$\begin{aligned} A_1(x, i) &= (a_i - b_i x)(1 - \gamma c_i)\gamma(x^\gamma - x^{-\gamma}) - \sum_{i \neq j} q_{ij}\gamma(c_j - c_i)(x^\gamma + x^{-\gamma}) \\ &= \gamma(1 - \gamma c_i)\left[\left(a_i - \sum_{j=1}^N q_{ij}c_j\right)x^\gamma - b_i x^{\gamma+1} + b_i x^{1-\gamma} - \left(a_i + \sum_{j=1}^N q_{ij}c_j\right)x^{-\gamma} + o(\gamma)\right]. \end{aligned}$$

To calculate $A_2(x, i)$, the following approximation is needed:

$$\begin{aligned} \sum_{i \neq j} q_{ij} \frac{c_j - c_i}{1 - \gamma c_i} &= \sum_{i \neq j} q_{ij}c_j + \sum_{i \neq j} q_{ij} \frac{\gamma c_i c_j - c_i}{1 - \gamma c_i} \\ &= \sum_{j=1}^N q_{ij}c_j + \sum_{i \neq j} q_{ij} \frac{\gamma c_j(c_j - c_i)}{1 - \gamma c_i}\gamma \\ &= \sum_{j=1}^N q_{ij}c_j + o(\gamma). \end{aligned}$$

Then, $A_2(x, i)$ is computed as

$$\begin{aligned} A_2(x, i) &= (1 - \gamma c_i)x^\gamma \sigma_i^\alpha \int_{-1}^0 \left((1 + y)^\gamma - 1 - \gamma y I_{\{|y| \leq \sigma_i K_i\}}\right) \frac{C_\alpha}{|y|^{1+\alpha}} \, \mathrm{d}y \\ &\quad + (1 - \gamma c_i)x^\gamma \sigma_i^\alpha \int_0^1 \left((1 + y)^\gamma - 1 - \gamma y I_{\{|y| \leq \sigma_i K_i\}}\right) \frac{C_\alpha}{|y|^{1+\alpha}} \, \mathrm{d}y \\ &= (1 - \gamma c_i)x^\gamma \sigma_i^\alpha \int_0^1 \left((1 - y)^\gamma + (1 + y)^\gamma - 2\right) \frac{C_\alpha}{y^{1+\alpha}} \, \mathrm{d}y. \end{aligned}$$

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Define $F_1(y) = (1-y)^\gamma + (1+y)^\gamma - 2$, where $y \in (0, 1)$. By the Taylor expansion of the function $F_1(y)$ in the interval (0,1), $F_1(y) = 2\sum_{n=1}^{+\infty} C_\gamma^{2n} y^{2n}$. Thus, it follows that

$$A_2(x, i) = (1-\gamma c_i)x^\gamma \sigma_i^\alpha \int_0^1 2\sum_{n=1}^{+\infty} C_\gamma^{2n} y^{2n} \frac{C_\alpha}{y^{1+\alpha}}\, \mathrm{d}y$$

$$= (1-\gamma c_i)x^\gamma \sigma_i^\alpha \frac{2\sum_{n=1}^{+\infty} C_\gamma^{2n} C_\alpha}{2n-\alpha}. \tag{8}$$

As for the term $A_3(x, i)$, splitting the integral into two terms, we have

$$A_3(x, i) = (1-\gamma c_i)x^{-\gamma} \sigma_i^\alpha \int_{-1}^1 \left((1+y)^{-\gamma} - 1 + \gamma y I_{\{|y|\le \sigma_i K_i\}}\right) \frac{C_\alpha}{|y|^{1+\alpha}}\, \mathrm{d}y$$

$$=: A_{31}(x, i) + A_{32}(x, i),$$

where

$$A_{31}(x, i) = (1-\gamma c_i)x^{-\gamma} \sigma_i^\alpha \int_{-1}^1 \left((1+y)^{-\gamma} - 1\right) \frac{C_\alpha}{|y|^{1+\alpha}}\, \mathrm{d}y,$$

$$A_{32}(x, i) = (1-\gamma c_i)x^{-\gamma} \sigma_i^\alpha \int_{-1}^1 \gamma y I_{\{|y|\le \sigma_i K_i\}} \frac{C_\alpha}{|y|^{1+\alpha}}\, \mathrm{d}y.$$

To estimate $A_{31}(x, i)$, by the Taylor expansion for the function $(1+y)^{-\gamma} - 1$ for $y \in (-1, 1)$, it follows that

$$A_{31}(x, i) = (1-\gamma c_i)x^{-\gamma} \sigma_i^\alpha \sum_{n=1}^{\infty} C_{-\gamma}^n \int_{-1}^1 \frac{y^n C_\alpha}{|y|^{1+\alpha}}\, \mathrm{d}y$$

$$= (1-\gamma c_i)x^{-\gamma} \sigma_i^\alpha \sum_{n=1}^{\infty} C_{-\gamma}^{2n} \int_{-1}^1 \frac{y^{2n-1} C_\alpha \mathrm{sgn}(y)}{|y|^\alpha}\, \mathrm{d}y$$

$$+ (1-\gamma c_i)x^{-\gamma} \sigma_i^\alpha \sum_{n=1}^{\infty} C_{-\gamma}^{2n-1} \int_{-1}^1 \frac{y^{2n-1} C_\alpha}{|y|^{1+\alpha}}\, \mathrm{d}y.$$

Note that

$$\int_{-1}^1 \frac{y^{2n-1} C_\alpha}{|y|^{1+\alpha}} dy = 0, \qquad \text{for all } n \in \mathbb{N}^+.$$

Hence, we have

$$A_{31}(x, i) = (1-\gamma c_i)x^{-\gamma} \sigma_i^\alpha \sum_{n=1}^{\infty} C_{-\gamma}^{2n} \int_{-1}^1 \frac{y^{2n-1} C_\alpha \mathrm{sgn}(y)}{|y|^\alpha}\, \mathrm{d}y$$

$$= 2(1-\gamma c_i)x^{-\gamma} \sigma_i^\alpha \sum_{n=1}^{\infty} C_{-\gamma}^{2n} \int_0^1 y^{2n} \frac{C_\alpha}{y^{1+\alpha}}\, \mathrm{d}y$$

$$= 2(1-\gamma c_i)x^{-\gamma} \sigma_i^\alpha \sum_{n=1}^{\infty} C_{-\gamma}^{2n} \frac{C_\alpha}{2n-\alpha}.$$

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Observing that $A_{32}(x,i)=(1-\gamma c_i)x^{-\gamma}\sigma_i^{\alpha}\int_{-1}^{1}\gamma y I_{\{|y|\le\sigma_i K_i\}}\frac{C_\alpha}{|y|^{1+\alpha}}\,\mathrm{d}y=0$, we get

$$A_3(x,i)=2(1-\gamma c_i)x^{-\gamma}\sigma_i^{\alpha}\sum_{n=1}^{\infty}C_{-\gamma}^{2n}\frac{C_\alpha}{2n-\alpha}. \tag{9}$$

Substituting the equalities (8) and (9) into (7) gives

$$\begin{aligned}\mathscr{A}V(x,i)=\gamma(1-\gamma c_i)\Bigg[&\left(a_i-\sum_{j=1}^{N}q_{ij}c_j+\sum_{n=1}^{+\infty}C_{\gamma}^{2n}\frac{2\sigma_i^{\alpha}C_\alpha}{\gamma(2n-\alpha)}\right)x^{\gamma}-b_ix^{\gamma+1}\\&+b_ix^{1-\gamma}-\left(a_i+\sum_{j=1}^{N}q_{ij}c_j-\sum_{n=1}^{\infty}C_{-\gamma}^{2n}\frac{2C_\alpha\sigma_i^{\alpha}}{\gamma(2n-\alpha)}\right)x^{-\gamma}\Bigg].\end{aligned}$$

Letting $\gamma\to 0$, we obtain

$$\lim_{\gamma\to 0}\sum_{n=1}^{+\infty}C_{\gamma}^{2n}\frac{2\sigma_i^{\alpha}C_\alpha}{\gamma(2n-\alpha)}=-\sum_{n=1}^{+\infty}\frac{C_\alpha\sigma_i^{\alpha}}{n(2n-\alpha)},$$

$$\lim_{\gamma\to 0}\sum_{n=1}^{+\infty}C_{-\gamma}^{2n}\frac{2C_\alpha\sigma_i^{\alpha}}{\gamma(2n-\alpha)}=\sum_{n=1}^{+\infty}\frac{C_\alpha\sigma_i^{\alpha}}{n(2n-\alpha)}.$$

It follows from these limits that

$$\begin{aligned}\mathscr{A}V(x,i)&=\gamma(1-\gamma c_i)\left(a_i-\sum_{j=1}^{N}q_{ij}c_j-\sum_{n=1}^{N}\frac{C_\alpha\sigma_i^{\alpha}}{n(2n-\alpha)}+o(\gamma)\right)x^{\gamma}\\&\quad-(1-\gamma c_i)b_ix^{\gamma+1}+(1-\gamma c_i)b_ix^{1-\gamma}\\&\quad-(1-\gamma c_i)\gamma\left(a_i-\sum_{n=1}^{+\infty}\frac{C_\alpha\sigma_i^{\alpha}}{n(2n-\alpha)}+\sum_{j=1}^{N}q_{ij}c_j+o(\gamma)\right)x^{-\gamma}\\&=\gamma(1-\gamma c_i)\left(a_i-\sum_{j=1}^{N}q_{ij}c_j-\sum_{n=1}^{N}\frac{C_\alpha\sigma_i^{\alpha}}{n(2n-\alpha)}+o(\gamma)\right)x^{\gamma}\\&\quad-(1-\gamma c_i)b_ix^{\gamma+1}+(1-\gamma c_i)b_ix^{1-\gamma}\\&\quad-(1-\gamma c_i)\gamma\,(\beta+o(\gamma))\,x^{-\gamma}.\end{aligned} \tag{10}$$

Therefore, there exists a sufficiently large $N_1>0$ such that, for all $x>N_1$ and $x<\frac{1}{N_1}$, $i\in\mathbb{S}$, it follows that

$$\mathscr{A}V(x,i)\le -1. \tag{11}$$

Again, by equality (10), for any $x\in\mathbb{R}^+$, $i\in\mathbb{S}$, there exists a positive constant C such that $\mathscr{A}V(x,i)\le C$. By Itô's formula [2], we get

$$V(X_t,r_t)=V(X_0,r_0)+\int_0^t\mathscr{A}V(X_s,r_s)\,\mathrm{d}s+M_t,$$

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where

$$M_t = \int_0^t \left(V(X_s + \sigma_{r_s} X_s z, r_s) - V(X_s, r_s) - V_x(X_s, r_s)\sigma_{r_s} X_s z I_{\{|z|<1\}}\right) \widetilde{N}^{r_s}(\mathrm{d}s, \mathrm{d}z).$$

Note that the process (M_t) is a local martingale. Let T_n be a localizing sequence, such that $M_{t\wedge T_n}$ is a martingale in t for any fixed n. By the Itô formula again, we have

$$V(X_{t\wedge T_n\wedge\tau_U}, r_{t\wedge T_n\wedge\tau_U}) = V(X_0, r_0) + \int_0^{t\wedge T_n\wedge\tau_U} \mathscr{A}V(X_s, r_s)\,\mathrm{d}s + M_{t\wedge T_n\wedge\tau_U}.$$

For any $(x_0, r_0) \in U^c \times \mathbb{S}$, it follows that

$$\mathbb{E}V(X_{t\wedge T_n\wedge\tau_U}, r_{t\wedge T_n\wedge\tau_U}) = V(X_0, r_0) + \mathbb{E}\int_0^{t\wedge T_n\wedge\tau_U} \mathscr{A}V(X_s, r_s)\,\mathrm{d}s. \tag{12}$$

Thus, applying the dominated convergence theorem yields

$$\lim_{n\to+\infty} \mathbb{E}\int_0^{t\wedge T_n\wedge\tau_U} \mathscr{A}V(X_s, r_s)\,\mathrm{d}s = \mathbb{E}\int_0^{t\wedge\wedge\tau_U} \mathscr{A}V(X_s, r_s)\,\mathrm{d}s. \tag{13}$$

Substituting (13) into (12) gives

$$\lim_{n\to+\infty} \mathbb{E}V(X_{t\wedge T_n\wedge\tau_U}, r_{t\wedge T_n\wedge\tau_U}) = V(X_0, r_0) + \mathbb{E}\int_0^{t\wedge\tau_U} \mathscr{A}V(X_s, r_s)\,\mathrm{d}s.$$

It follows from Fatou's lemma that

$$\begin{aligned}
\mathbb{E}V(X_{t\wedge\tau_U}, r_{t\wedge\tau_U}) &= \mathbb{E}\lim_{n\to+\infty} V(X_{t\wedge T_n\wedge\tau_U}, r_{t\wedge T_n\wedge\tau_U}) \\
&= \mathbb{E}\liminf_{n\to+\infty} V(X_{t\wedge T_n\wedge\tau_U}, r_{t\wedge T_n\wedge\tau_U}) \\
&\le \liminf_{n\to+\infty} \mathbb{E}V(X_{t\wedge T_n\wedge\tau_U}, r_{t\wedge T_n\wedge\tau_U}) \\
&= \lim_{n\to+\infty} \mathbb{E}V(X_{t\wedge T_n\wedge\tau_U}, r_{t\wedge T_n\wedge\tau_U}) \\
&= V(x, i) + \mathbb{E}\int_0^{t\wedge\tau_U} \mathscr{A}V(X_s, r_s)\,\mathrm{d}s,
\end{aligned}$$

which together with (11) implies that, for all $(X_0 = x, r_0 = i) \in U^c \times \mathbb{S}$, $\mathbb{E}(\tau_U \mid X_0 = x_0, r_0 = i_0) < V(x, i) < +\infty$. Thus, condition (ii) of Lemma 2 is verified. It follows from Lemma 2 that the desired result holds. □

Theorem 1 provides a sufficient condition to check the ergodicity of the model (1) for $1 < \alpha < 2$. It is a natural question whether, when α tends to 2, this condition simplifies the condition established in [13] for the Brownian motion case. The following corollary will answer this question.

Corollary 1. *Let Assumptions 1–4 hold. If α tends to 2, then the condition*

$$\sum_{i=1}^N \pi_i \left(a_i - \sum_{n=1}^\infty \frac{C_\alpha \sigma_i^\alpha}{n(2n-\alpha)}\right) > 0$$

reduces to $\sum_{i=1}^N \pi_i(a_i - \sigma_i^2) > 0$.

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Proof. Using the fact that $\lim_{x\to 0^+} x\Gamma(x) = 1$ yields $\lim_{\alpha\to 2^-} C_\alpha = 0$, $\lim_{\alpha\to 2^-} \frac{C_\alpha}{2-\alpha} = 1$. Therefore, it follows that, for any $i \in \mathbb{S}$,

$$\lim_{\alpha\to 2} \sum_{n=1}^{N} \frac{C_\alpha \sigma_i^\alpha}{n(2n-\alpha)} = \sigma_i^2.$$

This implies the desired result. □

Although for truncated stable processes corresponding to the Lévy jump measure given by Assumption 4 a sufficient condition is given in Theorem 1, it is worth considering what happens if the jump measure in Assumption 4 is replaced by Assumption 5. Next, we consider the jump process $\Delta Z_t =: Z_t - Z_{t^-}$ under Assumption 5.

Theorem 2. *Let Assumptions 1–3 and 5 hold. If*

$$\sum_{i=1}^{N} \pi_i \left(a_i - \sum_{n=1}^{+\infty} \frac{C_\alpha \sigma_i^\alpha}{n(2n-\alpha)} + \sigma_i^\alpha \frac{C_\alpha}{\alpha} \right) > 0,$$

then the SDE model (1) is ergodic and has a unique stationary distribution.

Proof. The proof is similar that of Theorem 1. For simplicity, we only provide the part where it differs. Using the Taylor formula we have, for $y \in (1, +\infty)$, $\gamma > 0$,

$$(1+y)^\gamma = \sum_{j=0}^{+\infty} C_\alpha C_\gamma^j y^{-j+\gamma}, \qquad (1+y)^{-\gamma} = \sum_{j=0}^{+\infty} C_\alpha C_\gamma^j y^{-j-\gamma}.$$

Hence, it follows that

$$I_1 := \int_1^{+\infty} \left((1+y)^\gamma - 1\right) \frac{C_\alpha \, \mathrm{d}y}{y^{1+\alpha}} = \sum_{j=0}^{+\infty} C_\gamma^j \frac{C_\alpha}{j+\alpha-\gamma} - \frac{C_\alpha}{\alpha}, \tag{14}$$

$$I_2 := \int_1^{+\infty} \left((1+y)^{-\gamma} - 1\right) \frac{C_\alpha \, \mathrm{d}y}{y^{1+\alpha}} = \sum_{j=0}^{+\infty} C_{-\gamma}^j \frac{C_\alpha}{j+\alpha+\gamma} - \frac{C_\alpha}{\alpha}. \tag{15}$$

Rewriting the limits in (14) and (15), we get $I_1 = I_2 = -\frac{C_\alpha}{\alpha} + o(\gamma)$.

Set the column vector $\overline{\mu} = (\overline{\mu}_1, \ldots, \overline{\mu}_N)^\top \in \mathbb{R}^N$ with

$$\overline{\mu}_i = \sum_{n=1}^{+\infty} \frac{C_\alpha \sigma_i^\alpha}{n(2n-\alpha)} - \sigma_i^\alpha \frac{C_\alpha}{\alpha} - a_i.$$

Define

$$\overline{\beta} = -\pi\overline{\mu} = \sum_{i=1}^{N} \pi_i \left(a_i - \sum_{n=1}^{+\infty} \frac{C_\alpha \sigma_i^\alpha}{n(2n-\alpha)} + \sigma_i^\alpha \frac{C_\alpha}{\alpha} \right).$$

By the conditions of Theorem 2, we have $\overline{\beta} > 0$. Using the same discussion as for Theorem 1 yields $\overline{\beta} = \sum_{j=1}^{N} q_{ij} c_j - \overline{\mu}_i$, $i \in \mathbb{S}$, where $c = (c_1, \ldots, c_N)$ is a solution of the Poisson equation $Qc = \overline{\mu} + \overline{\beta}\mathbf{1}$.

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Let $V(x, i)$ be the Lyapunov function defined in Theorem 1. Similar to the discussion of Theorem 1, and together with the equalities (14) and (15), it follows that

$$\begin{aligned}\mathscr{A}V(x,i) = \gamma(1-\gamma c_i)\left(a_i - \sum_{j=1}^{N} q_{ij}c_j - \sum_{n=1}^{N}\frac{C_\alpha \sigma_i^\alpha}{n(2n-\alpha)} - \sigma_1^\alpha \frac{C_\alpha}{\alpha} + o(\gamma)\right)x^\gamma \\ - (1-\gamma c_i)b_i x^{\gamma+1} + (1-\gamma c_i)b_i x^{1-\gamma} \\ - (1-\gamma c_i)\gamma\left(\overline{\beta} + o(\gamma)\right)x^{-\gamma}.\end{aligned}$$

Since $\overline{\beta} > 0$ and $b_i > 0$, it follows from Lemma 1 that the desired result holds. □

Next, we will rewrite Theorem 2 to compare with the case $\alpha = 2$. Let $D_\alpha = \sum_{n=2}^{+\infty}\frac{C_\alpha\sigma_i^\alpha}{n(2n-\alpha)} - \sigma_i^\alpha\frac{C_\alpha}{\alpha}$.

Corollary 2. *Let Assumptions 1–3 and 5 hold. If* $\sum_{i=1}^{N}\pi_i\left(a_i - \frac{C_\alpha\sigma_i^\alpha}{2-\alpha}\right) > D_\alpha$, *then the SDE model* (1) *is ergodic and has a unique stationary distribution.*

Proof. From Theorem 2, we can immediately see that

$$a_i - \sum_{n=1}^{+\infty}\frac{C_\alpha\sigma_i^\alpha}{n(2n-\alpha)} + \sigma_i^\alpha\frac{C_\alpha}{\alpha} = a_i - \frac{C_\alpha\sigma_i^\alpha}{2-\alpha} - D_\alpha.$$

Note that

$$\sum_{n=2}^{\infty}\frac{1}{n(2n-\alpha)} = -\frac{2}{\alpha}\sum_{n=2}^{\infty}\left(\frac{1}{2n} - \frac{1}{2n-\alpha}\right). \tag{16}$$

For $1 < \alpha < 2$, it follows that

$$\sum_{n=2}^{\infty}\left(\frac{1}{2n} - \frac{1}{2n-\alpha}\right) < -\frac{1}{2}. \tag{17}$$

Equations (16) and (17) yield that, for any $i \in \mathbb{S}$,

$$D_\alpha = \sigma_i^\alpha C_\alpha\left(-\frac{1}{\alpha} + \sum_{n=2}^{\infty}\frac{1}{n(2n-\alpha)}\right) > 0. \tag{18}$$

Consequently, Theorem 7 combined with (18) implies the desired result. □

From Corollary (2), letting α tend to 2 gives the following result.

Corollary 3. *Let Assumptions 1–3 and 5 hold. If* α *tends to 2 and* $\sum_{i=1}^{N}\pi_i(a_i - \sigma_i^2) > 0$, *then the solution of the SDE* (1) *is ergodic.*

Proof. By the proof of Corollary 1, we have $\lim_{\alpha\to 2^-}\frac{C_\alpha}{2-\alpha} = 1$. It follows from this and Corollary 7 that the desired result holds. □

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4. Extinction

In this section we find sufficient conditions such that the solution of the SDE (1) tends to zero almost surely. In the terminology of population dynamics, we shall derive under what conditions the population dynamics will go extinct.

Theorem 3. *Let Assumptions 1–4 hold. If* $\sum_{i=1}^{N} \pi_i \left(a_i - \sum_{n=1}^{+\infty} \frac{C_\alpha \sigma_i^\alpha}{n(2n-\alpha)} \right) < 0$, *then the SDE model* (1) *will be extinct with probability 1.*

Proof. The basic idea is to adopt the Lyapunov method together with the Poisson equation. We construct a Lyapunov function $V(x, i) : \mathbb{R}^+ \times \mathbb{S} \to \mathbb{R}$, that is,

$$\lim_{x \to 0^+} V(x, i) = 0, \qquad \lim_{x \to +\infty} V(x, i) = +\infty.$$

Then, we prove that, for any $(x, i) \in \mathbb{R}^+ \times \mathbb{S}$, $\lim_{t \to +\infty} \mathbb{E}[V(X_t, r_t) \mid X_0 = x, r_0 = i] = 0$, which implies the desired result. For simplicity, we use the same notation as the proof of Theorem 1. Recall that $\mu = (\mu_1, \mu_2, \ldots, \mu_N)^\top \in \mathbb{R}^N$, with $\mu_i = \sum_{n=1}^{+\infty} \frac{C_\alpha \sigma_i^\alpha}{n(2n-\alpha)} - a_i$. Thus,

$$-\beta = \pi \mu = -\sum_{i=1}^{N} \pi_i \left(a_i - \sum_{n=1}^{+\infty} \frac{C_\alpha \sigma_i^\alpha}{n(2n-\alpha)} \right).$$

By the conditions of Theorem 3, we have $\beta < 0$. Let $\mathbf{1} = (1, \ldots, 1)^\top$. Then, $\pi(-\mu - \beta \mathbf{1}) = \beta - \beta = 0$. Again from [16, Lemma A.12], this implies that the linear equation $Qc = -\mu - \beta \mathbf{1}$ has a solution $c = (c_1, c_2, \ldots, c_N)^\top \in \mathbb{R}^N$. Thus, we have

$$\beta = -\mu_i - \sum_{j=1}^{N} q_{ij} c_j, \qquad i \in \mathbb{S}.$$

For each $i \in \mathbb{S}$, consider the Lyapunov function $V : \mathbb{R}^+ \times \mathbb{S} \to \mathbb{R}$ such that $V(x, i) = (1 - \gamma c_i) x^\gamma$, where $\gamma \in (0, \alpha - 1)$ is sufficiently small that $1 - \gamma c_i > 0$ for each $i \in \mathbb{S}$. Using the same discussion as in Theorem 1, we obtain

$$\begin{aligned} \mathscr{A} V(x, i) &= \gamma (1 - \gamma c_i) \left(a_i - \sum_{j=1}^{N} q_{ij} c_j - \sum_{n=1}^{N} \frac{C_\alpha \sigma_i^\alpha}{n(2n-\alpha)} + o(\gamma) \right) x^\gamma \\ &\quad - (1 - \gamma c_i) b_i x^{\gamma+1} \\ &= \gamma (1 - \gamma c_i) \left(\beta + o(\gamma) \right) x^\gamma - (1 - \gamma c_i) b_i x^{\gamma+1}. \end{aligned}$$

Note that $\beta < 0$. It follows that, for any $(x, i) \in \mathbb{R}^+ \times \mathbb{S}$, $\mathscr{A} V(x, i) < 0$. Similar to (12) and (13), it can be shown that

$$0 \le \mathbb{E} V(X_t, r_t) \le V(x, i) + \mathbb{E} \int_0^t \mathscr{A} V(X_s, r_s) \, \mathrm{d}s.$$

By the non-negativity of $V(x, i)$, we immediately obtain the desired result. □

Next, we reconsider Theorem 3 if Assumption 4 is replaced by Assumption 5.

Theorem 4. *Let Assumptions 1–3 and 5 hold. If* $\sum_{i=1}^{N} \pi_i \left(a_i - \sum_{n=1}^{+\infty} \frac{C_\alpha \sigma_i^\alpha}{n(2n-\alpha)} + \sigma_i^\alpha \frac{C_\alpha}{\alpha} \right) < 0$, *then the SDE model* (1) *is extinct with probability 1.*

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Proof. The proof of this theorem is the same as Theorems 3 and 2, and is not detailed here. □

Remark 5. Letting $\alpha \to 2$ reproduces the conditions for Brownian motion.

5. Conclusion

We have focused on the population dynamics driven by two classes of truncated α-stable processes with Markovian switching. If the coefficient of the noise is small, the stochastic dynamic has a unique stationary distribution. However, if the coefficient is sufficiently large, this stochastic dynamic system will become extinct with probability 1. Also, Assumptions 4 and 5 show the effect of restricted or unrestricted immigration on the population dynamics.

Appendix A

To prove Lemma 2, we first prove a maximum principle for operators defined by (3). Assume for the following that $U \subset \mathbb{R}^+$ is open and bounded, with closure $\overline{U} \subset \mathbb{R}^+$. Let $U^c = \mathbb{R}^+ - U$. For simplicity, assume that $U = \left(\frac{1}{N_1}, N_1\right)$, where $N_1 > 0$. Consider the equation

$$\begin{cases} \mathscr{A}u(x, i) = 0, \text{in } (x, i) \in U \times \mathbb{S}, \\ u(x, i) = c_1, \text{ in } (x, i) \in (N_1, +\infty) \times \mathbb{S}, \\ u(x, i) = c_2, \text{ in } (x, i) \in (0, \frac{1}{N_1}) \times \mathbb{S}, \end{cases} \tag{19}$$

where c_1, c_2 are two non-negative constants and $c_1 \neq c_2$.

Lemma 3.

(i) *Let Assumptions 1–4 hold. If a smooth function $u(\cdot, i)$ is a solution to (19), then the maximum of u cannot be attained in the interior of U if u is not a constant function within $U \times \mathbb{S}$.*

(ii) *Let Assumptions 1–3 and 5 hold. If a smooth function $u(\cdot, i)$ is a solution to (19), then the maximum of u cannot be attained in the interior of U if u is not a constant function within $U \times \mathbb{S}$.*

Proof. For simplicity, we divide the proof into two steps.

Step 1. Assume that $\mathscr{A}u(x, i) > 0$ for all $(x, i) \in U \times \mathbb{S}$. Let us prove that the maximum of u cannot be attained in the interior. If $u(\cdot, \cdot)$ attains a maximum at $(x_0, i_0) \in U \times \mathbb{S}$, then

$$u_x(x_0, i_0) = 0. \tag{20}$$

Since the measure $\nu(\cdot, i)$ is symmetric, we can rewrite the integral term on the right-hand side of (3) as

$$I(x, i) := \text{p.v.} \int_{\mathbb{R}-\{0\}} (u(x + \sigma_i xz, i) - u(x, i))\, \nu(\mathrm{d}z, i),$$

where p.v. stands for the Cauchy principal value.

Noting that $u(x_0, i_0)$ is the maximum value on the domain $U \times \mathbb{S}$, we have

$$I(x_0, i_0) = \int_U (u(x_0 + \sigma_i x_0 z, i) - u(x_0, i_0))\, \nu(\mathrm{d}z, i_0) \leq 0. \tag{21}$$

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For the term, we have

$$Qu(x_0,\cdot)(i_0) = q_{i_0i_0}u(x_0,i_0) + \sum_{j\neq i_0} q_{i_0j}u(x_0,j)$$
$$\leq q_{i_0i_0}u(x_0,i_0) + \sum_{j\neq i_0} q_{i_0j}u(x_0,i_0) = 0. \tag{22}$$

It follows from (3) and (20)–(22) that $\mathscr{A}u(x_0,i_0)\leq 0$. This is a contradiction with $\mathscr{A}u(x_0,i_0)>0$.

Step 2. Now we consider u such that $\mathscr{A}u(x,i)\geq 0$ for all $x\in U$, $i\in\mathbb{S}$. To proceed, we consider two cases.

Case 1: Assumptions 1–4 hold. We define the function $u_\varepsilon(x,i)=u(x,i)+\varphi(x,i)$, where $\varphi(\cdot,i)\in C^2(\mathbb{R}^+)$ and $\varphi(x,i)=Kx^2$, where K is a positive constant to be determined. Since $\overline{U}$ is a bounded and closed set and $b(\cdot,i)$ is continuous on $\overline{U}$, there exists a constant $C>0$ such that, for any $x\in\overline{U}$, $i\in\mathbb{S}$, $|2xb(x,i)|<C$. Without loss of generality, we assume that $U=(\varepsilon_0,1/\varepsilon_0)$, where ε_0 is any positive real number. Choosing $K=\frac{(C+1)(2-\alpha)}{2\sigma_i^2\varepsilon_0^2C_\alpha\sigma_i^{\alpha-2}}$, we get, for any $x\in U$, $i\in\mathbb{S}$,

$$\begin{aligned}\mathscr{A}u_\varepsilon(x,i) &= \mathscr{A}u(x,i) + b(x,i)\varphi_x(x,i)\\ &\quad + \int_{|z|<\frac{1}{\sigma_i}} (\varphi(x+\sigma_ixz,i)-\varphi(x,i)-\varphi_x(x,i)\sigma_ixz)\frac{C_\alpha\,\mathrm{d}z}{|z|^{1+\alpha}}\\ &= \mathscr{A}u(x,i) + 2xb(x,i) + 2K\sigma_i^2x^2\frac{C_\alpha\sigma_i^{\alpha-2}}{2-\alpha}\\ &\geq \mathscr{A}u(x,i) - C + 2K\sigma_i^2x^2\frac{C_\alpha\sigma_i^{\alpha-2}}{2-\alpha}\geq 1.\end{aligned}$$

Now, the desired result follows from Step 1.

Case 2: Assumptions 1–3 and 5 hold. For any $\varepsilon>0$, define the function $u_\varepsilon(x,i)=u(x,i)+\varphi(x,i)$, where $\varphi(\cdot,i)\in C^2(\mathbb{R}^+)$ and

$$\varphi(x,i)=\begin{cases}x^2, & \text{if } 0\leq x<2R,\\ 8R^2, & \text{if } x>2R+1,\end{cases}$$

where R is a constant to be determined. Hence, we have, for any $x\in U$, $i\in\mathbb{S}$,

$$\begin{aligned}\mathscr{A}u_\varepsilon(x,i) &= \mathscr{A}u(x,i) + b(x,i)\varphi_x(x,i)\\ &\quad + C_\alpha\sigma_i^\alpha x^\alpha\int_{B_R(0)\bigcap(-1,+\infty)}(\varphi(x+z,i)-\varphi(x,i)-\varphi_x(x,i)z)\frac{\mathrm{d}z}{|z|^{1+\alpha}}\\ &\quad + C_\alpha\sigma_i^\alpha x^\alpha\int_{1\leq z\leq R}\varphi_x(x,i)\frac{\mathrm{d}z}{|z|^{1+\alpha}} + \int_{B_R^c(0)\bigcap(-1,+\infty)}(\varphi(x+z)-\varphi(x))\,\frac{\mathrm{d}z}{|z|^{1+\alpha}}>0.\end{aligned}$$

Note that, for $1<\alpha<2$, $x\in U$. For any $U=\left(\varepsilon_0,\frac{1}{\varepsilon_0}\right)$, the term

$$\begin{aligned}I &:= C_\alpha\sigma_i^\alpha x^\alpha\int_{z>-1}(\varphi(x+z,i)-\varphi(x,i)-\varphi_x(x,i)z)\frac{\mathrm{d}z}{|z|^{1+\alpha}}\\ &\geq 2\inf_{x\in U}C_\alpha\sigma_i^\alpha x^\alpha\int_{z>-1}z^2\frac{\mathrm{d}z}{|z|^{1+\alpha}}=+\infty.\end{aligned}$$

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Set $K = \inf_{x\in U, i\in\mathbb{S}} b(x,i)\varphi_x(x,i)$. Thus, there exists a sufficiently large R such that $I > |K| + 1$. Since sufficiently large R, $x \in U \subset \mathbb{R}^+$, we have $\varphi(x+z,i) - \varphi(x,i) > 0$ for all $z \in B_{R^c} \bigcap (-1, +\infty)$, and $\varphi_x > 0$. Hence, it follows that $\mathscr{A}u_\varepsilon(x,i) > 1$. Applying the result of Step 1, the maximum of u_ε is attained on $\partial U \times \mathbb{S}$, say at (x_ε, i). Since ∂U is compact, choosing a convergent subsequence of x_ε as ε tends to 0 through a sequence, it follows that the maximum of u is attained on ∂U. Finally, if $\mathscr{A}u(x,i) = 0$ for $x \in U$, $i \in \mathbb{S}$, then applying the above results to u and $-u$, we obtain that u attains both maximum and minimum on $\partial U \times \mathbb{S}$. □

Next, we will prove the uniqueness of the stationary distribution of the process (X_t, r_t). Lemma 2 can be viewed as a degenerate noise coefficient matrix of [5]. By the maximum principle, following almost the same argument as [17, Theorem 4.10] together with the solution to the Dirichlet problem (cf. [1]), we can prove the existence of the stationary distribution of the process (X_t, r_t). To proceed, we introduce some notation. Let (x, i) be the initial point of the process (X_t, r_t), and $y \in \mathbb{R}^+$ be any point $(x \neq y)$. Set $k > \max\{|x|, |y| + 1\}$. Without loss of generality, assume that $0 < \varepsilon < 1$. Choose c such that $0 < c < k$ and $\overline{B}_c(0) \cap B_\varepsilon(y) = \emptyset$. Define the stopping times $\{\tau_n\}_{n\geq 1}$ inductively as follows. Let $\tau_0 = 0$. For $n \in N^+$, set

$$\tau_{2n+1} = \inf\{s > \tau_{2n} : X_s \notin B_k(0)\},$$
$$\tau_{2n+2} = \inf\{s > \tau_{2n-1} : (X_s, r_s) \text{ hits the set } \overline{B}_c(0) \times \{i\}\}.$$

Proof of Lemma 2. We need to show the uniqueness of the invariant measure for the process (X_t, r_t). This result is a modification of [17, Theorems 4.1, 4.2; Corollary 4.4]. To proceed, the positive recurrence gives $\mathbb{E}_{x,i}(\tau_{2n} - \tau_{2n-2}) = \mathbb{E}_{x,i}\tau_2 < +\infty$. Hence, by the classical strong law of large numbers,

$$\frac{\tau_{2n}}{n} = \frac{\sum_{j=1}^n (\tau_{2n} - \tau_{2(n-1)})}{n} \to \mathbb{E}_{x,i}\tau_2,$$

almost surely.

Let f be a bounded real-valued measurable function on $\mathbb{R}^+ \times \mathbb{S}$. For $n \in \mathbb{N}^+$, set $Y_n = \int_{\tau_{2n}}^{\tau_{2n+2}} f(X_s, r_s)\,\mathrm{d}s$. Similar to [4, Proposition 12.1], we can prove that the sequences of the random variables $\{Y_n\}$ are independent and identically distributed. Then, we can see that, applying the strong law of large numbers to the sequence $\{Y_n\}_{n=1}^\infty$, we have

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=1}^n Y_j = \mathbb{E}_{x,i}Y_0 = \mathbb{E}_{x,i}\int_0^{\tau_2} f(X_s, r_s)\,\mathrm{d}s.$$

Similar to [4, Theorem 8.14], we have

$$\lim_{n\to+\infty} \frac{1}{\tau_{2n}}\int_0^{\tau_{2n}} f(X_s, r_s)\,\mathrm{d}s = \frac{1}{t}\int_0^t f(X_s, r_s)\,\mathrm{d}s.$$

Note that

$$\frac{1}{\tau_{2n}}\int_0^{\tau_{2n}} f(X_s, r_s)\,\mathrm{d}s = \frac{n}{\tau_{2n}}\frac{1}{n}\sum_{j=1}^n Z_j \to \frac{1}{\mathbb{E}_{x,i}\tau_2}\mathbb{E}_{x,i}\int_0^{\tau_2} f(X_s, r_s)\,\mathrm{d}s$$

with probability 1. Hence, we have

$$\mathbb{P}\left\{\lim_{t\to+\infty}\frac{1}{t}\int_0^t f(X_s, r_s)\,\mathrm{d}s = \sum_{j\in\mathbb{S}}\int_{\mathbb{R}^+} f(y,i)\mu(\mathrm{d}y, j)\right\} = 1. \tag{23}$$

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If $f(\cdot, i)$ is a bounded function with respect to x, then

$$\frac{1}{t}\int_0^t \mathbb{E}_{x,i} f(X_s, r_s)\,\mathrm{d}s = \mathbb{E}_{x,i}\left(\frac{1}{t}\int_0^t f(X_s, r_s)\,\mathrm{d}s\right).$$

By (23) and the dominated convergence theorem, we have

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}_{x,i} f(X_s, r_s)\,\mathrm{d}s = \sum_{j\in\mathbb{S}}\int_{\mathbb{R}^+} f(y, j)\mu(\mathrm{d}y, j).$$

In particular, for any $A \in \mathbb{B}(\mathbb{R}^+)$, $j \in \mathbb{S}$, taking $f(x, j) = I_{A\times j}$, we have

$$\frac{1}{t}\int_0^t \mathbb{P}(s, (x, i), A \times j)\,\mathrm{d}s \to \mu(A, j).$$

Last, we prove the uniqueness. Let $\mu_1(\cdot, \cdot)$ be another stationary distribution. Then

$$\sum_{i\in\mathbb{S}}\int_{\mathbb{R}^+} \mathbb{P}(s, (x, i), (A, j))\mu_1(\mathrm{d}x, i) = \mu_1(A, j). \tag{24}$$

Integrating (24) with respect to s from 0 to t, according to the limit (23), we have, for any $j \in \mathbb{S}$, $\mu_1(A, j) = \mu(A, j)$, which completes the proof.

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