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Infinitely divisible probabilities on $SL(2, \mathbb{C})$ are continuously embedded

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For any locally compact Hausdorff topological group G, let $M(G)$ denote the topological semigroup of all probability measures on G, where $M(G)$ is given the weak topology and the multiplication is convolution.

A measure $\mu \in M(G)$ is called *infinitely divisible* if and only if, for each natural number n, there exists $\nu \in M(G)$ such that $\nu^n = \mu$. A measure $\mu \in M(G)$ is said to be *continuously embedded* if and only if there is a continuous homomorphism $r \to \lambda_r$ of $\mathbb{R}^+$ (the positive reals) into $M(G)$ such that $\mu = \lambda_1$. The *embedding problem* is the problem of determining the class $\mathscr{E}$ of those locally compact groups G which have the property that any $\mu \in M(G)$ which is infinitely divisible is also continuously embedded. A number of authors have contributed to this problem in recent years, but it still remains unsolved in general. The concept of *strong root compactness* of the group G (see 3.1.10 of (1)) is the only known tool of general application that is available up to now for showing that a given group belongs to the class $\mathscr{E}$, but certainly this property fails to hold for many (connected) Lie groups, and in particular it fails for the groups $SL(2, \mathbb{C})$ and $SL(2, \mathbb{R})$. Nevertheless these groups do in fact belong to the class $\mathscr{E}$, as we now proceed to show. Unfortunately our proof is of rather an ad hoc nature, depending on the small dimension of the group, and it is not clear that it can be extended to other (connected) Lie groups.

For any locally compact G and $\mu \in M(G)$, we write $S(\mu)$ for the support of μ and $G(\mu)$ for the smallest closed subgroup of G that contains $S(\mu)$.

Lemma 1. *Let G be a locally compact group and let $\mu, \nu \in M(G)$ such that $\nu^n = \mu$. Then $G(\mu) \subseteq G(\nu) \subseteq N(G(\mu))$, where $N(G(\mu))$ is the normalizer of $G(\mu)$ in G. Further for any $a \in S(\nu)$, $a^n \in S(\mu)$ and $S(\nu) \subseteq aG(\mu)$.*

Proof. Recall from (1.2.1) of (1) that

$$S(\mu) = \overline{\underbrace{S(\nu)\, S(\nu) \dots S(\nu)}_{n}},$$

from which it is immediate that $G(\mu) \subseteq G(\nu)$ and that for $a \in S(\nu)$, $a^n \in S(\mu)$.

Now let $y \in S(\mu)$, $a \in S(\nu)$ and observe that

$$aya^{-1} = aya^{n-1}a^{-n} \in \overline{\underbrace{S(\nu)\, S(\nu) \dots S(\nu)}_{2n}}\, G(\mu) \subseteq G(\mu).$$

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Similarly $a^{-1}ya \in G(\mu)$, hence $S(\nu) \subseteq N(G(\mu))$, and so $G(\nu) \subseteq N(G(\mu))$.

If $f: N(G(\mu)) \to N(G(\mu))/G(\mu)$ is the canonical projection, then $f(\nu)^n = 1$ and so $f(\nu)$ is a point mass. Hence $S(\nu) \subseteq aG(\mu)$ for any $a \in S(\nu)$.

For any locally compact group G, and $A \subseteq G$, we write $\mathscr{L}(A, G)$ for the set of all y in G such that for every neighbourhood U of y, there is a compact set K (depending on U) such that $U \cap aKa^{-1} \neq \phi$, for all $a \in A$. We note that $\mathscr{L}(A, G)$ is always a closed subgroup of G. For $C \subseteq M(G)$ we write $\mathscr{F}(C, G)$ for the set of all $\lambda \in M(G)$ such that there exists $\nu \in M(G)$ such that $\lambda\nu = \nu\lambda \in C$.

LEMMA 2. *Let $\{\lambda_n : n \geqslant 1\}$ be a sequence in $\mathscr{F}(C, G)$, where C is a compact subset of $M(G)$, and let $\{x_n : n \geqslant 1\}$ be a sequence in G such that $\{\lambda_n x_n : n \geqslant 1\}$ is relatively compact. Then there is a subsequence $A = \{x_{n_r} : r \geqslant 1\}$ and some $\lambda \in C$, such that $S(\lambda) \subseteq \mathscr{L}(A, G)$.*

Proof. For each λ_n there exists ν_n such that $\lambda_n \nu_n = \nu_n \lambda_n = \mu_n \in C$, hence $\{x_n^{-1}\nu_n : n \geqslant 1\}$ is relatively compact by theorem 2·1 of chapter III of (3), and so $\{x_n^{-1}\mu_n x_n : n \geqslant 1\}$ is relatively compact. Since C is compact, there exists $\lambda \in C$ and a sequence $\{\mu_{n_r} : r \geqslant 1\}$ such that $\mu_{n_r} \to \lambda$. Suppose $y \in S(\lambda)$, let U be a compact neighbourhood of y, then $\lambda(U) > 0$. As $\mu_{n_r} \to \lambda$, there is some $\delta > 0$ and some r_0 such that $\mu_{n_r}(U) > \delta (\forall r \geqslant r_0)$. By compactness, there is some compact set $K \subseteq G$ such that $\mu_n(x_n K x_n^{-1}) > 1 - \delta$ $(\forall n \geqslant 1)$, hence

$$\forall r \geqslant r_0, \quad x_{n_r} K x_{n_r}^{-1} \cap U \neq \phi.$$

So if $K' = K \cup (\cup_{r=1}^{r_0} x_{n_r}^{-1} U x_{n_r})$, we have

$$x_{n_r} K' x_{n_r}^{-1} \cap U \neq \phi \quad (\forall r \geqslant 1)$$

which implies that $y \in \mathscr{L}(A, G)$, where $A = \{x_{n_r} : r \geqslant 1\}$.

LEMMA 3. *Let $G = SL(V)$, where V is a finite-dimensional real or complex vector space. If $A \subseteq G$ is unbounded, there is a proper linear subspace W of V (dependent on A) such that $x(W) \subseteq W$, for all $x \in \mathscr{L}(A, G)$.*

Proof. Let $\{x_n : n \geqslant 1\}$ be a sequence in A which has no convergent subsequence. Passing to a subsequence if necessary, we may assume, in view of proposition 3 of (2), that V has a proper linear subspace W such that every limit of every convergent subsequence of every sequence of the form $\{x_n z_n x_n^{-1} : n \geqslant 1\}$, where $\{z_n : n \geqslant 1\}$ is any bounded sequence in G, leaves W invariant.

Now suppose $a \in \mathscr{L}(A, G)$, then for any neighbourhood U of a there is a compact set K such that for all $n \geqslant 1$, $U \cap x_n K x_n^{-1} \neq \phi$. Hence there is a convergent sequence $\{k_r : r \geqslant 1\}$ in K, and a subsequence $\{x_{n_r} : r \geqslant 1\}$ such that $\{x_{n_r} k_r x_{n_r}^{-1} : r \geqslant 1\}$ converges to some $v \in U$. From the above paragraph it follows that $v \in H_W = \{x \in G : x(W) \subseteq W\}$, which means that every neighbourhood of a meets H_W. As H_W is closed, the result follows.

For $C \subseteq M(G)$, we write

$$R(C, G) = \bigcup_{n \geqslant 1} \{\nu^k : \nu \in M(G), \nu^n \in C, 1 \leqslant k \leqslant n\}$$

and we note that $R(C, G) \subseteq \mathscr{F}(C, G)$.

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LEMMA 4. *Let $G = SL(V)$, let $\mu \in M(G)$ and suppose that $G(\mu)$ acts irreducibly on V. Then $\mathscr{F}(\mu, G)$ is compact, and so $R(\mu, G)$ is relatively compact.*

If μ is also infinitely divisible over G, then μ is continuously embedded.

Proof. If $\{\lambda_n : n \geqslant 1\}$ is a sequence in $\mathscr{F}(\mu, G)$ then by theorem 2·2 of chapter III of (3) there is a sequence $\{x_n : n \geqslant 1\}$ in G such that $\{\lambda_n x_n : n \geqslant 1\}$ is relatively compact. By Lemma 2, there is a subsequence $A = \{x_{n_r} : r \geqslant 1\}$ such that $S(\mu) \subseteq \mathscr{L}(A, G)$, hence $G(\mu) \subseteq \mathscr{L}(A, G)$. By Lemma 3, A must be bounded, otherwise $G(\mu)$ leaves a proper subspace W invariant. Hence $\{\lambda_n : n \geqslant 1\}$ has a convergent subsequence, and so $\mathscr{F}(\mu, G)$ is compact.

The remaining statement follows as in the proof of proposition 11 of (2).

LEMMA 5. *Let $G = SL(2, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let H be a non-trivial subgroup of G. Then either*

(i) *H acts irreducibly on $\mathbb{F}^2$, or*

(ii) *$H = \{\pm I\}$, the centre of G, or*

(iii) *H is conjugate to a subgroup H_1 of G such that $N(H_1)$ lies inside the subgroup of upper-triangular matrices, or*

(iv) *H is conjugate to a subgroup H_2 of the diagonal subgroup of G, and $N(H_2) \subseteq K$, where K is the subgroup*

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ab = 0 = cd\right\}.$$

Proof. If H does not act irreducibly on $\mathbb{F}^2$, there is a one-dimensional subspace V such that $h(V) \subseteq V$, for all $h \in H$. If every element of $N(H)$ also leaves V invariant, then we clearly have case (iii), if this is not the case then there is some $k \in N(H)$ such that $k(V) \nsubseteq V$. This means that if v is a non-zero element of V, then $\{v, k(v)\}$ is a basis for $\mathbb{F}^2$, and since $k \in N(H)$, we have $k^{-1}hk \in H$, $\forall h \in H$. Hence $h(k(V)) \subseteq k(V)$, $\forall h \in H$, which means that $\{v, k(v)\}$ are eigenvectors for each $h \in H$. Hence H is conjugate to a subgroup H_2 of the diagonal group. An elementary computation now shows that either $H_2 = \{\pm I\}$ or else $N(H_2) \subseteq K$.

From now on, we use G_1 for the group of 2×2 upper-triangular matrices in $SL(2, \mathbb{F})$.

LEMMA 6. *If G_1 has the embedding property, so does $SL(2, \mathbb{F})$.*

Proof. Let $G = SL(2, \mathbb{F})$, suppose $\mu \in M(G)$ is infinitely divisible on G, then by Lemma 4, μ is continuously embedded unless $G(\mu)$ is reducible. By Lemmas 1 and 5, either μ is infinitely divisible on a subgroup conjugate to G_1, hence is continuously embedded by our hypothesis, or (*a*) $G(\mu) = \{\pm I\}$ or (*b*) some conjugate of μ (by an element of G) is infinitely divisible on the group K (of lemma 5) and supported on the diagonal subgroup. In case (*b*), we note that D has index 2 in K, where D is the diagonal subgroup, so if $\nu \in M(K)$ such that $\nu^{2n} = \mu$, then $S(\nu^2) \subseteq D$, and in fact μ is infinitely divisible on D. But $D \cong \mathbb{F}^*$ so is strongly root compact and hence μ is continuously embedded.

In case (*a*), for each $n \geqslant 1$ we pick $\nu_n \in M(G)$ such that $\nu_n^n = \mu$, and we note that if

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$x_n \in S(\nu_n)$, then by Lemma 1, $S(\nu_n) \subseteq \{\pm x_n\}$. Let A be a maximal compact subgroup of G, then $\exists y_n \in G$ such that $y_n x_n y_n^{-1} \in A$, and note that $\{\pm I\} \subseteq A$. Hence

$$S(y_n \nu_n y_n^{-1}) \subseteq A,$$

and since $(y_n \nu_n y_n^{-1})^n = \mu$, we see that μ is infinitely divisible on A. Since A is compact and so strongly root compact, we conclude that μ is continuously embedded.

From this point on we concentrate attention on the group G_1. We define $p: G_1 \to \mathbb{F}^*$ by

$$p\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a,$$

and note that p is a homomorphism whose kernel we denote by N, so $N \cong \mathbb{F}$. We denote by H the subgroup of G_1 generated by N and $-I$, and note that $H \cong \mathbb{F} \oplus \mathbb{Z}_2$. We denote by ϕ the homomorphism πp of G_1 onto $\mathbb{F}^*/\{\pm I\}$, where π is the natural homomorphism of $\mathbb{F}^*$ onto $\mathbb{F}^*/\{\pm I\}$.

LEMMA 7. *If $A \subseteq H$ is not bounded, then $\mathscr{L}(A, G_1) \subseteq H$.*

Proof. An argument as in the second paragraph of the proof of Lemma 3 shows that it suffices to prove that if $z_n \to z$ in G_1 and $\{h_n : n \geqslant 1\}$ is a sequence in H such that $h_n z_n h_n^{-1} \to l$ in G_1, with $\{h_n : n \geqslant 1\}$ having no convergent subsequence, then l belongs to H. But the latter assertion is easily shown by an elementary matrix computation.

COROLLARY. *Suppose $C \subseteq M(G_1)$ is compact and for each $\lambda \in C$, $S(\lambda) \not\subseteq H$. Then $R(C, G_1)$ is relatively compact in $M(G_1)$.*

Proof. Suppose $R(C, G_1)$ is not relatively compact and select a sequence $\{\lambda_n : n \geqslant 1\}$ in $R(C, G_1)$ which has no convergent subsequence. As usual we can find $\{x_n : n \geqslant 1\}$ in G_1 such that $\{\lambda_n x_n : n \geqslant 1\}$ is relatively compact, and we note that $\{p(\lambda_n) : n \geqslant 1\}$ is a subset of $R(p(C), \mathbb{F}^*)$, so is relatively compact by 3.1.13 of (1) and strong root compactness of $\mathbb{F}^*$. Hence $\{p(x_n) : n \geqslant 1\}$ is relatively compact, which implies that there is a sequence $\{h_n : n \geqslant 1\}$ in N such that $\{\lambda_n h_n : n \geqslant 1\}$ is relatively compact.

But $\{h_n : n \geqslant 1\}$ has no convergent subsequence, so passing to a subsequence if necessary, we conclude by Lemmas 2 and 7 that there exists $\lambda \in C$ with

$$S(\lambda) \subseteq \mathscr{L}(A, G_1) \subseteq H, \quad \text{where} \quad A = \{h_n : n \geqslant 1\}.$$

But we are given that $S(\lambda) \not\subseteq H$, and this contradiction leads to the conclusion that $R(C, G_1)$ is indeed relatively compact.

We have now done enough to finish the argument when the field is real.

THEOREM 8. *$SL(2, \mathbb{R})$ has the embedding property.*

Proof. In view of Lemma 7, Corollary, and Lemma 6, it suffices to consider an infinitely divisible $\mu \in M(G_1)$ such that $S(\mu) \subseteq H$. By Lemma 1, if $\nu \in M(G_1)$ such that $\nu^n = \mu$, then $\phi(\nu) \in \mathbb{R}^*/\{\pm I\}$ and $\phi(\nu)^n = 1$. Since $\mathbb{R}^+ \cong \mathbb{R}^*/\{\pm I\}$ and the only periodic element of $\mathbb{R}^+$ is 1, we conclude that $\phi(\nu) = 1$, which means that $S(\nu) \subseteq H$. So μ is infinitely divisible on H and since H is strongly root compact, this is sufficient to show that μ is continuously embedded.

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The complex case requires a little more work. From this point on it is assumed the field is complex.

LEMMA 9. *Let $\mu \in M(G_1)$ with $S(\mu) \subseteq H$ and suppose $\nu \in M(G_1)$ such that $\nu^n = \mu$ but $S(\nu) \not\subseteq H$. Then for each $y \in N$, $(\nu y)^n = \mu$.*

Proof. Let $S = C^*/\{\pm I\}$, we have $\phi\colon G_1 \to S$ and $\ker \phi = H$. For each $x \in S$ we have the automorphism θ_x of H (and of N) given by

$$\forall h \in H, \quad \theta_x(h) = yhy^{-1}, \quad \text{where} \quad y \in H \quad \text{s.t.}\ \phi(y) = x$$

and as H is abelian, this is independent of whichever y we select. As $S(\nu) \not\subseteq H$, we have $\phi(\nu) \neq 1$ and by Lemma 1 $\phi(\nu)^n = 1$. Hence $\phi(\nu) = a = \phi(b)$,

$$b = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

and $e^{ni\theta} = \pm 1$, but $e^{i\theta} \neq \pm 1$. We note that the effect of θ_a on $N \cong \mathbb{C}$ is to multiply by $e^{2i\theta}$, and $e^{2i\theta} \neq 1$. So the transformation $(I - \theta_a)$ on N is multiplication by $(1 - e^{2i\theta})$, and hence is invertible on N. But on N

$$(I + \theta_a + \ldots + \theta_a^{n-1})(I - \theta_a) = I - \theta_a^n = 0,$$

whence

$$(I + \theta_a + \ldots + \theta_a^{n-1})(x) = 0 \quad (\forall x \in N). \tag{1}$$

The measure $\lambda = b^{-1}\nu$ is supported on H, and

$$\begin{aligned} \mu = \nu^n &= (b\lambda)^n = (b\lambda b^{-1})(b^2\lambda b^{-2})\ldots(b^n\lambda b^{-n})\, b^n \\ &= z(b\lambda y b^{-1})(b^2\lambda y b^{-2})\ldots(b^n\lambda y b^{-n})\, b^n \\ &= z(b\lambda y)^n = z(\nu y)^n, \end{aligned}$$

where

$$\begin{aligned} z^{-1} &= (byb^{-1})(b^2yb^{-2})\ldots(b^nyb^{-n}) \\ &= 1 \quad \text{by} \quad (1). \end{aligned}$$

Hence $\mu = (\nu y)^n$ as required.

LEMMA 10. *Let $\lambda \in M(G_1)$ and $k, n \in N$ such that $S(\lambda^k) \not\subseteq H$ and $S(\lambda^{kn}) \subseteq H$. Then for each $h \in N$ there exists $t \in N$ such that $(\lambda t)^k = \lambda^k h$.*

Proof. We follow the notation of the proof of Lemma 9. Let $\mu = \lambda^{kn}$ and let $\nu = \lambda^k$. Select

$$d = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

such that $\phi(d) = \phi(\lambda)$ and note that $e^{ikn\theta} = \pm 1$ but $e^{ik\theta} \neq \pm 1$. Let $\gamma = d^{-1}\lambda$, then $\gamma \in M(H)$, and for every $t \in N$,

$$\begin{aligned} \nu = \lambda^k &= (d\gamma)^k \\ &= d^k(d^{-k+1}\gamma d^{k-1})(d^{-k+2}\gamma d^{k-2})\ldots(d^{-1}\gamma d)\,\gamma \\ &= d^k(d^{-k+1}\gamma t d^{k-1})(d^{-k+2}\gamma t d^{k-2})\ldots(d^{-1}\gamma t d)\,\gamma t h^{-1} \\ &= (d\gamma t)^k h^{-1}, \end{aligned}$$

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where

$$h = (I + \theta_c^{-1} + \theta_c^{-2} + \ldots + \theta_c^{-k+1})(t), \quad \text{with} \quad c = \phi(d).$$
$$= \theta_c^{-k+1}(I + \theta_c + \ldots + \theta_c^{k-1})(t). \tag{2}$$

But on N we have

$$(1 - \theta_c)(1 + \theta_c + \ldots + \theta_c^{k-1}) = 1 - \theta_c^k,$$

and $1 - \theta_c^k$ is an invertible transformation on N, because $e^{2ik\theta} \neq 1$. Hence

$$(1 + \theta_c + \ldots + \theta_c^{k-1})$$

is invertible, so for any given $h \in N$, the equation (2) can be solved for t. Then clearly $\nu h = (d\gamma t)^k$, hence $\nu h = (\nu t)^k$, as required.

LEMMA 11. *Suppose $\mu \in M(G_1)$, $m \in \mathbb{N}$, and for each $k \geqslant 1$, there exists $\lambda_k \in M(G_1)$ such that $S(\lambda^k) \not\subseteq H$ and $(\lambda_k)^{km} = \mu$. Then there is a measure $\nu \in M(G_1)$ such that ν is infinitely divisible, $\nu^m = \mu$ and $R(\nu, G_1)$ is relatively compact in $M(G_1)$.*

Proof. Let $\nu_k = \lambda_k^k$, then $\nu_k \in R(\mu, G_1)$ and as usual we can find $\{h_k : k \geqslant 1\}$ in G_1 such that $\{\nu_k h_k : k \geqslant 1\}$ is relatively compact. Using the homomorphism $p : G \to C^*$ and the strong root compactness of C^* we deduce that the sequence $\{h_k : k \geqslant 1\}$ may be assumed to lie in N. Let $C = \overline{\{\nu_k h_k : k \geqslant 1\}}$, then C is a compact set in $M(G_1)$ and for each $k \geqslant 1$, Lemma 9 shows that $(\nu_k h_k)^m = \nu_k^m = \mu$, whence $\nu^m = \mu$, for all $\nu \in C$. Furthermore $(\lambda_k)^k = \nu_k$, so by Lemma 10 there exists $t_k \in N$ such that $(\lambda_k t_k)^k = \nu_k h_k \in C$, hence $\forall k \geqslant 1$

$$R_k(C) = \{\alpha \in M(G_1) : \alpha^k \in C\} \neq \phi.$$

Note that by Lemma 1, each $\nu_k h_k$ is supported entirely on just one of at most $(m-1)$ different H-cosets ($\nu_k h_k$ cannot be supported on H by our assumption), from which we conclude that $S(\nu) \cap H = \phi$, for each $\nu \in C$. It follows from Lemma 7, Corollary, that $R(C, G_1)$ is relatively compact, and so $R_k(C)$ is relatively compact, for all $k \geqslant 1$.

Let $D_k = \{\alpha^k : \alpha \in R_k(C)\}$, then D_k is compact and clearly $\{D_k : k \geqslant 1\}$ has the finite intersection property, so $\bigcap_{k=1}^{\infty} D_k \neq \phi$. If we now select $\nu \in \bigcap_{k=1}^{\infty} D_k$, then clearly ν is infinitely divisible, $\nu^m = \mu$, and $R(\nu, G_1) \subseteq R(C, G_1)$ is relatively compact, as required.

THEOREM 12. *$SL(2, \mathbb{C})$ has the embedding property.*

Proof. By Lemma 7, Corollary, and Lemma 6, it suffices to consider an infinitely divisible $\mu \in M(G_1)$ such that $S(\mu) \subseteq H$. If μ is infinitely divisible on H, then μ is continuously embedded because H is strongly root compact. If not then there is some $m \geqslant 2$ such that no mth-root of μ in $M(G_1)$ is supported on H, and so $\exists$ for each $k \geqslant 1$ some $\nu_k \in M(G_1)$ such that $\nu_k^{km} = \mu$, but $S(\nu_k^k) \not\subseteq H$. Then if we select $\nu \in M(G_1)$ as in Lemma 11, we see that ν is continuously embedded by an argument as in proposition 11 of (2). So $\mu = \nu^m$ is continuously embedded.

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