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Improved estimates for continuous data dependence in linear elastodynamics

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1. *Introduction*

In a previous paper [**6**], the present authors established estimates for the continuous dependence of the solution on various data in the initial boundary value problem of linear elastodynamics on a bounded region of space. The main conclusion concerned continuous dependence on the body-force, but also it was shown how this result could be used to derive continuous dependence on the initial data, elasticities, boundary data and initial geometry. The method adopted was based upon logarithmic convexity arguments and hence led naturally to continuity in the sense of Hölder on compact sub-intervals of time. A special feature of the study entailed the lack of any sign-definiteness conditions on the elasticities which, of course, in the absence of any *a priori* constraint on the solution always gives rise to an ill-posed problem. (See, for instance, the comprehensive survey by Payne [**10**].)

Nonetheless, the results obtained in [**6**] are unsatisfactory in many respects. For instance, a higher degree of regularity than is strictly necessary is required of the solution and elasticities, while the perturbations in the solution and data are measured by somewhat weak norms. Such criticisms are probably less important when dealing with only classical or smooth solutions, but it is well known that in dynamical problems shocks and other singularities may evolve and so even for linear problems ultimate concern must always be with the behaviour of weak solutions. Of course, the technique of logarithmic convexity can be generalized to include treatment of the abstract problem, but deficiencies of the kind we have noted remain and accordingly improvements of the results established in [**6**] must probably be sought by other means.

The purpose of the present paper is therefore to reconsider the problem (still without any sign-definiteness assumption on the elasticities and still defined on a bounded spatial region) and to show how under suitable constraints a completely different approach based upon Lagrange identities enables improvements to be achieved. We examine continuous dependence on the body-force, initial data, elasticities, density and initial geometry, placing particular emphasis on the latter. Each set of data is discussed separately, which produces sharper individual estimates than if, following the procedure in [**6**], we were to deduce dependence on other data from the estimate giving dependence of the solution on the body-force. It does not follow, however, that we can derive by direct superposition a comprehensive

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estimate simultaneously furnishing continuous dependence on all constituent data since the solution for each component problem is not necessarily stabilized by a common constraint set. Furthermore, such sets are non-linear.

In each estimate, continuity is established in the sense of Hölder and we are able to relax many of the regularity and other assumptions introduced in [6]. However, in the first instance, our estimates are valid only on the half-interval of existence, and not on all compact sub-intervals. The extension of the interval over which our results hold requires the introduction of extra constraints, with the notion of continuity involved becoming successively weaker until in the limit, when the interval of validity coincides with that of maximal existence, Hölder continuity of the type used in [4, 6, 7] is reached.

The Lagrange identity was first systematically used in continuum physics by Brun [1] to discuss uniqueness and related topics in the linear dynamic theories of thermoelasticity and viscoelasticity. (Other applications to uniqueness include treatments by Naghdi and Trapp [9] and by Green [3].) The technique was later extended by Levine [8] to weak solutions and in part used by him to prove equipartition of energy. In the extension to weak solutions, the Lagrange identity method possesses a distinct advantage over logarithmic convexity since it is unnecessary separately to postulate conservation of energy. Moreover, the extension need not be restricted to a Hilbert space (cf. Payne [11]). On the other hand, the Lagrange identity appears to be limited to a consideration of first and second order linear autonomous self-adjoint differential systems whereas the techniques involving logarithmic convexity may be applied to more general classes of equations.

The Lagrange identity has been previously used by Payne [12] to obtain estimates giving dependence on initial data and initial geometry, similar to the results established here. However, for dependence on the initial geometry he assumed the elasticities to be negative-definite, whereas our relaxation of any definiteness assumption whatsoever renders this section of the analysis considerably more difficult. By combining the Lagrange identity with the method of weighted energy, Galdi, Knops and Rionero [2] were able to demonstrate continuous dependence upon initial data for unbounded regions. Their approach may be used to extend similarly the arguments of this paper. The Lagrange identity had been earlier utilized by Knops and Payne [7] to establish continuous dependence on initial data for solutions with constrained asymptotic behaviour. This constraint differs markedly from those usually imposed in the stabilization of ill-posed problems and which involve the boundedness of integrals over space-time. This is the kind introduced here.

Continuous dependence results are important because of the inevitable error that arises in both numerical computation and the physical measurement of data. It is vital to know the magnitude of the effect of such errors on the actual solution. Furthermore, with regard to the particular theory of elasticity, since positive-definiteness of the elasticities cannot be accurately established by laboratory or other requirements, it is reasonable to abandon this classical hypothesis in any fundamental investigation. At the very least, in endeavouring to determine by measurement whether the elasticities are positive-definite we clearly cannot incorporate this property as an assumption into any theory underlying the measurements.

The basic equations and notation used in our study, together with the derivation

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of the Lagrange identities in the forms suitable for our needs, are described in §2. For convenience, we forego an abstract formulation of the problem, and throughout the paper we confine attention to classical solutions although the techniques developed may be extended to weak solutions in an obvious manner using, for instance, the treatment by Levine [**8**]. In general, the Lagrange identity method is simple to use, and this facility is illustrated in §3 where separate estimates are determined for the continuous dependence of the solution on the body-force, initial data, elasticities and density. This section also acquaints the reader with the main steps of the argument and describes how the interval of dependence may be extended in the case of joint dependence on body-force and initial data. The extension in the case of other data is accomplished similarly. The discussion of dependence on initial geometry is contained in §4 . The derivation is considerably more complex, involving several subsidiary estimates, but apart from its detailed intricacy, in broad outline it is the same as that of §3.

We adopt the standard summation and comma notation and as already stated always assume the existence of a classical solution on some maximum time interval.

2. *Notation, basic equations and the Lagrange identities*

Let Ω, a bounded region of $\mathbb{R}^n$ ($n = 2, 3$) with smooth boundary $\partial\Omega$, be occupied by a linear elastic material of density $\rho(x) \geqslant \rho_0 > 0$, and elasticities $c_{ijkl}(x)$ where $x \in \Omega$. With respect to a fixed set of cartesian axes, the components $u_i(x,t)$ of the displacement at the point x and at time t are well-known to satisfy

$$(c_{ijkl}\, u_{k,l})_{,j} + \rho f_i = \rho \ddot{u}_i, \quad ((x,t) \in \Omega \times (0,T)), \tag{2·1}$$

$$u_i = 0, \qquad ((x,t) \in \partial\Omega \times (0,T)), \tag{2·2}$$

$$u_i(x,0) = g_i(x), \qquad \dot{u}_i(x,0) = h_i(x) \quad (x \in \Omega), \tag{2·3}$$

where $(0,T)$ is the maximal interval of existence, $f_i(x,t)$ is the body-force per unit mass, g_i and h_i are prescribed functions and a superposed dot indicates partial differentiation with respect to time. Boundary conditions other than (2·2) could be included in the following discussion but for simplicity are not considered. The elasticities are assumed to be continuously differentiable and to possess the major symmetry specified by

$$c_{ijkl} = c_{klij}. \tag{2·4}$$

They are not required to be sign-definite. Condition (2·4) renders the initial boundary value problem formally self-adjoint, but in the absence of positive-definite elasticities, the problem is well-known to be ill-posed. (See, for instance, Payne [**10**].)

We assume that on the interval $(0,T)$ a smooth solution exists to (2·1)–(2·3) subject to (2·4). This assumption implies that the given functions f_i, g_i and h_i are all sufficiently smooth and moreover satisfy appropriate continuity and compatibility conditions on $\partial\Omega$ at $t = 0$.

Later, in §4, when considering continuous dependence on initial geometry, we shall slightly enlarge the domain over which the initial boundary value problem (2·1)–(2·3) is defined.

We derive the basic Lagrange identity associated with (2·1) and then deduce various forms required in the subsequent discussion. It is convenient, however, to

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introduce first the inner product of two vector functions $v_i(x,t)$, $w_i(x,t)$, by means of the definition

$$(v(t_1), w(t_2)) = \int_\Omega \rho v_i(x,t_1)\, w_i(x,t_2)\, dx. \tag{2·5}$$

The corresponding norm is denoted by $\|v(t)\|$.

Let $v_1(x,t)$ and $v_2(x,t)$ be two solutions to the initial boundary value problem (2·1)–(2·3) corresponding respectively to distinct body-forces $f_1(x,t)$ and $f_2(x,t)$ and possibly different initial data. Then we may use the respective equations (2·1), followed by an integration by parts and the symmetry (2·4) to obtain the basic *Lagrange identity*

$$\int_{t_1}^{t} [(v_2(\eta), f_1(\eta)) - (v_1(\eta), f_2(\eta))]\, d\eta = [(v_2(\eta), v_{1,\eta}(\eta)) - (v_1(\eta), v_{2,\eta}(\eta))]\Big|_{t_1}^{t}, \tag{2·6}$$

where $0 \leqslant t_1 < t < T$.

Special choices of the functions v_1, v_2 lead to particular forms of the basic identity. Thus, on setting

$$v_1(x,t) = u(x,t), \qquad v_2(x,t) = \frac{\partial u}{\partial t}(x,t) \quad (0 \leqslant t < T), \tag{2·7}$$

where $u(x,t)$ is the solution to (2·1)–(2·3) with body-force $f(x,t)$, we have by uniqueness (cf. Brun [1]) that

$$f_1(x,t) = f(x,t), \quad f_2(x,t) = \frac{\partial f}{\partial t}(x,t). \tag{2·8}$$

Substitution into (2·6) and use of (2·1) followed by integration by parts leads immediately to the expression for conservation of energy:

$$K(t,t) + V(t,t) = K(t_1,t_1) + V(t_1,t_1) + 2\int_{t_1}^{t} (u_{,\eta}(\eta), f(\eta))\, d\eta \quad (0 \leqslant t_1 < t < T), \tag{2·9}$$

in which

$$K(t_1,t_2) = (\dot{u}(t_1), \dot{u}(t_2)), \tag{2·10}$$

$$V(t_1,t_2) = \int_\Omega c_{ijkl}\, u_{i,j}(x,t_1)\, u_{k,l}(x,t_2)\, dx. \tag{2·11}$$

Next, on putting for fixed t,

$$v_1(x,\tau) = u(x,\tau), \qquad v_2(x,\tau) = \frac{\partial u}{\partial \tau}(x, 2t-\tau) \quad (0 \leqslant \tau \leqslant 2t < T), \tag{2·12}$$

and observing that again by uniqueness we have

$$f_1(x,\tau) = f(x,\tau), \quad f_2(x,\tau) = \frac{\partial f}{\partial t}(x, 2t-\tau), \tag{2·13}$$

it follows by substitution in (2·6) together with appropriate integrations that we obtain the identity

$$K(t,t) - V(t,t) = K(2t-t_1,t_1) - V(2t-t_1,t_1) - \int_{t_1}^{2t-t_1} (u_{,\eta}(\eta), f(2t-\eta))\, d\eta \quad (0 \leqslant t_1 < 2t \leqslant T+t_1). \tag{2·14}$$

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We remark that elimination of the potential energy $V(t,t)$ between (2·9) and (2·14) (with $t_1 = 0$) forms the basis of the uniqueness proof due to Brun [1].

To derive the final form of the basic identity required in the sequel, we put

$$v_1(x,\tau) = u(x,\tau), \qquad v_2(x,\tau) = u(x,2t-\tau) \quad (0 \leqslant \tau < 2t \leqslant T), \tag{2·15}$$

which by uniqueness implies that

$$f_1(x,\tau) = f(x,\tau), \qquad f_2(x,\tau) = f(x,2t-\tau) \quad (0 \leqslant \tau < 2t \leqslant T). \tag{2·16}$$

Hence the identity (2·6) becomes

$$2(u(t),\dot{u}(t)) = \int_{t_1}^{t} [(u(2t-\eta),f(\eta)) - (u(\eta),f(2t-\eta))]\,d\eta + (u(2t-t_1),\dot{u}(t_1)) + (u(t_1),\dot{u}(2t-t_1)) \quad (0 \leqslant t_1 < 2t \leqslant T+t_1). \tag{2·17}$$

The various forms (2·9), (2·14) and (2·17) of the basic Lagrange identity (2·6) are all needed in the subsequent derivation of estimates for continuous dependence. Other forms are presented by Tartar [**13**].

3. *Continuous dependence on body-force, initial data, elasticities and density*

In order to illustrate the essential ideas of our approach, we derive in this section estimates for continuous dependence of the solution on body-force, initial data, elasticities and density. Continuous dependence on the latter three sets of data could be deduced from the estimate for continuous dependence on the body-force (cf. [**6**]), but the results in many respects are less satisfactory than the separate *ab initio* calculations undertaken here. However, as remarked in the Introduction, because the constraints are non-linear, superposition of the separate estimates is not always valid. We compare our estimates with those obtained in [**6**] by means of logarithmic convexity arguments and conclude that the main differences between the respective estimates are to be found in the measure of continuity and in the time-interval of validity. The approach based upon logarithmic convexity yields continuity valid on all compact sub-intervals of the maximal interval of existence $[0,T)$, while the present method leads in the first instance to continuity only on the half interval $[0,T/2]$. One task is therefore to extend the interval of continuous dependence. This may be achieved by various iterative procedures, which are chiefly characterized by the imposition of further constraints and a successive weakening of the notion of continuity occurring in each iterated estimate. We present in detail one such iterative scheme and then briefly mention others in order to convey the broad features of the approach. Several other such procedures will become readily apparent to the reader. We also describe a hybrid device which combines an inequality based upon logarithmic convexity with our estimates derived from the Lagrange identities. The general discussion concerning the extension of the interval of dependence is restricted to continuous dependence jointly on the body-force and initial data since, in principle, all other data may be treated in similar fashion.

We also show how, in the case of continuous dependence on the body-force, we may recover from our basic results the Hölder continuity estimates previously derived by logarithmic convexity arguments. Again, the discussion may be adapted to include other data.

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Apart from the features just mentioned, other differences between the two methods concern the constraint sets and regularity assumptions. These vary according to the type of data under consideration and hence are discussed in detail as they arise.

3·1. *Continuous dependence on body-force*

We commence our discussion by considering continuous dependence on body-force alone, so that we suppose homogeneous initial data and thus put

$$u_i(x,0) = \dot{u}_i(x,0) = 0. \tag{3·1·1}$$

We then see that (2·17) with $t_1 = 0$ becomes

$$2(u(t), \dot{u}(t)) = \int_0^t [(u(2t-\eta), f(\eta)) - (u(\eta), f(2t-\eta))]\, d\eta, \tag{3·1·2}$$

which after application of Schwarz's inequality leads to

$$2(u(t), \dot{u}(t)) \leqslant \left(\int_0^{2t} \|u(\eta)\|^2 \, d\eta \right)^{\frac{1}{2}} \left(\int_0^{2t} \|f(\eta)\|^2 \, d\eta \right)^{\frac{1}{2}}. \tag{3·1·3}$$

Now we suppose that $0 < 2t \leqslant T$ and that the solution to our problem lies in the constraint set defined by

$$\int_0^T \|u(\eta)\|^2 \, d\eta \leqslant M_1^2, \tag{3·1·4}$$

where M_1 is some positive constant. We then let $0 < 2s \leqslant T$ and integrate (3·1·3) over $(0, s)$ to obtain

$$\|u(s)\|^2 \leqslant s \left(\int_0^T \|u(\eta)\|^2 \, d\eta \right)^{\frac{1}{2}} \left(\int_0^T \|f(\eta)\|^2 \, d\eta \right)^{\frac{1}{2}} \tag{3·1·5}$$

$$\leqslant \tfrac{1}{2} T M_1 \left(\int_0^T \|f(\eta)\|^2 \, d\eta \right)^{\frac{1}{2}} \quad (0 < 2s \leqslant T) \tag{3·1·6}$$

which is the required continuous dependence estimate. We note that when the constraint set (3·1·4) is replaced by

$$\max_{[0,\,T]} \|u(t)\| \leqslant M_2, \tag{3·1·7}$$

where M_2 is a positive constant, the corresponding continuous dependence estimate is given by

$$\|u(s)\|^2 \leqslant \tfrac{1}{3} T^{\frac{3}{2}} M_2 \left(\int_0^T \|f(\eta)\| \, d\eta \right)^{\frac{1}{2}} \quad (0 < 2s \leqslant T). \tag{3·1·8}$$

Further continuous dependence estimates involving slightly different norms and associated constraint sets may be derived from the inequality

$$\|u(s)\|^2 \leqslant \frac{1}{2} \left(\int_0^{2s} (2s-\eta) \, \|u(\eta)\|^2 \, d\eta \right)^{\frac{1}{2}} \left(\int_0^{2s} (2s-\eta) \, \|f(\eta)\|^2 \, d\eta \right)^{\frac{1}{2}},$$

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which is obtained by integrating (3·1·2) over $(0, s)$ and then applying Schwarz's inequality. The treatment is now analogous to (3·1·5) and provides estimates for solutions lying in the constraint set given, for example, by

$$\int_0^T (T-\eta)\,\|u(\eta)\|^2\,d\eta \leqslant M_3^2,$$

where M_3 is a positive constant.

We remark that the estimates (3·1·6) and (3·1·8) hold only on the half-interval $[0, T/2]$, even though the constraints (3·1·4), (3·1·6) are defined on the whole interval $[0, T]$. Furthermore, the body-force is measured by the L_2-norm over $\Omega \times [0, T]$, while the solution itself is measured by the L_2-norm over Ω. This contrasts with the result obtained by logarithmic convexity arguments [**6**] which in the present notation is given by

$$\int_0^t \|u(\eta)\|^2\,d\eta \leqslant e^{\frac{1}{4}}(1+T^4k)^{\delta}T^{4(1-\delta)}M_1^{2\delta}\left(\int_0^T \|f(\eta)\|^2\,d\eta\right)^{(1-\delta)} \quad (0 \leqslant t < T), \quad (3\cdot1\cdot9)$$

where $\delta(t) = t/T$ satisfies $0 \leqslant \delta(t) \leqslant 1$, the constant k is such that

$$\int_0^T \|f(\eta)\|^2\,d\eta \leqslant kM_1^2,$$

and the solution lies in the constraint set (3·1·4). Thus, (3·1·9) uses a weaker norm to measure the solution and moreover involves a weaker notion of Hölder continuity than does (3·1·6) or (3·1·8). Indeed, we show that an estimate of type (3·1·9) may be derived from (3·1·5), but meanwhile let us further observe that (3·1·9) remains valid on all compact sub-intervals of $[0, T)$ and not just the half-interval $[0, T/2]$ over which (3·1·6) and (3·1·8) so far have been shown to hold. Extension of the interval within the context of the present arguments is possible by a variety of methods subject to the imposition of additional constraints. Similar techniques apply to the extension of the interval of continuous dependence in the case of initial data and hence further discussion is postponed to a combined treatment in §3·3 of both sets of data.

We continue by describing how (3·1·9) may be derived from (3·1·5) but, of course, the derivation holds only on $[0, T/2]$. Thus, integration of (3·1·5) over $[0, T/2]$ gives

$$\int_0^{T/2} \|u(\eta)\|^2\,d\eta \leqslant (T^2/8)\left(\int_0^T \|u(\eta)\|^2\,d\eta\right)^{\frac{1}{2}}\left(\int_0^T \|f(\eta)\|^2\,d\eta\right)^{\frac{1}{2}} \quad (3\cdot1\cdot10)$$

which apart from the multiplicative constant corresponds to (3·1·9) with $\delta(t) = \frac{1}{2}$. Next, on replacing T by $T/2$ in (3·1·10) we obtain

$$\begin{aligned}\int_0^{T/4} \|u(\eta)\|^2\,d\eta &\leqslant (T^2/32)\left(\int_0^{T/2} \|u(\eta)\|^2\,d\eta\right)^{\frac{1}{2}}\left(\int_0^{T/2} \|f(\eta)\|^2\,d\eta\right)^{\frac{1}{2}} \\ &\leqslant (T^3/64\sqrt{2})\left(\int_0^T \|u(\eta)\|^2\,d\eta\right)^{\frac{1}{4}}\left(\int_0^T \|f(\eta)\|^2\,d\eta\right)^{\frac{1}{4}}\left(\int_0^{T/2} \|f(\eta)\|^2\,d\eta\right)^{\frac{1}{2}} \quad (3\cdot1\cdot11) \\ &\leqslant (T^3/64\sqrt{2})\left(\int_0^T \|u(\eta)\|^2\,d\eta\right)^{\frac{1}{4}}\left(\int_0^T \|f(\eta)\|^2\,d\eta\right)^{\frac{3}{4}}, \quad (3\cdot1\cdot12)\end{aligned}$$

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which corresponds to (3·1·9) with $\delta(t) = \frac{1}{4}$. Indeed, repeated iteration for $0 \leqslant 2t \leqslant T$ easily leads to

$$\int_0^t \|u(\eta)\|^2 d\eta \leqslant (T^2/8)^{2(1-2^{-(n+1)})} \left(\int_0^T \|u(\eta)\|^2 d\eta\right)^{2^{-(n+1)}} \left(\int_0^{2^{n+1}t} \|f(\eta)\|^2 d\eta\right)^{2^{-(n+1)}} \ldots \left(\int_0^{2t} \|f(\eta)\|^2 d\eta\right)^{\frac{1}{2}} \quad (3\cdot1\cdot13)$$

$$\leqslant (T^2/8)^{2(1-2^{-(n+1)})} \left(\int_0^T \|u(\eta)\|^2 d\eta\right)^{2^{-(n+1)}} \left(\int_0^T \|f(\eta)\|^2 d\eta\right)^{(1-2^{-(n+1)})} \quad (0 < 2t \leqslant T), \quad (3\cdot1\cdot14)$$

where n is the largest integer such that $2^{n+1}t \leqslant T$. The similarity between (3·1·14) and (3·1·9) is obvious, although we remark that the multiplicative constant in (3·1·14) is independent of the constant k and is simpler than that of (3·1·9). Indeed, the constant given in (3·1·14) is an upper bound to the exact value which, however, is straightforward to determine. We emphasize that the interval of dependence holding in (3·1·14) is still $[0, T/2]$. This apparent limitation of our approach is underlined by reference to (3·1·11) which shows that the identical vanishing of f on $[0, 2t]$ implies the identical vanishing of the solution on $[0, t]$. Yet the uniqueness theorem of Brun [1] states that the solution vanishes identically on $[0, 2t]$ and hence an estimate is needed which is valid for $T/2 < t < T$. We shall show in §3·3 that the use of a stronger constraint set enables the interval of dependence to be extended beyond $[0, T/2]$, thus suggesting that constraint and interval are mutually dependent, a relaxation of one requiring a compensating strengthening of the other.

3·2. *Continuous dependence on initial data*

Continuous dependence on initial data can be discussed by means of the transformation (cf. [6], §4)

$$w_i(x,t) = \int_0^t (t-\eta)\, u_i(x,\eta)\, d\eta, \quad (3\cdot2\cdot1)$$

since it may easily be shown that w_i satisfies (2·1) and (2·2) subject to homogeneous initial data and body-force given by

$$\rho f_i(x,t) = t\dot{u}_i(x,0) + u_i(x,0). \quad (3\cdot2\cdot2)$$

The problem is thus reduced to the case of continuous dependence on the body-force discussed in §3·1. Estimates obtained by this means, however, are necessarily poor in view of the transformation (3·2·1) which obscures the precise nature of the dependence on the actual solution u_i. Instead, we study the problem afresh and determine estimates analogous to those derived in §3·1 which considerably improve those presented in [6]. Although our treatment is similar in part to that employed by [2] in the discussion of continuous dependence of the solution on initial data for unbounded spatial regions, we follow more closely the modified arguments due to Payne [11]. The present authors [4, 7] were the first to derive continuous dependence on the initial data, but used the method of logarithmic convexity. A comparison with these earlier results is presented below.

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Thus, let us consider (2·1)–(2·3) with body-force absent. The Lagrange identity (2·17) becomes

$$2(u(t), \dot{u}(t)) = (\dot{u}(0), u(2t)) + (u(0), \dot{u}(2t)) \quad (0 < 2t \leqslant T), \tag{3·2·3}$$

which on integration over $(0, s)$, $0 < 2s \leqslant T$, gives

$$\|u(s)\|^2 = \tfrac{1}{2}(u(0), u(2s)) + \tfrac{1}{2}\|u(0)\|^2 + \frac{1}{2}\int_0^{2s} (u(\eta), \dot{u}(0))\, d\eta. \tag{3·2·4}$$

We assume that the solution belongs to the constraint set defined by (3·1·7). Then, by means of Schwarz's inequality, we conclude from (3·2·4) that

$$\|u(s)\|^2 \leqslant \tfrac{1}{2}\|u(0)\|\,\|u(2s)\| + \tfrac{1}{2}\|u(0)\|^2 + \tfrac{1}{2}\|\dot{u}(0)\| \int_0^{2s} \|u(\eta)\|\, d\eta \quad (0 < 2s \leqslant T), \tag{3·2·5}$$

which on using (3·1·7) gives the continuous dependence estimate

$$\|u(s)\|^2 \leqslant \tfrac{1}{2}\|u(0)\|^2 + \tfrac{1}{2}M_2[\,\|u(0)\| + T\|\dot{u}(0)\|\,] \quad (0 < 2s \leqslant T). \tag{3·2·6}$$

Alternatively, we may constrain the solution to lie in the set defined by (3·1·4) and then inequality (3·2·5) becomes after an integration over $(0, s)$:

$$\int_0^s \|u(\eta)\|^2\, d\eta \leqslant \tfrac{1}{2}s\|u(0)\|^2 + \tfrac{1}{4}\|u(0)\| \int_0^{2s} \|u(\eta)\|\, d\eta + \tfrac{1}{2}\|\dot{u}(0)\| \int_0^s \int_0^{2\eta} \|u(\tau)\|\, d\tau\, d\eta \quad (0 < 2s \leqslant T), \tag{3·2·7}$$

$$\leqslant \tfrac{1}{4}T\|u(0)\|^2 + \tfrac{1}{4}T^{\frac{1}{2}}M_1\left[\|u(0)\| + \frac{T}{\sqrt{2}}\,\|\dot{u}(0)\|\right] \quad (0 < 2s \leqslant T). \tag{3·2·8}$$

The last inequality provides a second estimate for continuous dependence on initial data.

The continuous dependence estimate previously established by the authors [7] using logarithmic convexity is given by

$$\|u(t)\|^2 \leqslant M_4^{2t/T}\|u(0)\|^{2(1-t/T)} \quad (0 \leqslant t < T) \tag{3·2·9}$$

and is valid on compact sub-intervals of $[0, T]$ for solutions in the constraint set defined by

$$\|u(T)\|^2 \leqslant M_4^2, \tag{3·2·10}$$

where M_4 is a positive constant. The derivation of (3·2·9) depends upon the conservation of energy, expressed by (2.9) with zero body-force. For weak solutions, energy conservation must be either separately postulated or derived from the weak form of the Lagrange identity (cf. Levine [8]). Furthermore, the estimate (3·2·9) is valid only for initial data with non-positive total energy, whereas no such restriction applies to either (3·2·6) or (3·2·8). A continuous dependence estimate similar to, but less simple than, (3·2·9) is derived in [7] for initial data with positive total energy, while the estimate derived in [4] is likewise similar to (3·2·9), except that the norm measuring the initial data additionally contains terms involving $\dot{u}_i(x, 0)$, $u_{i,j}(x, 0)$ and the magnitude of elasticities. Conservation of energy remains essential for both estimates.

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In the sense, therefore, that the estimates established here do not require conservation of energy, hold for general initial data and employ less stringent Hölder continuity, they represent an improvement on those of [**4**, **6**, **7**]. Nevertheless, the constraints necessary to stabilize the solution are stronger than required in the earlier analysis and, as we have already remarked, the estimates so far determined are valid only on the half-interval $[0, T/2]$ instead of on all compact sub-intervals of $[0, T)$. In the next section, techniques based on Lagrange identities are used to extend the interval on which there is continuous dependence on both body-force and initial data.

We conclude this section by noting that an iterative procedure may be given that partially reproduces the estimate (3·2·9) obtained from logarithmic convexity arguments. The interval of dependence is still $[0, T/2]$. Accordingly, from (3·2·5) we have

$$\|u(T/2)\|^2 \leqslant \tfrac{1}{2}\|u(0)\|\,\|u(T)\| + \tfrac{1}{2}\|u(0)\|^2 + \tfrac{1}{2}\|\dot{u}(0)\| \int_0^T \|u(\eta)\|\,d\eta, \tag{3·2·11}$$

which on inserting in (3·2·5) evaluated at $s = T/4$ leads to, for example,

$$\|u(T/4)\|^2 \leqslant \frac{1}{2\sqrt{2}}\|u(0)\|^{\frac{3}{2}}\|u(T)\|^{\frac{1}{2}} + \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)\|u(0)\|^2 + \frac{1}{2\sqrt{2}}\|u(0)\|\,\|\dot{u}(0)\|^{\frac{1}{2}}\left(\int_0^T \|u(\eta)\|\,d\eta\right)^{\frac{1}{2}} + \tfrac{1}{2}\|\dot{u}(0)\|\left(\int_0^T \|u(\eta)\|\,d\eta\right). \tag{3·2·12}$$

We may then appeal to the constraint (3·1·7) in order to derive a continuous dependence estimate for $t = T/4$. Iteration then establishes an estimate for any $t = [0, T/2]$. An alternative estimate involving the constraint (3·1·4) may be obtained by a similar iterative procedure based on (3·2·7) and is analogous to the result (3·1·4) for dependence on body-force. Obviously other estimates may be derived in line with the comment given immediately after (3·1·8).

3·3. *Extension of the interval of dependence for body-force and initial data*

We have seen that estimates (3·1·6), (3·1·8), (3·2·6) and (3·2·8) have been established only on the half-interval $[0, T/2]$. By subjecting the solution to additional constraints it is possible to extend the interval on which continuous dependence holds. This may be accomplished in several ways, some of which are indicated here.

We suppose that body-force and initial data are both non-zero. An application of Schwarz's inequality to the right-side of (2.17) with $t_1 \geqslant 0$ then leads to the inequality

$$2(u(t), \dot{u}(t)) \leqslant \left(\int_{t_1}^{2t-t_1} \|u(\eta)\|^2\,d\eta\right)^{\frac{1}{2}}\left(\int_{t_1}^{2t-t_1} \|f(\eta)\|^2\,d\eta\right)^{\frac{1}{2}} + (\dot{u}(t_1), u(2t-t_1)) + (u(t_1), \dot{u}(2t-t_1)) \quad (0 \leqslant t_1 \leqslant 2t \leqslant T+t_1). \tag{3·3·1}$$

We hold t_1 fixed and after a further application of Schwarz's inequality to the second term on the right, we integrate (3·3·1) with respect to t over the interval (t_1, s), where $2t \leqslant 2s \leqslant T+t_1$, to obtain

$$\|u(s)\|^2 \leqslant (s-t_1)\left(\int_0^T \|u(\eta)\|^2\,d\eta\right)^{\frac{1}{2}}\left(\int_0^T \|f(\eta)\|^2\,d\eta\right)^{\frac{1}{2}} + \tfrac{1}{2}\|\dot{u}(t_1)\|\int_{t_1}^{2s-t_1} \|u(\eta)\|\,d\eta + \tfrac{1}{2}\|u(t_1)\|\,\|u(2s-t_1)\| + \tfrac{1}{2}\|u(t_1)\|^2 \quad (0 \leqslant 2t_1 < 2s \leqslant T+t_1), \tag{3·3·2}$$

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which generalizes (3·1·5) and (3·2·5). Indeed, on setting $t_1 = 0$ and imposing the constraint (3·1·7) we may immediately conclude from (3·3·2) that

$$\|u(s)\|^2 \leqslant \tfrac{1}{2}T^{\frac{3}{2}}M_2\left(\int_0^T \|f(\eta)\|^2\,d\eta\right)^{\frac{1}{2}} + \tfrac{1}{2}M_2[\,\|u(0)\| + T\|\dot{u}(0)\|\,] + \tfrac{1}{2}\|u(0)\|^2 \quad (0 < 2s \leqslant T), \tag{3·3·3}$$

thus deriving a continuous dependence on $[0, T/2]$.

In order to extend the interval of dependence, we return to (3·3·1), set $2t - t_1 = T$ and integrate with respect to t_1 over $(0, s)$, to get

$$2\left\|u\left(\frac{T+s}{2}\right)\right\|^2 \leqslant 2\|u(T/2)\|^2 + s\left(\int_0^T \|u(\eta)\|^2\,d\eta\right)^{\frac{1}{2}}\left(\int_0^T \|f(\eta)\|^2\right)^{\frac{1}{2}} + \|u(T)\|\,[\,\|u(s)\| + \|u(0)\|\,] + \|\dot{u}(T)\| \int_0^s \|u(\eta)\|\,d\eta. \tag{3·3·4}$$

We let $0 \leqslant s \leqslant T/2$, and use (3·3·3) to bound $\|u(s)\|$ on the right side of (3·3·4). On supposing that the solution satisfies the constraint

$$\max_{[0,\,T]} \|u(t)\| + \|\dot{u}(T)\| \leqslant M_5, \tag{3·3·5}$$

for positive constant M_5 we derive the estimate

$$2\left\|u\left(\frac{T+s}{2}\right)\right\|^2 \leqslant \tfrac{3}{2}T^{\frac{3}{2}}M_5\left(\int_0^T \|f(\eta)\|^2\,d\eta\right)^{\frac{1}{2}} + \tfrac{1}{2}T^{\frac{3}{4}}M_5^{\frac{3}{2}}(\sqrt{2}+T)\left(\int_0^T \|f(\eta)\|^2\,d\eta\right)^{\frac{1}{4}} + M_5(\,\|u(0)\| + T\|\dot{u}(0)\|\,) + \frac{1}{\sqrt{2}}M_5^{\frac{3}{2}}(\,\|u(0)\| + T\|\dot{u}(0)\|\,)^{\frac{1}{2}}(1+\tfrac{1}{2}T) + M_5\|u(0)\|\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}}T\right) + \|u(0)\|^2, \tag{3·3·6}$$

which together with (3·3·3) yields continuous dependence on $[0, 3T/4]$ in the set (3·3·5). We now repeat the operation. Thus, in (3·3·4) we set $T/2 \leqslant s \leqslant 3T/4$ and use (3·3·6) to bound appropriate terms on the right of (3·3·4) and hence obtain continuous dependence on $[3T/4, 7T/8]$. By recursion, we thus have continuous dependence on $[(1-2^{-(n-1)})T, (1-2^{-n}T)]$ and therefore on $[0, (1-2^{-n})T]$, for $n = 1, 2, 3, \ldots$. We note that each successive iteration involves a weaker sense of Hölder continuity.

Other recursive procedures are possible. For instance, let us estimate the term $\|\dot{u}(t_1)\|$ in the right side of (3·3·2) by eliminating the potential energy $V(t, t)$ between the identities (2.9) and (2.14). We have, for $t_1 \geqslant 0$,

$$2\|\dot{u}(t)\|^2 = \|\dot{u}(0)\|^2 + V(0,0) + (\dot{u}(t_1), \dot{u}(2t-t_1)) - V(2t-t_1, t_1) + 2\int_0^t (u_{,\eta}(\eta), f(\eta))\,d\eta - 2\int_{t_1}^{2t-t_1} (u_{,\eta}(\eta), f(\eta))\,d\eta \tag{3·3·7}$$

$$\leqslant \|\dot{u}(0)\|^2 + |V(0,0)| + \|\dot{u}(t_1)\|\,\|\dot{u}(2t-t_1)\| - V(2t-t_1, t_1) + 4\max_{[0,\,T]} \|\dot{u}(t)\| \int_0^T \|f(\eta)\|\,d\eta \quad (0 \leqslant t_1 \leqslant 2t \leqslant T + t_1). \tag{3·3·8}$$

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Now, from definition (2·11) we obtain on integrating by parts and using (2·1)

$$-V(2t-t_1,t_1) = (\ddot{u}(2t-t_1), u(t_1)) - (f(2t-t_1), u(t_1)) \tag{3·3·9}$$

$$\leqslant [\|\ddot{u}(2t-t_1)\| + \|f(2t-t_1)\|]\,\|u(t_1)\|. \tag{3·3·10}$$

Let us suppose that the solution lies in the constraint set defined by

$$\max_{[0,T]} \|u(t)\| + \max_{[0,T]} \|\dot{u}(t)\| + \|\ddot{u}(T)\| \leqslant M_6 \tag{3·3·11}$$

for some positive constant M_6. Then with the help of (3·3·10) and (3·3·11), the inequalities (3·3·2) and (3·3·8) respectively become,

$$\|u(s)\|^2 \leqslant (s-t_1)T^{\frac{1}{2}}M_6\left(\int_0^T \|f(\eta)\|^2\,d\eta\right)^{\frac{1}{2}} + M_6(s-t_1)\,\|\dot{u}(t_1)\| + \tfrac{1}{2}\|u(t_1)\|\,M_6 + \tfrac{1}{2}\|u(t_1)\|^2 \quad (0 \leqslant 2t_1 < 2s \leqslant T+t_1), \tag{3·3·12}$$

$$2\|\dot{u}(t)\|^2 \leqslant \|\dot{u}(0)\|^2 + |V(0,0)| + M_6\left[\|u(t_1)\| + \|\dot{u}(t_1)\| + 4\int_0^T \|f(\eta)\|\,d\eta\right] + \|u(t_1)\|\,\|f(2t-t_1)\| \quad (0 \leqslant t_1 < 2t = T+t_1). \tag{3·3·13}$$

We extend the basic interval of continuous dependence by iteration of the inequalities (3·3·12), (3·3·13). We already have an estimate for $\|u(s)\|^2$ on $[0,T/2]$. Hence, we set $t_1 = T/2$ in (3·3·12) and use (3·3·3) to bound the terms involving $\|u(T/2)\|$ on the right side of (3·3·12). We determine the bound for $\|\dot{u}(T/2)\|$ from (3·3·13) with $t_1 = 0$, $t = T/2$, and hence (3·3·12) provides the continuous dependence estimate on $[T/2, 3T/4]$. More generally, in (3·3·12) we take $t_1 = (1-2^{-n})T$, so that

$$(1-2^{-n})T < s \leqslant (1-2^{-(n+1)})T \quad (n = 1,2,3,\ldots),$$

and assume that by previous iterations, a bound is known for $\|u((1-2^{-n})T)\|$. To estimate $\|\dot{u}((1-2^{-n})T)\|$, we set $t_1 = (1-2^{-(n-1)})T$, $t = (1-2^{-n})T$, in (3·3·13) to get

$$2\|\dot{u}((1-2^{-n})T)\|^2 \leqslant \|\dot{u}(0)\|^2 + |V(0,0)| + M_6\left[\|u((1-2^{-(n-1)})T)\| + \|\dot{u}((1-2^{-(n-1)})T)\| + 4\int_0^T \|f(\eta)\|\,d\eta\right] + \|u((1-2^{-(n-1)})T)\|\,\|f(T)\|, \tag{3·3·14}$$

and thus we have recursive relations yielding continuous dependence on $[0,(1-2^{-n})T]$, for $n = 1,2,3,\ldots$. Once again, at each iteration there is a weakening of the continuity concept.

Remark 1. We may alternatively express the potential energy in (3·3·9) by

$$-V(2t-t_1,t_1) = (u(2t-t_1), \ddot{u}(t_1)) - (f(t_1), u(2t-t_1)),$$

and by suitable modification of the constraint set obtain a different estimate of continuous dependence. Yet again, either in the last expression or in (3·3·9) the later results of §4 may be used to estimate $\|\ddot{u}(t)\|$ in terms of initial data.

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Remark 2. Integration by parts in the last two terms in (3·3·7) leads to an upper bound for $\|\dot{u}(t)\|^2$ independent of $\max \|\dot{u}(t)\|$ over $[0,T]$, but involving the body-force and its first time-derivative. This enables the constraint to be replaced by

$$\max_{[0,T]} \|u(t)\| + \|\dot{u}(T)\| + \|\ddot{u}(T)\| \leqslant M_7$$

for positive constant M_7.

Remark 3. Instead of using (2·9) and (2·14) to estimate the kinetic energy, we could apply (3·2·2) to successively higher time derivatives of the solution. The result combined with (3·3·12) again produces recursive relations for a continuous dependence estimate on $[0, (1-2^{-n})T]$ for $n = 1, 2, \ldots$.

Remark 4. A further iterative scheme may be derived with the help of logarithmic convexity properties. We illustrate the argument for dependence on initial data only. It has been shown in [4] under the assumption of energy conservation that the function $G(t)$, defined by

$$G(t) = \ln [e^{t^2}\{\|u(t)\|^2 + Q\}] \tag{3·3·15}$$

where

$$Q = 4 \max [\{K(0,0) + V(0,0)\}, 0], \tag{3·3·16}$$

is convex on $[0,T]$, so that on letting $0 \leqslant t_1 < T$ we have

$$2G\left(\frac{T+t_1}{2}\right) \leqslant G(t_1) + G(T). \tag{3·3·17}$$

Hence we obtain

$$\left\| u\left(\frac{T+s}{2}\right)\right\|^2 \leqslant e^{\frac{1}{4}(T-s)^2}[(\|u(s)\|^2 + Q)(\|u(T)\|^2 + Q)]^{\frac{1}{2}}, \tag{3·3·18}$$

which forms the basis of an iterative scheme analogous to that based on (3·3·6). Now, however, the constraint set is given by (3·1·7).

Remark 5. Iterative procedures may be developed involving weaker norms (over space-time) of the solution. One such scheme based on a time integration of (3·3·2) leads to continuous dependence estimates analogous to the family of estimates just derived, but with suitably modified constraint sets. A second scheme may be derived as follows. We commence with (3·3·1) expressed as

$$2(u(t), \dot{u}(t)) \leqslant \left(\int_0^T \|u(\eta)\|^2 d\eta\right)^{\frac{1}{2}} \left(\int_0^T \|f(\eta)\|^2 d\eta\right)^{\frac{1}{2}} + (u(t-\tau), \dot{u}(t+\tau)) + (\dot{u}(t-\tau), u(t+\tau)), \tag{3·3·19}$$

which on integration with respect to τ from $T/4$ to $T-t$ leads to

$$\left[\left(\frac{3T}{4} - t\right)\|u(t)\|^2\right]_{.t} + \|u(t)\|^2 = \left(\frac{3T}{4} - t\right)\left(\int_0^T \|u(\eta)\|^2 d\eta\right)^{\frac{1}{2}} \left(\int_0^T \|f(\eta)\|^2 d\eta\right)^{\frac{1}{2}} + (u(T), u(2t-T)) - (u(t-T/4), u(t+T/4)) + 2\int_{T/4}^{T-t} (u(t+\eta), \dot{u}(t-\eta))\, d\eta. \tag{3·3·20}$$

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Next, we integrate (3·3·20) with respect to t over the interval $(T/2, 3T/4)$ and use the constraint set

$$\max_{[0,T]} \|u(t)\| + \max_{[0,T]} \|\dot{u}(t)\| \leqslant M_8 \tag{3·3·21}$$

to obtain the dependence estimate

$$\int_{T/2}^{3T/4} \|u(\eta)\|^2 \, d\eta \leqslant T/2 \|u(T/2)\|^2 + \frac{T^{\frac{5}{2}}}{32} M_8 \left(\int_0^T \|f(\eta)\|^2 \, d\eta \right)^{\frac{1}{2}}$$
$$+ \tfrac{1}{2} M_8 \int_0^{T/2} \|u(\eta)\| \, d\eta + M_8 \int_{T/4}^{T/2} \|u(\eta)\| \, d\eta + 2M_8 \int_{T/2}^{3T/4} \int_{t-T/4}^{2t-T} \|u(2t-\eta)\| \, d\eta \, dt. \tag{3·3·22}$$

We may now use any of the previous inequalities valid on $[0, T/2]$ to provide an estimate for terms involving $\|u(s)\|$ for $0 \leqslant s \leqslant T/2$, which after substitution in (3·3·22) enables an estimate in terms of initial data to be derived. Alternatively, we may treat the kinetic energy arising in the last term on the right of (3·3·22) by means of inequality (3·3·13), whereupon the constraint set (3·3·21) may be replaced by (3·3·11). Further extension of the interval of continuous dependence follows by iteration in an obvious manner.

3·4. *Continuous dependence on the elasticities*

We turn to a consideration of how the solution is affected by a variation in the elasticities. Thus, let $u^{(1)}, u^{(2)}$ be two solutions to the initial boundary value problem (2·1)–(2·3) corresponding to different elasticities $c^{(1)}_{ijkl}, c^{(2)}_{ijkl}$, respectively, but with the same density, body-force, initial data and boundary conditions. Now, however, we do not require the elasticities $c^{(2)}_{ijkl}$ to satisfy the symmetry condition (2·4). On putting

$$u_i(x,t) = u^{(1)}_i(x,t) - u^{(2)}_i(x,t), \tag{3·4·1}$$

$$c_{ijkl} = c^{(1)}_{ijkl} - c^{(2)}_{ijkl}, \tag{3·4·2}$$

and subtracting the respective equations, we obtain

$$(c^{(1)}_{ijkl} u_{k,l})_{,j} + (c_{ijkl} u^{(2)}_{k,l})_{,j} = \rho \ddot{u}_i, \quad ((x,t) \in \Omega \times [0,T)), \tag{3·4·3}$$

$$u_i = 0, \quad ((x,t) \in \partial\Omega \times [0,T)), \tag{3·4·4}$$

$$u_i(x,0) = \dot{u}_i(x,0) = 0, \quad (x \in \Omega). \tag{3·4·5}$$

We recognize (3·4·3) as the equation of motion for a material with elasticities $c^{(1)}_{ijkl}$ subject to an appropriate body-force. Hence by substitution in the identity (2·17), with $t_1 = 0$,we are led to

$$2(u(t), \dot{u}(t)) = \int_0^t \int_\Omega [u_i(2t-\eta) (c_{ijkl} u^{(2)}_{k,l}(\eta))_{,j} - u_i(\eta) (c_{ijkl} u^{(2)}_{k,l}(2t-\eta))_{,j}] \, d\eta \, dx, \tag{3·4·6}$$

where dependence on the argument x is henceforth suppressed. We introduce the definitions

$$C = \max_{\Omega} (c_{ijkl} c_{ijkl})^{\frac{1}{2}}, \tag{3·4·7}$$

$$\|\nabla u(t)\|^2 = \int_\Omega u_{i,j}(x,t) \, u_{i,j}(x,t) \, dx, \tag{3·4·8}$$

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and after integration by parts, repeatedly apply the Cauchy–Schwarz inequality to obtain from (3·4·6) the following sequence of inequalities, valid for $0 < 2t \leqslant T$:

$$\begin{aligned}
2(u(t), \dot{u}(t)) &\leqslant C\left(\int_0^{2t} \|\nabla u(\eta)\|^2 \, d\eta\right)^{\frac{1}{2}} \left(\int_0^{2t} \|\nabla u^{(2)}(\eta)\|^2 \, d\eta\right)^{\frac{1}{2}} \\
&= C\left(\int_0^{2t} \|\nabla u^{(1)}(\eta) - \nabla u^{(2)}(\eta)\|^2 \, d\eta\right)^{\frac{1}{2}} \left(\int_0^{2t} \|\nabla u^{(2)}(\eta)\|^2 \, d\eta\right)^{\frac{1}{2}} \\
&\leqslant C(1+\alpha)^{\frac{1}{2}} \left(\int_0^{2t} \left\{\|\nabla u^{(1)}(\eta)\|^2 + \frac{1}{\alpha} \|\nabla u^{(2)}(\eta)\|^2\right\} d\eta\right)^{\frac{1}{2}} \left(\int_0^{2t} \|\nabla u^{(2)}(\eta)\|^2 \, d\eta\right)^{\frac{1}{2}} \\
&\leqslant C(1+\alpha)^{\frac{1}{2}} \left[\tfrac{1}{2}\beta\gamma \left(\int_0^{2t} \|\nabla u^{(1)}(\eta)\|^2 \, d\eta\right)^2 \right. \\
&\quad + (1-\beta) \left(\int_0^{2t} \|\nabla u^{(1)}(\eta)\|^2 \, d\eta\right) \left(\int_0^{2t} \|\nabla u^{(2)}(\eta)\|^2 \, d\eta\right) \\
&\quad \left. + \left(\frac{1}{\alpha} + \frac{\beta}{2\gamma}\right) \left(\int_0^{2t} \|\nabla u^{(2)}(\eta)\|^2 \, d\eta\right)^2\right]^{\frac{1}{2}} \\
&\leqslant C\left[\frac{(1+\alpha)(1-\beta)}{2}\right]^{\frac{1}{2}} \left[\int_0^{2t} \|\nabla u^{(1)}(\eta)\|^2 + \int_0^{2t} \|\nabla u^{(2)}(\eta)\|^2 \, d\eta\right], \qquad (3\cdot4\cdot9)
\end{aligned}$$

where α, β (< 1) and γ are positive constants which in the last inequality are chosen to satisfy

$$\gamma = (1-\beta)/\beta, \quad \alpha = 2(1-\beta)/(1-2\beta). \qquad (3\cdot4\cdot10)$$

The minimum value of the multiplicative factor in (3·4·9) is given by $\frac{1}{2}(1+\sqrt{2})C$. We suppose that the solutions $u^{(1)}, u^{(2)}$ belong to the constraint set given by

$$\int_0^T \|\nabla u^{(\alpha)}(\eta)\|^2 \, d\eta \leqslant M_9^2 \quad (\alpha = 1, 2) \qquad (3\cdot4\cdot11)$$

where M_9 is a positive constant. An integration of (3·4·9) over $(0, s)$, where $0 < 2s \leqslant T$, then leads to the continuous dependence estimate

$$\|u(s)\|^2 \leqslant (1+\sqrt{2}) M_9^2 sC \leqslant \tfrac{1}{2} T M_9^2 (1+\sqrt{2}) C \quad (0 < 2s \leqslant T). \qquad (3\cdot4\cdot12)$$

Alternatively we may use the constraint set

$$\int_0^T \int_\Omega [u_{i,j}^{(\alpha)}(x, \eta) \, u_{i,j}^{(\alpha)}(x, \eta)]^2 \, dx \, d\eta \leqslant M_{10}^2 \quad (\alpha = 1, 2), \qquad (3\cdot4\cdot13)$$

where M_{10} is a positive constant, to recover from (3·4·6) the continuous dependence estimate

$$\|u(s)\|^2 \leqslant \frac{1}{3}\left(1 + \frac{1}{\sqrt{2}}\right)^{\frac{1}{2}} T^{\frac{3}{2}} M_{10} \left(\int_\Omega c_{ijkl} \, c_{ijkl} \, dx\right)^{\frac{1}{2}} \quad (0 < 2s \leqslant T), \qquad (3\cdot4\cdot14)$$

in which we have used Schwarz's inequality and again integrated over $(0, s)$.

The same general comments made in §3·1 for the body-force apply equally to the relationship between the estimates (3·4·12), (3·4·14) and the corresponding results derived in Corollary 3·1 of [6] which also do not require the elasticities $c_{ijkl}^{(2)}$ to be symmetric. The various recursive schemes described in §§3·1, 3·2, 3·3 continue to hold, although is should be noted that when extending the interval of dependence,

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the iterations for intervals beyond $[0,T/2]$ introduce additional terms involving, for example, the first time-derivative of the solution (cf. 3·3·1). Furthermore, it is worth emphasizing that the constraint sets (3·4·11), (3·4·13) are less stringent than that imposed in [**6**] which requires the set to be composed of functions whose first and *second* spatial derivatives are bounded in $L_2(\Omega\times[0,T])$. Again, the perturbed solution in [**6**] is measured in $L_2(\Omega\times(0,T))$, and not $L_2(\Omega)$ as now required, although as just noted the estimates of [**6**] may be derived from the ones established here, at least on $[0,T/2]$.

3·5. *Continuous dependence on density*

The methods developed in the previous sections may be easily applied to the determination of estimates for continuous dependence on the density. We let $u^{(1)}, u^{(2)}$ be solutions to the initial boundary value problem corresponding to different density functions $\rho_1(x), \rho_2(x)$ respectively, but with all other data the same, including body-force per unit mass. Then on setting

$$u_i(x,t) = u_i^{(1)}(x,t) - u_i^{(2)}(x,t), \tag{3·5·1}$$

$$\rho(x) = \rho_2(x) - \rho_1(x), \tag{3·5·2}$$

we have that subtraction of the respective equations (2·1) yields

$$(c_{ijkl}\,u_{k,l})_{,j} + \rho(\ddot{u}_i^{(2)} - f_i) = \rho_1\,\ddot{u}_i \quad ((x,t)\in\Omega\times[0,T)), \tag{3.5.3}$$

with homogeneous boundary and initial data. We note from (3·5·3) that the perturbation u_i occurs in a material of density ρ_1 subject to body force $(\rho/\rho_1)(\ddot{u}_i^{(2)} - f_i)$ per unit mass. Suitable modification of identity (2·17) therefore leads to the expression

$$\begin{aligned} 2(u(t),\dot{u}(t)) &= \int_0^t\int_\Omega \rho[u_i(2t-\eta)\,u_{i,\eta\eta}^{(2)}(\eta) - u_i(\eta)\,u_{i,\eta\eta}^{(2)}(2t-\eta)\,dx\,d\eta \\ &\quad + \int_0^t\int_\Omega \rho[u_i(2t-\eta)f_i(\eta) + u_i(\eta)f_i(2t-\eta)]\,dx\,d\eta \qquad (3·5·4) \\ &= -\int_0^t\int_\Omega \rho[u_{i,\eta}(2t-\eta)\,u_{i,\eta}^{(2)}(\eta) + u_{i,\eta}(\eta)\,u_{i,\eta}^{(2)}(2t-\eta)]\,dx\,d\eta \\ &\quad - \int_\Omega \rho u_i(2t)\,u_{i,t}^{(2)}(0)\,dx + \int_0^t\int_\Omega \rho[u_i(2t-\eta)f_i(\eta) + u_i(\eta)f_i(2t-\eta)]\,dx\,d\eta \qquad (3·5·5) \end{aligned}$$

where we have integrated by parts to obtain (3·5·5). The first integral on the right of (3·5·5) may be treated in similar manner to that used in the derivation of (3·4·9) while the second integral may be estimated according to the discussion of §3·1. Hence we have

$$\begin{aligned} 2(u(t),\dot{u}(t)) \leqslant \max_\Omega |\rho/\rho_1| \Big\{ &\tfrac{1}{2}(1+\sqrt{2})\left[\int_0^{2t}\|\dot{u}^{(1)}(\eta)\|^2\,d\eta + \int_0^{2t}\|\dot{u}^{(2)}(\eta)\|^2\,d\eta\right] \\ &+ \sqrt{2}\,\|\dot{u}^{(2)}(0)\|\,[\,\|u^{(1)}(2t)\| + \|u^{(2)}(2t)\|\,]^{\frac{1}{2}} \\ &+ \left(\int_0^{2t}\|u(\eta)\|^2\,d\eta\right)^{\frac{1}{2}}\left(\int_0^{2t}\|f(\eta)\|^2\,d\eta\right)^{\frac{1}{2}}\Big\} \quad (0<2t\leqslant T), \end{aligned} \tag{3·5·6}$$

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where in the definition of the norm we must now take the density function to be ρ_1 (cf. 2·5).

We impose on the solutions the constraint

$$\int_0^T [\|u^{(\alpha)}(\eta)\|^2 + \|\dot{u}^{(\alpha)}(\eta)\|^2]\,d\eta \leqslant M_{11}^2 \quad (\alpha = 1, 2), \tag{3·5·7}$$

and then, as usual, integrate (3·5·6) over $(0, s)$, $0 < 2s \leqslant T$ to conclude

$$\|u(s)\|^2 \leqslant \max|\rho/\rho_1|\,[M_{11}T^{\frac{1}{2}}(1+\sqrt{2})/2 + \sqrt{2}\|\dot{u}^{(2)}(0)\|]M_{11}T^{\frac{1}{2}} \quad (0 < 2s \leqslant T), \tag{3·5·8}$$

which provides a continuous dependence estimate.

It is straightforward to derive an estimate analogous to (3·4·14) and to generate recursion relations corresponding to those described in §§3·1, 3·3, so that in particular continuous dependence can be extended to the interval $[0, (1-2^{-n})T]$, $n = 1, 2, \ldots$.

We remark that the constraint on $\|\dot{u}^{(\alpha)}(t)\|$, $\alpha = 1, 2$ may be eliminated on using the weaker norm $\|v(t)\|^2$, where

$$v_i(x, t) = \int_0^t u_i(x, \eta)\,d\eta,$$

to measure the solution. The resulting equations of motions continue to be in the form given by (3·5·3) but with the body-force term replaced by

$$\rho(\ddot{v}_i^{(2)} - f_i - h_i),$$

where h_i is the common value of the initial velocity. The constraint (3·5·7) then becomes

$$\int_0^T [\|v^{\alpha}(\eta)\|^2 + \|u^{\alpha}(\eta)\|^2]\,d\eta \leqslant M_{13}^2 \quad (\alpha = 1, 2),$$

with corresponding alterations to (3·5·8) and subsequent estimates.

Finally, let us note that an alternative analysis may be based on (3·5·4). An application of Schwarz's inequality yields

$$2(u(t), \dot{u}(t)) \leqslant \max_{\Omega}|\rho/\rho_1| \left(\int_0^{2t} \|u(\eta)\|^2\,d\eta\right)^{\frac{1}{2}} \left(\int_0^{2t} \|\ddot{u}^{(2)}(\eta)\|^2\,d\eta\right)^{\frac{1}{2}} \quad (0 < 2t \leqslant T).$$

We may locate the perturbation u (or equivalently, the component solutions $u^{(1)}, u^{(2)}$) in the constraint class (3·1·4), but then a bound is needed for the second integral on the right in terms of the data. This is derived in the course of calculations described in the next section. (See the final remark at the end of §4.) Integration over $(0, s)$ then leads, as before, to the continuous dependence estimate.

4. *Continuous dependence on initial geometry*

4·1. This final section considers continuous dependence of the solution upon the initial geometry. It is unlikely that measurement of initial data can occur simultaneously throughout Ω at a given instant of time, say $t = 0$. Rather, the measurement of data will take place during a given interval of time about the instant $t = 0$, so that at a definite point in Ω the data is measured at an actual time depending upon the position of that point. It is of interest therefore to estimate the error in the solution thus introduced, and in fact we show that for suitably constrained solutions

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the error is proportional to the maximal time interval over which the data is measured.

After the treatment given in [6], Payne [12] presented a preliminary version of the arguments developed here but only for negative-definite elasticities. The continued relaxation by us of any sign-definiteness assumption means that the present calculations encounter considerably more difficulty.

Let $Q(T)$ be the space-time cylinder enclosed by the boundary $\partial\Omega$, the plane $t = T$ and the surface Σ defined by $t = \epsilon f(x)$, $x \in \Omega$, where $|f(x)| < 1$ and $\epsilon > 0$. We suppose that $u_i(x,t)$ is the true solution whose initial data is measured on the surface Σ, while $v(x,t)$ is the solution supposing data to be specified at $t = 0$. Hence, when no body-force acts, u_i and v_i respectively satisfy

$$(c_{ijkl}\, u_{k,l})_{,j} = \rho \ddot{u}_i \quad ((x,t) \in Q(T)), \tag{4·1·1}$$

$$u_i = 0 \quad ((x,t) \in \partial Q \backslash (\Sigma \cup \{t = T\})), \tag{4·1·2}$$

$$u_i(x,t) = g_i(x), \qquad u_i(x,t) = h_i(x) \quad ((x,t) \in \Sigma), \tag{4·1·3}$$

$$(c_{ijkl}\, v_{k,l})_{,j} = \rho \ddot{v}_i \quad ((x,t) \in \Omega \times (0,T)), \tag{4·1·4}$$

$$v_i = 0 \quad ((x,t) \in \partial\Omega \times (0,T)), \tag{4·1·5}$$

$$v_i(x,0) = g_i(x), \qquad \dot{v}_i(x,0) = h(x) \quad (x \in \Omega), \tag{4·1·6}$$

where we also assume that the elasticities obey the major symmetry condition

$$c_{ijkl} = c_{klij}. \tag{4·1·7}$$

We further assume that the functions g_i and h_i are everywhere sufficiently smooth on $\bar{\Sigma}$ (or on $\partial\Omega$ at $t = 0$), and that the differential equation (4·1·1) holds in an appropriate weak sense on Σ (cf. for instance the derivation of (4·3·16) below). These assumptions are reasonable for processes that have commenced at times well before the data has begun to be measured. We additionally suppose that Σ is a non-characteristic surface of the differential equation (4·1·1), so that the matrix whose components are given by

$$A_{ik} = [\rho n_\eta^2 \delta_{ik} - c_{ijkl}\, n_j\, n_l], \tag{4·1·8}$$

is positive-definite on Σ, where n_i, n_η are the spatial and time components, respectively, of the unit outward normal n on Σ. Finally, we assume that

$$n_\eta \leqslant -k < 0 \quad ((x,t) \in \Sigma), \tag{4·1·9}$$

where k is a positive constant.

Our problem, ill-posed due to the lack of any sign-definiteness assumption on the elasticities, may be stabilized by restricting the solutions to lie in a constraint set, here chosen to be defined by

$$\max_{(x,t) \in Q(T)} \|u(t)\|^2 + \max_{(x,t) \in \Omega \times [0,T]} \|v(t)\|^2 \leqslant M_{12}^2, \tag{4·1·10}$$

for a positive constant M_{12}.

We first state and prove the main result of this section which contains the estimate needed to establish continuous dependence on initial geometry. The proof relies upon a lemma which is discussed in §4·3.

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4·2. *Continuous dependence estimate*

The basic estimate, expressed in terms of the function w_i defined by

$$w_i(x,t) = v_i(x,t) - u_i(x,t), \tag{4·2·1}$$

is contained in the following Proposition.

PROPOSITION 4·2. *Suppose u_i, v_i are solutions of* (4·1·1)–(4·1·3), (4·1·4)–(4·1·6) *respectively and lie in the constraint set given by* (4·1·10). *Then under the assumptions listed in* §4·1, *we have*

$$\|w(t)\|^2 \leqslant KM_{12}^2 \epsilon^{\frac{1}{2}} \quad (0 < \epsilon \leqslant t < (T+\epsilon)/2), \tag{4·2·2}$$

where K is some (computable) positive constant.

Inequality (4·2·2) clearly establishes continuous dependence on the initial geometry, although again continuity is restricted to a sub-interval. Nevertheless, as regards the norm in which the solution is measured, inequality (4·2·2) represents an improvement on the previous result obtained in [**6**], which in the present notation is given by

$$\int_0^t \|W(\eta)\|^2 \, d\eta \leqslant K_4 M_{13}^{2\delta} \epsilon^{1-\delta} \quad (t \in (0,T)) \tag{4·2·3}$$

where $0 < \delta(t) < 1$, W_i is defined by

$$W_i(x,t) = \int_0^t (t-\eta)\, w_i(x,\eta)\, d\eta \tag{4·2·4}$$

and K_4, M_{13} are positive constants. It is supposed that W_i belongs to the constraint set defined by (3·1·4) and that the second time-derivatives of u_i are in $L_2(Q(T)\backslash\Omega \times [0,T])$. Clearly, the assumptions required for the present estimate (4·2·2) are less stringent than those needed for (4·2·3).

Proof of Proposition 4·2. We observe that the function $w_i(x,t)$, defined by (4·2·1), satisfies on appropriate domains the same differential equation and boundary conditions as either the solutions u_i or v_i. A Lagrange identity in the form of (2·17) is therefore available for w_i. Thus, on taking the base interval of time to be (ϵ,τ) instead of (t_1,t) as in (2·17), we have

$$2(w(\tau), \dot{w}(\tau)) = (w(2\tau-\epsilon), \dot{w}(\epsilon)) + (\dot{w}(2\tau-\epsilon), w(\epsilon)). \tag{4·2·5}$$

Integration of (4·2·5) over (ϵ,t), where $\epsilon \leqslant t < (T+\epsilon)/2$, thus leads to

$$\|w(t)\|^2 = \tfrac{1}{2}\|w(\epsilon)\|^2 + \tfrac{1}{2}(w(\epsilon), w(2t-\epsilon)) + \frac{1}{2}\int_\epsilon^{2t-\epsilon} (\dot{w}(\epsilon), w(\eta))\, d\eta, \tag{4·2·6}$$

so that by Schwarz's inequality we obtain

$$\|w(t)\|^2 \leqslant \tfrac{1}{2}\|w(\epsilon)\|^2 + \tfrac{1}{2}\|w(\epsilon)\|\,\|w(2t-\epsilon)\| + \tfrac{1}{2}T^{\frac{1}{2}}\|\dot{w}(\epsilon)\| \left[\int_\epsilon^{2t-\epsilon} \|w(\eta)\|^2\, d\eta\right]^{\frac{1}{2}}, \tag{4·2·7}$$

which, since the solutions u_i and v_i are assumed to lie in the constraint class (4·1·10), yields

$$\|w(t)\|^2 \leqslant \tfrac{1}{2}\|w(\epsilon)\|^2 + \tfrac{1}{2}M_{12}\|w(\epsilon)\| + \frac{1}{\sqrt{2}} M_{12} T \|\dot{w}(\epsilon)\| \quad (\epsilon \leqslant t < (T+\epsilon)/2). \tag{4·2·8}$$

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Lemma 4·2 furnishes bounds for $\|w(\epsilon)\|$ and $\|\dot{w}(\epsilon)\|$ in terms of ϵ (see (4·3·5), (4·3·6) respectively) and hence insertion of these estimates into (4·2·8) gives

$$\|w(t)\|^2 \leqslant M_{12}^2 \epsilon^{\frac{1}{2}} \left[\tfrac{1}{2} K_1 \epsilon^{\frac{1}{2}} + \tfrac{1}{2} K_1^{\frac{1}{2}} + \frac{1}{\sqrt{2}} T K_2^{\frac{1}{2}} \right], \tag{4·2·9}$$

thus proving the Proposition. ∎

It remains to state and prove Lemma 4·2.

4.3. *Estimates for* $\|w(\epsilon)\|$, $\|\dot{w}(\epsilon)\|$.

We first continue the solutions u_i, v_i and their time-derivatives as follows:

$$\tilde{u}_i(x,t) = \begin{cases} u_i(x,t) & (\epsilon f(x) < t < T, \quad x \in \Omega), \\ g_i(x) & (t \leqslant \epsilon f(x), \quad x \in \Omega), \end{cases} \tag{4·3·1}$$

$$\tilde{v}_i(x,t) = \begin{cases} v_i(x,t) & (0 < t < T, \quad x \in \Omega), \\ g_i(x) & (t \leqslant 0, \quad x \in \Omega), \end{cases} \tag{4·3·2}$$

$$\hat{u}_i(x,t) = \begin{cases} \dot{u}_i(x,t) & (\epsilon f(x) < t < T, \quad x \in \Omega), \\ h_i(x) & (t \leqslant \epsilon f(x), \quad x \in \Omega), \end{cases} \tag{4·3·3}$$

$$\hat{v}_i(x,t) = \begin{cases} \dot{v}_i(x,t) & (0 < t < T, \quad x \in \Omega), \\ h_i(x) & (t \leqslant 0, \quad x \in \Omega). \end{cases} \tag{4·3·4}$$

Then for fixed $x \in \Omega$, we see that at $t = \epsilon$ we have $w_i = \tilde{w}_i$, $\dot{w}_i = \hat{w}$ where

$$\tilde{w}_i(x,t) = \tilde{u}_i(x,t) - \tilde{v}_i(x,t), \; \hat{w}_i(x,t) = \hat{u}_i(x,t) - \hat{v}_i(x,t).$$

We are now ready to state and prove the Lemma.

LEMMA 4·2. *Let u_i, v_i satisfy the conditions of Proposition 4·1. Then the function w_i defined by* (4·2·1) *satisfies the estimates*

$$\|w(\epsilon)\|^2 \leqslant K_1 M_{12}^2 \epsilon, \tag{4.3.5}$$

$$\|\dot{w}(\epsilon)\|^2 \leqslant K_2 M_{12}^2 \epsilon, \tag{4·3·6}$$

where K_1 and K_2 are positive constants dependent on the initial data.

Proof. We first treat the bound (4·3·6). The derivation of the bound (4·3·5) for $\|w(\epsilon)\|$ will then follow similarly.

We have already seen for fixed $x \in \Omega$ that $\dot{w}_i(x,\epsilon) = \hat{w}_i(x,\epsilon)$. Upon further noting for fixed x that $\hat{w}_i(x,-\epsilon) = 0$, we clearly have

$$\begin{aligned} \|\dot{w}(\epsilon)\|^2 &= \|\hat{w}(\epsilon)\|^2 \\ &= 2 \int_{-\epsilon}^{\epsilon} (\hat{w}(\eta), \hat{w}_{,\eta}(\eta))\, d\eta \\ &= 2 \left[\int_{-\epsilon}^{\epsilon} \|\hat{w}(\eta)\|^2 d\eta \int_{-\epsilon}^{\epsilon} \|\hat{w}_{,\eta}(\eta)\|^2 d\eta \right]^{\frac{1}{2}}, \end{aligned} \tag{4·3·7}$$

which by virtue of Poincaré's inequality

$$\int_{-\epsilon}^{\epsilon} \|\hat{w}(\eta)\|^2 d\eta \leqslant \left(\frac{4\epsilon}{\pi}\right)^2 \int_{-\epsilon}^{\epsilon} \|\hat{w}_{,\eta}(\eta)\|^2 d\eta \tag{4·3·8}$$

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becomes

$$\|\dot{w}(\epsilon)\|^2 \leqslant \frac{8\epsilon}{\pi}\int_{-\epsilon}^{\epsilon}\|\hat{w}_{,\eta}(\eta)\|^2\,d\eta. \tag{4·3·9}$$

On the other hand, we also have

$$\|\hat{w}_{,\eta}(\eta)\|^2 \leqslant 2[\,\|\hat{u}_{,\eta}(\eta)\|^2+\|\hat{v}_{,\eta}(\eta)\|^2], \tag{4·3·10}$$

so that after appeal to definitions (4·3·3), (4·3·4) for $\hat{u}_i, \hat{v}_i$, we may conclude that

$$\int_{-\epsilon}^{\epsilon}\|\hat{w}_{,\eta}(\eta)\|^2\,d\eta \leqslant 2\left[\int_\Omega\int_{\epsilon f(x)}^{\epsilon}\rho(x)\,u_{i,\eta\eta}(x,\eta)\,u_{i,\eta\eta}(x,\eta)\,d\eta\,dx+\int_0^{\epsilon}\|v_{,\eta\eta}(\eta)\|^2\,d\eta\right]. \tag{4·3·11}$$

The proof is complete once the right side of the last inequality is bounded in terms of the data and the constraint constant M_{12}. We consider in detail the first integral on the right omitting in the subsequent discussion the dependence of functions upon their arguments. We clearly have

$$\int_\Omega\int_{\epsilon f}^{\epsilon}\rho u_{i,\eta\eta}\,u_{i,\eta\eta}\,d\eta\,dx \leqslant \int_\Omega\int_{\epsilon f}^{T}\rho\left(\frac{T-\eta}{T-\epsilon}\right)^4 u_{i,\eta\eta}\,u_{i,\eta\eta}\,d\eta\,dx \tag{4·3·12}$$

$$= \tfrac{1}{2}(I_1+I_2), \tag{4·3·13}$$

where the integrals I_1, I_2 are defined to be

$$I_1 = \int_\Omega\int_{\epsilon f}^{T}\left(\frac{T-\eta}{T-\epsilon}\right)^4\{\rho u_{i,\eta\eta}\,u_{i,\eta\eta}+c_{ijkl}\,u_{i,\eta j}\,u_{k,\eta l}\}\,d\eta\,dx, \tag{4·3·14}$$

$$I_2 = \int_\Omega\int_{\epsilon f}^{T}\left(\frac{T-\eta}{T-\epsilon}\right)^4\{\rho u_{i,\eta\eta}\,u_{i,\eta\eta}-c_{ijkl}\,u_{i,\eta j}\,u_{k,\eta l}\}\,d\eta\,dx. \tag{4·3·15}$$

We next express I_1 in the equivalent form

$$I_1 = -\frac{1}{5(T-\epsilon)^4}\int_\Omega\int_{\epsilon f}^{T}[(T-\eta)^5]_{,\eta}\{\rho u_{i,\eta\eta}\,u_{i,\eta\eta}+c_{ijkl}\,u_{i,\eta j}\,u_{k,\eta l}\}\,d\eta\,dx,$$

integrate by parts with respect to both time and space, and use the differential equation (4·1·1), to obtain

$$\begin{aligned} I_1 &= -\frac{1}{5(T-\epsilon)^4}\int_\Sigma(T-\eta)^5\{\rho u_{i,\eta\eta}\,u_{i,\eta\eta}+c_{ijkl}\,u_{i,\eta j}\,u_{k,\eta l}\}\,n_\eta\,dS \\ &\quad+\frac{2}{5(t-\epsilon)^4}\int_\Omega\int_{\epsilon f}^{T}(T-\eta)^5\{\rho u_{i,\eta\eta}\,u_{i,\eta\eta\eta}+c_{ijkl}\,u_{i,\eta j}\,u_{k,\eta\eta l}\}\,d\eta\,dx \\ &= -\frac{1}{5(T-\epsilon)^4}\int_\Sigma(T-\eta)^5\{(\rho u_{i,\eta\eta}\,u_{i,\eta\eta}+c_{ijkl}\,u_{i,\eta j}\,u_{k,\eta l})\,n_\eta-2c_{ijkl}\,u_{k,\eta l}\,u_{i,\eta\eta}\,n_j\}\,dS \end{aligned} \tag{4·3·16}$$

where n is the unit outward normal on the surface Σ defined by $t=\epsilon f(x)$, $x\in\Omega$.

Justification for the last calculation is immediate for classical solutions, but for weak solutions extra differentiability is required to that imposed by the definition of the solution. What is needed is that in a suitable weak sense second and third derivatives must exist and derivatives of solutions must be solutions.

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In order to obtain a bound in terms of the data, we decompose I_1 into the sum of integrals

$$I_1 = J_1 + J_2 + J_3, \tag{4·3·17}$$

where

$$J_1 = \frac{1}{5(T-\epsilon)^4}\int_\Sigma (T-\eta)^5 c_{ijkl}\, u_{i,\eta\eta}\left(n_j\frac{\partial}{\partial\eta} - n_\eta\frac{\partial}{\partial x_j}\right)u_{k,l}\, dS, \tag{4·3·18}$$

$$J_2 = \frac{1}{5(T-\epsilon)^4}\int_\Sigma (T-\eta)^5 \{\rho u_{i,\eta\eta}\, u_{i,\eta\eta} - c_{ijkl}\, u_{i,\eta j}\, u_{k,\eta l}\}\, n_\eta\, dS, \tag{4·3·19}$$

$$J_3 = -\frac{2}{5(T-\epsilon)^4}\int_\Sigma (T-\eta)^5 c_{ijkl,j}\, u_{i,\eta\eta}\, u_{k,l}\, n_\eta\, dS. \tag{4·3·20}$$

We next decompose the integrand appearing in J_2 into normal and tangential components on Σ, and in this way obtain

$$J_2 = \frac{1}{5(T-\epsilon)^4}\int_\Sigma (T-\eta)^5 \{\rho n_\eta^2\,\delta_{ik} - c_{ijkl}\, n_j\, n_l\}\, n_\eta^3\frac{\partial^2 u_i}{\partial n^2}\frac{\partial^2 u_k}{\partial n^2}\, dS + D_2, \tag{4·3·21}$$

where D_2 (and more generally D_i, $i = 1, 2, \ldots$) denotes terms depending solely upon the data. Similar decompositions for J_1 and J_3 lead to

$$J_1 = \frac{2}{5(T-\epsilon)^4}\int_\Sigma (T-\eta)^5 c_{ijkl}\frac{\partial^2 u_i}{\partial n^2}\left(n_j\frac{\partial}{\partial\eta} - n_\eta\frac{\partial}{\partial x_j}\right)u_{k,l}\, n_\eta^2\, dS + D_1, \tag{4·3·22}$$

$$J_3 = -\frac{2}{5(T-\epsilon)^4}\int_\Sigma (T-\eta)^5 c_{ijkl,j}\frac{\partial^2 u_i}{\partial n^2}\, u_{k,l}\, n_\eta^3\, dS + D_3.$$

We observe that the terms

$$\left(n_j\frac{\partial}{\partial\eta} - n_\eta\frac{\partial}{\partial x_j}\right)u_{k,l} \quad \text{and} \quad u_{k,l} \tag{4·3·23}$$

appearing in the integrands of J_1, and J_3 are completely known from the data specified on Σ.

Now we recall assumption (4·1·9) together with the assumption that Σ is a non-characteristic surface of the differential equation (4·1·1) so that the matrix (4·1·8) is positive-definite on Σ. It then follows from the arithmetic-geometric mean inequality that terms in the integrands of J_1 and J_3 involving second order normal derivatives may be dominated by those appearing in the integrand of J_2. Hence, we are able to bound I_1 in terms of the data alone.

In order to derive a bound for the term I_2, we integrate by parts and employ the differential equation (4·1·1) to express I_2 in the equivalent form

$$I_2 = \frac{1}{2(T-\epsilon)^4}\int_\Sigma (T-\eta)^4 \{\rho u_{i,\eta\eta}\, u_{,\eta}\, n_\eta - c_{ijkl}\, u_{i,\eta}\, u_{k,l\eta}\, n_j\}\, dS$$
$$+\frac{2}{(T-\epsilon)^4}\int_\Omega\int_{\epsilon f}^T (T-\eta)^3 \rho u_{i,\eta\eta}\, u_{i,\eta}\, d\eta\, dx, \tag{4·3·24}$$

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which after some algebraic rearrangement and a time integration becomes

$$I_2 = \frac{1}{2(T-\epsilon)^4}\int_\Sigma (T-\eta)^4 c_{ijkl}u_{i,\eta}\left(n_\eta\frac{\partial}{\partial x_j}-n_j\frac{\partial}{\partial\eta}\right)u_{k,l}\,dS$$
$$+\frac{1}{(T-\epsilon)^4}\int_\Sigma (T-\eta)^3\rho u_{i,\eta}u_{i,\eta}\,dS+\frac{3}{(T-\epsilon)^4}\int_\Omega\int_{\epsilon f}^T (T-\eta)^2\rho u_{i,\eta}u_{i,\eta}\,d\eta\,dx$$
$$+\frac{1}{2(T-\epsilon)^4}\int_\Sigma (T-\eta)^4 c_{ijkl,j}u_{i,\eta}u_{k,l}n_\eta\,dS. \tag{4·3·25}$$

As we have seen already in the discussion of the integral I_1, all the integrals over Σ in the expression (4·3·25) involve terms depending only on the data. Thus we have

$$I_2 = P + D_4, \tag{4·3·26}$$

where
$$P = \frac{3}{(T-\epsilon)^4}\int_\Omega\int_{\epsilon f}^T (T-\eta)^2\rho u_{i,\eta}u_{i,\eta}\,d\eta\,dx, \tag{4·3·27}$$

and, as before, D_4 indicates data terms. We now set

$$P = P_1 + P_2,$$

where
$$P_1 = \frac{3}{2(T-\epsilon)^4}\int_\Omega\int_{\epsilon f}^T (T-\eta)^2\{\rho u_{i,\eta}u_{i,\eta}+c_{ijkl}u_{i,j}u_{k,l}\}\,d\eta\,dx, \tag{4·3·28}$$

$$P_2 = \frac{3}{2(T-\epsilon)^4}\int_\Omega\int_{\epsilon f}^T (T-\eta)^2\{\rho u_{i,\eta}u_{i,\eta}-c_{ijkl}u_{i,j}u_{k,l}\}\,d\eta\,dx. \tag{4·3·29}$$

The integral P_1 may be treated in exactly the same manner as I_1 and hence may be bounded in terms of data. To estimate P_2, we integrate by parts in (4·3·29) to obtain successively

$$P_2 = \frac{3}{2(T-\epsilon)^4}\int_\Sigma (T-\eta)^2\{\rho u_i u_{i,\eta}n_\eta - c_{ijkl}u_{k,l}u_i n_j\}\,dS$$
$$+\frac{3}{(T-\epsilon)^4}\int_\Omega\int_{\epsilon f}^T (T-\eta)\rho u_i u_{i,\eta}\,d\eta\,dx$$
$$=\frac{3}{2(T-\epsilon)^4}\int_\Omega\int_{\epsilon f}^T \rho u_i u_i\,d\eta\,dx + D_5$$
$$\leqslant \frac{3(T+\epsilon)}{2(T-\epsilon)^4}M_{12}^2 + D_5, \tag{4·3·30}$$

where in the last line we have used the constraint (4·1·10).

We have thus succeeded in establising upper bounds for I_1 and I_2 in terms of the data and the constraint constant M_{12}. Upon inserting the various estimates into (4·3·13) we conclude that

$$\int_\Omega\int_{\epsilon f}^T \rho u_{i,\eta\eta}u_{i,\eta\eta}\,d\eta\,dx \leqslant \frac{3(T+\epsilon)}{4(T-\epsilon)^4}M_{12}^2 + D_6, \tag{4·3·31}$$

and therefore on supposing that the data term D_6 satisfies

$$D_6 \leqslant k_1 M_{12}^2, \tag{4·3·32}$$

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for some positive constant k_1, we obtain the final estimate

$$\int_\Omega \int_{\epsilon f}^{T} \rho u_{i,\eta\eta} u_{i,\eta\eta} \, d\eta \, dx \leqslant K_3 M_{12}^2 \tag{4·3·33}$$

with K_3 a positive constant.

In order to obtain the corresponding bound for the second integral $\int_0^\epsilon \|v_{,\eta\eta}(\eta)\|^2 \, d\eta$ occurring in (4·3·11), we simply replace the surface Σ by $t = 0$ in the previous calculations and note that on $t = 0$ we have $n_t = -1$, $n_j = 0$. The requisite bound is thus of the same form as (4·3·33).

On substituting these estimates into (4·3·11) and (4·3·9) we arrive at the given bound (4·3·6) for $\|\dot{w}(\epsilon)\|^2$.

We remark that for the constraint set (4·1·10) we could have taken the factor $[(T-\eta)/(T-\epsilon)]^3$ instead of $[(T-\eta)/(T-\epsilon)]^4$ in the initial inequality (4·3·12).

We next consider the derivation of the bound (4·3·5) for $\|w(\epsilon)\|^2$. Indeed, this bound may be easily determined by broadly repeating the procedure just described leading to the estimate for $\|\dot{w}(\epsilon)\|^2$. In terms of the continuations (4·3·1), (4·3·2) for the solutions u_i, v_i respectively, the inequalities analogous to (4·3·9) become

$$\begin{aligned} \|w(\epsilon)\|^2 &\leqslant \frac{8\epsilon}{\pi} \int_{-\epsilon}^{\epsilon} \|\tilde{w}_{,\eta}(\eta)\|^2 \, d\eta \\ &\leqslant \frac{16\epsilon}{\pi} \left[\int_\Omega \int_{\epsilon f}^{\epsilon} \rho u_{i,\eta} u_{i,\eta} \, d\eta \, dx + \int_0^\epsilon \|v_{,\eta}(\eta)\|^2 \, d\eta \right]. \end{aligned} \tag{4·3·34}$$

The integrals on the right of (4·3·34) may be treated by means of the techniques used to bound the integral P in (4·3·27), and so we may immediately establish the required estimate (4·3·5). The proof of the lemma is thus complete.

One final remark deserves mention. Let us set $\epsilon = 0$ and consider the inequality

$$\int_0^t \int_\Omega \rho u_{i,\eta\eta} u_{i,\eta\eta} \, dx \, d\eta \leqslant \frac{16}{T^4} \int_0^T \int_\Omega (T-\eta)^4 \rho u_{i,\eta\eta} u_{i,\eta\eta} \, dx \, d\eta \quad (0 < t < T/2).$$

We may thus employ our procedure to derive a bound for the integral on the right in terms of the initial data and the constraint constant M_{12} provided the solution u_i lies in the constraint set (4·1·10). Such a result is of importance elsewhere. In particular it may be combined with arguments analogous to those of §3.5 to derive continuous dependence of the solution upon the density. (See the Remark at the end of §3.5.)

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