Euro. Jnl of Applied Mathematics (2011), *vol.* 22, *pp.* 151–168. © Cambridge University Press 2011
doi:10.1017/S0956792510000367

A Mellin transform solution to a second-order pantograph equation with linear dispersion arising in a cell growth model

B. VAN BRUNT[1] and G. C. WAKE[2]

[1]*Institute of Fundamental Sciences, Massey University Manawatu, Private Bag 11-222, Palmerston North 4442, New Zealand*
[2]*Centre for Mathematics-in-Industry, Institute of Information and Mathematical Sciences, Massey University Auckland, Private Bag 102-904, NSMC, Auckland 0745, New Zealand*
email: g.c.wake@massey.ac.nz

(*Received 11 June 2010; revised 2 December 2010; accepted 2 December 2010; first published online 5 January 2011*)

In this paper we study the probability density function solutions to a second-order pantograph equation with a linear dispersion term. The functional equation comes from a cell growth model based on the Fokker–Planck equation. We show that the equation has a unique solution for constant positive growth and splitting rates and construct the solution using the Mellin transform.

Key words: Functional differential equations, existence and uniqueness, exact solutions; steady-size distributions

1 Introduction

The first-order pantograph equation,

$$y'(x) + by(x) = \lambda y(\alpha x), \tag{1.1}$$

where $b > 0$, $\lambda \neq 0$ and $\alpha > 1$ are constants, has been studied by numerous researchers (e.g. [6,14,16,18]) because of a growing number of applications ranging from the collection of current in an electric locomotive [8,20] to modelling the growth of tumour cells [2]. In certain applications, such as a cell growth model of Hall and Wake ([11,12]), an absorption probability problem originating in waiting line theory [10] and light absorption in the Milky Way [1], the function y represents a probability density function (pdf). In these problems, $\lambda = b\alpha$, and the boundary conditions are

$$\lim_{x\to 0^+} y(x) = 0, \quad \lim_{x\to\infty} y(x) = 0, \tag{1.2}$$

along with the normalising condition

$$\int_0^\infty y(x)\,dx = 1. \tag{1.3}$$

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

Part of the reason for the frequent appearance of this equation is that it arises as a consequence from the first-order Markov process,

$$Z_k = \frac{1}{\alpha} Z_{k-1} + X_k$$

(cf. [1, 4, 5, 10, 19]).

It is well known that given an initial value

$$y(0) = y_0,$$

equation (1.1) has an infinite number of solutions (cf. [16]). In contrast, there is a unique solution to equation (1.1) that satisfies conditions (1.2) and (1.3). The boundary-value problem is well posed, and it is of note that the condition $y \geqslant 0$ is not imposed, yet the solution satisfies this condition.

The second-order pantograph equations, where the unknown function is required to be a pdf, arise in the cell growth model of Hall and Wake when the effects of dispersion are incorporated into the model. The general equation is the Fokker–Planck equation in time and one spatial variable (the size of the cell as measured by the DNA content). The Fokker–Planck equation comes about by treating the cell-division process as a stochastic process with variance σ^2. By following the standard transformations (see [9]), the Langevin equation becomes the Fokker–Planck equation for the resulting evolving (in time t) probability distribution $n(x, t)$ of cells structured by size x. If one looks for steady size distributions (separable solutions) for this Fokker–Planck equation, then a resulting equation is of the pantograph type. This model is detailed in [3] in connexion with a model of plankton cell growth. The key equation is

$$(D(x)y(x))'' - (g(x)y(x))' + \alpha^2 B(\alpha x)y(\alpha x) - (B(x) + \Lambda)y(x) = 0, \tag{1.4}$$

where $D(= \sigma^2/2)$ is the dispersion, g is the rate of growth, B is the rate at which cells divide, $\alpha > 1$ is the splitting constant (usually 2) and Λ is a constant that arises from separation of variables. The function y represents a pdf for the cell size, i.e. for $0 \leqslant x_1 \leqslant x_2$ the probability of a cell of size z to be between x_1 and x_2 is

$$P(x_1 < z < x_2) = \int_{x_1}^{x_2} y(\xi)\, d\xi.$$

In this model it is assumed that the mortality is constant and hence this term does not appear in equation (1.4), although it does appear in the time-dependent factor of the solution to the Fokker–Planck equation (cf. [3]). Since y must be a pdf, $y \in L^1[0, \infty)$. In fact, the condition $By \in L^1[0, \infty)$ is also imposed. The boundary conditions for this model come from the integrability conditions and are

$$\lim_{x \to \infty} y(x) = 0, \tag{1.5}$$

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

and the zero flux conditions

$$\lim_{x\to\infty}\{(D(x)y(x))' - g(x)y(x)\} = 0, \tag{1.6}$$

$$\lim_{x\to 0^+}\{(D(x)y(x))' - g(x)y(x)\} = 0. \tag{1.7}$$

In addition, the condition $By \in L^1[0,\infty)$ restricts the value of the constant Λ. Integrating equation (1.4) from 0 to ∞ gives

$$\Lambda = (\alpha - 1)\int_0^\infty B(x)y(x)\,dx. \tag{1.8}$$

The simplest version of the second-order problem is when the dispersion, growth rate and splitting rate are constant. This choice leads to an equation of the form

$$y''(x) - ay'(x) - by(x) = b\alpha y(\alpha x), \tag{1.9}$$

where a and b are positive constants. Solutions to this equation have been studied in [17] and [24]. As with the first-order case, the problem has a unique solution and this solution is non-negative so that y is a pdf. The related equation

$$y''(x) - ay'(x) - by(x) = \lambda y(\alpha x),$$

was studied by van Brunt *et al.* [23] as the singular Sturm–Liouville problem with λ as the eigenvalue parameter. Under the assumption of exponential decay, the first eigenvalue is $\lambda = b\alpha$, and the zeros of the higher eigenfunctions behave much like the zeros of eigenfunctions from the classical Sturm–Liouville problem. The first eigenfunction has no positive zeros, the second one has precisely one positive zero etc. This work gives some insight into the positivity of solutions to cell growth models.

Pantograph equations with non-constant coefficients have received less attention than the constant coefficient case, partially due to the difficulty of obtaining solutions. Hall and Wake [13] studied equations of the form

$$y'(x) + bx^n y(x) = b\alpha^{n+1}x^n y(\alpha x) \tag{1.10}$$

in a model of cell growth with exponential growth. For the second-order case, the plankton model [3] uses a generalised function for the rate of cells splitting. Some isolated examples of the pantograph-type equations with non-constant coefficients outside the cell growth model have been studied by Kim [17].

In this paper we look at a class of second-order pantograph equations of the form (1.4) with non-constant dispersion. In particular, we look at the case $D(x) = Dx$ and the other coefficients are constant. Here, D is a positive constant. We show that the problem is well posed and derive a solution for the pdf. The pdf is a generalised version of the K-distribution that has found applications in non-Gaussian light scattering configurations among other fields [15]. A key tool in this investigation is the Mellin transform, which can be used to characterise solutions that are generated by solutions to the homogeneous (non-functional) equation. The technique is a useful tool for solving certain pantograph-type equations and it provides some insight into the resulting distributions. The Mellin

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

transform also provides a convenient method for obtaining the asymptotic behaviour of solutions to the pantograph equation, as $x \to 0^+$ and $x \to \infty$. The Mellin transform is best suited for obtaining the solutions of the equations above involving $y(\alpha x)$. It has been acknowledged that the transformation $x = e^z$, $w(z) = y(x)$, would transform it into an equation with a term $w(z + \ln(\alpha))$, where the non-local effect is advanced as $\ln(\alpha) > 0$. This could then be solved using the more usual method of the Laplace transforms as in [12, 13], but the details are technically more difficult than what we do get using the Mellin transform.

In the next section we note a straightforward connexion between the Mellin transform of solutions and the use of a homogeneous solution generator to get the pdf. A simple case of the first-order problem is studied in Section 3 to introduce ideas and formulae that will be used in the Section 5. Some qualitative properties of solutions to second-order case are given in Section 4. In Section 5, the pdf solutions are derived.

2 A Mellin transform characterisation of solutions

The pdf solution to equation (1.1) is a series of the form

$$y(x) = \sum_{k=0}^{\infty} a_k e^{-b\alpha^k x},$$

and the pdf solution to equation (1.9) is of the form

$$y(x) = \sum_{k=0}^{\infty} b_k e^{-r\alpha^k x},$$

where r is the positive root of

$$r^2 + ar - b = 0,$$

and the pdf solution to equation (1.10) is of the form

$$y(x) = \sum_{k=0}^{\infty} c_k e^{-b\alpha^{k(n+1)} \frac{x^{n+1}}{n+1}}.$$

Here, the a_k, b_k and c_k are constants that are determined recursively. The Dirichlet series solutions for these equations have a feature in common: They are all generated by the homogeneous solution to the ordinary differential equation. For instance, the homogeneous equation

$$y'(x) + by(x) = 0$$

has a solution $y_h(x) = e^{-bx}$, and the pdf solution to equation (1.1) is of the form

$$\sum_{k=0}^{\infty} a_k y_h(\alpha^k x).$$

Similar comments apply to the other equations, although the solution to the homogeneous equation for (1.9) that decays as $x \to \infty$ must be used for convergence. If L is a general

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

linear first- or second-order operator, the question naturally arises, under what conditions does the functional equation

$$Ly = r(x)y(\alpha x)$$

has a solution of the form

$$\sum_{k=0}^{\infty} c_k y_h(\alpha^k x), \tag{2.1}$$

where c_k is a constant and y_h is a solution to

$$Ly = 0.$$

Solutions of the form (2.1) can be characterised through the Mellin transform. These solutions are special in the Mellin transform space because the transform can be expressed as a product of the transform of y_h and a function that is holomorphic in the variable α^{-s}, where s is the transform variable. For this class of solutions, the effect of the functional term is separated from that of the homogeneous operator.

Let f be a function defined on $[0, \infty)$ such that $f(x) = O(x^a)$ as $x \to 0^+$ and $f(x) = O(x^b)$ as $x \to \infty$. Recall that the Mellin transform of f is defined as

$$M[f(x); s] = \int_0^{\infty} x^{s-1} f(x)\, dx,$$

for s in the fundamental strip $-a < \mathrm{Re}\, s < -b$. An important property of this transform is the relation

$$M[f(\alpha x); s] = \frac{1}{\alpha^s} M[f(x); s]. \tag{2.2}$$

If a function y can be represented on $[0, \infty)$ by a series uniformly convergent on $[0, \infty)$ of the form

$$y(x) = \sum_{k=0}^{\infty} c_k f(\alpha^k x), \tag{2.3}$$

then the series can be integrated term by term and relation (2.2) can be used to get

$$M[y(x); s] = M[f(x); s] \sum_{k=0}^{\infty} \frac{c_k}{\alpha^{ks}}, \tag{2.4}$$

where s is in the intersection of the fundamental strip for f and the domain of absolute convergence (a half plane) for the Dirichlet series. Assuming that this intersection is non-empty, functions y that can be represented in the form (2.3) have the Mellin transforms consisting of the product of $M[f(x); s]$ with a function holomorphic in α^{-s}. Alternatively, if y has the Mellin transform of the form (2.4) for s in some fundamental strip, then the series can be inverted term by term to get a representation of y in the form (2.3). In this manner, the Mellin transform can be used to characterise solutions of the form (2.1) arising from the pantograph-type equations.

Finally, we note that the Mellin transforms of the form (2.4) have a concrete interpretation when the functions involved are pdfs. If X_1 and X_2 are independent random variables with pdfs f and q, respectively, and h denotes the pdf for the product $X_1 X_2$,

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

then h is given by the Mellin convolution as

$$h(x) = \int_0^\infty \frac{1}{\xi} f\left(\frac{x}{\xi}\right) q(\xi)\, d\xi.$$

If H, F and Q denote the Mellin transforms of h, f and q, respectively, then

$$H(s) = F(s)G(s)$$

(cf. [7,21]).

3 First-order problem

The basic pantograph equation

$$y'(x) + y(x) = \alpha y(\alpha x) \tag{3.1}$$

along with the conditions (1.2) and (1.3) provide a simple framework to illustrate some of the commments/observations of the previous section, and also to introduce a function that will be used in Section 5. For succinctness, let $M(s) = M[y(x); s]$. Applying the Mellin transform to equation (3.1) and using the boundary conditions yields the difference equation

$$-(s-1)M(s-1) + M(s) = \frac{1}{\alpha^{s-1}} M(s), \tag{3.2}$$

along with the condition

$$M(1) = 1. \tag{3.3}$$

The homogeneous equation is

$$y_h'(x) + y_h(x) = 0,$$

and the corresponding transform equation is

$$-(s-1)F(s-1) + F(s) = 0, \tag{3.4}$$

where $F(s) = M[y_h(x); s]$. Evidently, $y_h(x) = e^{-x}$ is a solution to the homogeneous problem, so we can use $F(s) = \Gamma(s)$. We seek a solution to (3.2) of the form

$$M(s) = F(s)Q(s). \tag{3.5}$$

Substituting this form into equation (3.2) and using (3.4) give

$$Q(s-1) = \left(1 - \frac{1}{\alpha^{s-1}}\right) Q(s), \tag{3.6}$$

which has a solution

$$Q(s) = C \prod_{k=0}^{\infty} \left(1 - \frac{1}{\alpha^{k+s}}\right), \tag{3.7}$$

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

where C is a constant. A solution to (3.2) is therefore

$$M(s) = C\Gamma(s)\prod_{k=0}^{\infty}\left(1-\frac{1}{\alpha^{k+s}}\right), \tag{3.8}$$

and condition (3.3) implies that

$$C = \prod_{k=0}^{\infty}\left(1-\frac{1}{\alpha^{k+1}}\right)^{-1}. \tag{3.9}$$

The Mellin transform (3.8) can be inverted conveniently once the infinite product is put in the form of an infinite series. This conversion is done using the Euler identity

$$\prod_{k=0}^{\infty}(1+zq^k) = 1+\sum_{k=1}^{\infty}\frac{q^{k(k-1)/2}z^k}{\prod_{j=1}^{k}(1-q^j)}, \tag{3.10}$$

which is valid for $|q|<1$ and $z\in\mathbb{C}$. Let $q=1/\alpha$ and $z=-1/\alpha^s$, then

$$Q(s) = C\left(1+\sum_{k=1}^{\infty}\frac{p_k}{\alpha^{ks}}\right), \tag{3.11}$$

where for $k\geqslant 1$,

$$p_k = \frac{(-1)^k}{\alpha^{k(k-1)/2}\prod_{j=1}^{k}(1-\alpha^{-j})}.$$

The Dirichlet series (3.11) is uniformly convergent in the right half plane and the transform $M(s)$ can thus be inverted term by term to get

$$y(x) = C\sum_{k=0}^{\infty}p_k e^{-\alpha^k x}, \tag{3.12}$$

where $p_0=1$. The above solution corresponds to that given by Hall and Wake [12], which was derived using the Laplace transforms. They also proved that the solution was unique and non-negative.

The solution methods using the Laplace or Mellin transforms are equivalent for this problem. The factored form of the Mellin transform is interesting because F contains all the information about the generating pdf (Poisson) and Q contains the information about the splitting process. All the effects of the functional term in the Mellin transform space are concentrated in Q. The function Q inverts to the pdf

$$q(x) = C\sum_{k=0}^{\infty}p_k\delta(\alpha^k(x-1)),$$

where δ is the Dirac delta function.

The pdf (3.12) is an example of the general Erlang distribution, which as Morgan [19] notes also appears in queuing theory, reliability theory and psychology. The Mellin transform approach makes it clear how to change the generating distribution. For instance,

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

suppose that we wish a gamma distribution instead of Poisson, but also wish to keep the same splitting process. We simply choose F to be the Mellin transform of the gamma distribution; more interestingly, we might ask what type of pantograph equation produces this distribution. A solution to the differential equation

$$y'(x) + \left(1 - \frac{p}{x}\right) y(x) = 0,$$

where p is a constant, is $y_h(x) = x^p e^{-x}$. Proceeding as with the previous example, we find that the Mellin transform of the solution to the pantograph equation

$$y'(x) + \left(1 - \frac{p}{x}\right) y(x) = \alpha y(\alpha x)$$

is

$$M(s) = C \frac{\Gamma(s+p)}{\Gamma(p+1)} \prod_{k=0}^{\infty} \left(1 - \frac{1}{\alpha^{k+s}}\right),$$

where C is as defined in (3.9). The pdf solution to this pantograph equation is thus

$$y(x) = \frac{C}{\Gamma(p+1)} \sum_{k=0}^{\infty} p_k \alpha^{kp} x^p e^{-\alpha^k x}.$$

The constant C is itself of interest because of its connexions with partition functions and various special functions. Methods for determining C were discussed by Hall and Wake [12] and later by Morgan [19]. Both note that the Euler pentagonal number theorem can be invoked to show that

$$C^{-1} = \sum_{-\infty}^{\infty} (-1)^k \alpha^{-k(3k+1)/2},$$

which converges faster than the infinite product. Morgan notes, however, that the series is not well conditioned because it involves the subtraction of large numbers. Both also note that complete elliptic integrals can be used to get C. Yet another representation is

$$C = \alpha^{1/24} \left(\frac{1}{2} \vartheta_1' \left(0, \frac{1}{\sqrt{\alpha}}\right)\right)^{1/3},$$

where $\vartheta_1(z, q)$ is a Jacobi theta function and $'$ denotes d/dz. Doubtless, there are many more representations. For the connoisseur of special functions, we note, following Morgan, that C can be expressed in terms of the Dedekind eta function

$$\eta(\tau) = \bar{q}^{1/24} \prod_{k=1}^{\infty} (1 - \bar{q}^k),$$

where

$$\bar{q} = e^{2\pi i \tau}.$$

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

In terms of this function

$$C = \alpha^{1/24}\eta\left(\frac{-\log\alpha}{2\pi i}\right).$$

Morgan uses this function to approximate C in an efficient manner for $\alpha < e^{2\pi}$. He also proves the interesting functional relation

$$\frac{C(\alpha^2)}{C^2(\alpha)} = \vartheta_4(0,\alpha),$$

where

$$C(\alpha) = \prod_{k=0}^{\infty}\left(1-\frac{1}{\alpha^{k+1}}\right)^{-1},$$

and $\vartheta_4(z,q)$ is another Jacobi theta function.

4 A second-order problem with linear dispersion: qualitative results

The second-order cell growth model has been studied only for the simplest cases when the dispersion is constant. In this section we study equation (1.4) for the case $D(x) = Dx$ and constant rates of growth and splitting. Here, D, g and B are positive constants. In this case equation (1.4) reduces to

$$xy''(x) - ay'(x) - by(x) = -b\alpha y(\alpha x), \tag{4.1}$$

where

$$a = \left(\frac{g}{D} - 2\right),$$

and

$$b = \frac{B\alpha}{D}.$$

Since g and B are positive, $b > 0$ and $a > -2$.

We look for a classical solution to equation (4.1) that is also a pdf. Let $\mathbb{D}^2(0,\infty)$ denote the space of functions that are twice differentiable in $(0,\infty)$. It will be shown in this section that if a solution exists that is in $L^1[0,\infty)$ such that the normalising condition (1.3) is satisfied, then it is unique and positive for $x > 0$.

Let

$$\Delta(x) = \int_x^{\infty} y(\xi)\,d\xi.$$

If $y \in L^1[0,\infty)$, then this function is well defined for $x \geqslant 0$; if y is a pdf, then Δ corresponds the compliment of the cumulative distribution function. Integrating equation (4.1) from x to ∞ gives

$$x\Delta''(x) - (a+1)\Delta'(x) = b(\Delta(x) - \Delta(\alpha x)). \tag{4.2}$$

The next lemma shows that Δ has no extrema in $(0,\infty)$.

Lemma 4.1 *Suppose that $\Delta \in \mathbb{D}^2(0,\infty)$ is a non-trivial solution to equation (4.2) such that $\Delta(x) \to 0$ as $x \to \infty$. Then, Δ does not have extrema in $(0,\infty)$.*

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

Proof Suppose that Δ has a positive local maximum at $x_1 > 0$. Then $\Delta'(x_1) = 0$, $\Delta''(x_1) \leqslant 0$, and equation (4.2) implies

$$\Delta(x_1) - \Delta(\alpha x_1) \leqslant 0.$$

Since $\Delta(x) \to 0$ as $x \to \infty$, there must be another positive local maximum at some point $x_2 \geqslant \alpha x_1$ and

$$\Delta(x_1) \leqslant \Delta(x_2).$$

The argument can be repeated with x_2 to establish another local maximum at $x_3 \geqslant \alpha x_2$ such that $\Delta(x_2) \leqslant \Delta(x_3)$, and it is plain that a sequence $\{x_k\}$ can be constructed such that $x_k \to \infty$ as $k \to \infty$ and

$$0 < \Delta(x_1) \leqslant \Delta(x_k).$$

The above inequality contradicts the condition that $\Delta(x) \to 0$ as $x \to \infty$ and therefore Δ cannot have a positive local maximum.

The above argument can be applied to $-\Delta$ to show that Δ cannot have a negative local minimum in $(0, \infty)$. The absence of these extrema precludes the existence of negative local maxima and positive local minima in $(0, \infty)$. Since Δ is non-trivial, points $x > 0$ at which $\Delta(x) = 0$ cannot be extrema without introducing non-zero extrema. □

Corollary 4.2 *Under the conditions of Lemma 4.1,* $\Delta'(x) \neq 0$ *for all* $x > 0$.

Proof Suppose that $\Delta'(x_0) = 0$ for some $x_0 > 0$. Lemma 4.1 indicates that Δ has no extrema in $(0, \infty)$; therefore, $\Delta''(x_0) = 0$. Equation (4.2) implies

$$\Delta(x_0) = \Delta(\alpha x_0),$$

and since there are no extrema, we conclude that $\Delta(x) = \Delta(x_0)$ for all $x \in [x_0, \alpha x_0]$. Thus we have $\Delta'(\alpha x_0) = 0$, so that the argument can be repeated. In this manner we see that $\Delta(x) = \Delta(x_0)$ for all $x \in [x_0, \infty)$. Since $\Delta(x) \to 0$ as $x \to \infty$, we have $\Delta(x) = 0$ for all $x \in [x_0, \infty)$.

If $x \in [x_0/\alpha, x_0]$, then equation (4.2) implies

$$x\Delta''(x) - (a+1)\Delta(x) = 0,$$

and the continuity of Δ provides the initial conditions

$$\Delta(x_0) = \Delta'(x_0) = 0.$$

Picard's theorem shows that the unique solution to this initial-value problem is the trivial solution; hence, $\Delta(x) = 0$ for all $x \in [x_0/\alpha, \infty)$. The argument can be repeated k times to show that $\Delta(x) = 0$ for all $x \in [x_0/\alpha^k, \infty)$. Therefore, $\Delta(x) = 0$ for all $x \in (0, \infty)$. By hypothesis Δ is not the trivial solution and therefore $\Delta'(x) \neq 0$ for any $x > 0$. □

The above results can be used to establish a unique and positive result so that a solution to the problem is a pdf.

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

Theorem 4.3 *If there exists a solution* $y \in L^1[0,\infty) \cap \mathbb{D}^2(0,\infty)$ *to equation (4.1) that satisfies (1.3), then the solution is unique. Moreover,* $y(x) > 0$ *for all* $x > 0$.

Proof Suppose that y and w are two solutions in $L^1[0,\infty) \cap \mathbb{D}^2(0,\infty)$ and let $z = y - w$. Then $z \in L^1[0,\infty) \cap \mathbb{D}^2[0,\infty)$ satisfies equation (4.1) and

$$\lim_{x\to 0^+} \int_x^\infty z(x)\,dx = 0.$$

Let

$$\hat{\delta}(x) = \int_x^\infty z(\xi)\,d\xi.$$

Then $\hat{\delta}$ is a solution to equation (4.2) that satisfies the conditions of Lemma 4.1. Now, $\hat{\delta}$ is continuous, $\hat{\delta}(0) = 0$, $\hat{\delta}(x) \to 0$ as $x \to \infty$ and $\hat{\delta}$ cannot have any extrema in $(0,\infty)$; therefore $\hat{\delta}(x) = 0$ for all $x \geqslant 0$. This shows that $\hat{\delta}'(x) = -z(x) = 0$ for all $x > 0$ and therefore $y = w$.

The positivity of y follows immediately from Corollary 4.2. Specifically, $\Delta(0) = 1$ and $\Delta'(x) \neq 0$ for all $x > 0$. Since $\Delta(x) \to 0$ as $x \to \infty$, the function Δ is monotone strictly decreasing and therefore $y(x) = -\Delta'(x) > 0$ for all $x > 0$. □

The next result will be used in Section 5 to show that the pdf solutions are either unimodal or do not have local maxima in $(0,\infty)$.

Theorem 4.4 *Suppose* $y \in L^1[0,\infty) \cap \mathbb{D}^2(0,\infty)$ *is a solution to equation (4.1) that satisfies (1.3). Suppose further that* y *is holomorphic on the positive real axis and there is an* $\hat{x}$ *such that* $y'(x) < 0$ *for all* $x > \hat{x}$. *Then* y *has at most one local maximum in* $(0,\infty)$.

Proof Since y is holomorphic on the positive real axis, so is y'. It is clear that y cannot be a constant function and consequently y' is not identically zero on the positive real axis. The identity theorem from complex analysis shows that if y' has any zeros on this axis, they must be isolated. This means that any extrema for the solution y in $(0,\infty)$ must be isolated (e.g. no intervals where the function is constant).

Suppose that y has two local maxima in $(0,\infty)$. Since y must be positive in this interval, there must be a positive local minimum between the maxima. Let x_{m_1} denote a point at which y has a local minimum, and let $x_{M_1} > x_{m_1}$ be the closest point beyond x_{m_1} at which y has a local maximum. Then,

$$y(x_{m_1}) < y(x_{M_1}).$$

Now, $y''(x_{m_1}) \geqslant 0$ and $y'(x_{m_1}) = 0$; therefore, equation (4.1) implies

$$\alpha y(\alpha x_{m_1}) \leqslant y(x_{m_1}).$$

Similarly, at x_{M_1}

$$y(x_{M_1}) \leqslant \alpha y(\alpha x_{M_1}).$$

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

The above inequalities show that

$$y(\alpha x_{m_1}) \leqslant y(\alpha x_{M_1}),$$

where $\alpha x_{m_1} < \alpha x_{M_1}$. Since y cannot be asymptotic to any positive number, there must be another local maximum at some point $x_{M_2} > \alpha x_{M_1}$ and therefore another local minimum at some point $x_{m_2} > x_{M_1}$. The same argument can be applied to the minimum at x_{m_2} to deduce the existence of another local minimum. We can thus construct a sequence of local minima $\{x_{m_k}\}$ such that $x_{m_k} \to \infty$ as $k \to \infty$ and $y'(x_{m_k}) = 0$ for all $k \in \mathbb{N}$. This, however, contradicts the assumption that $y'(x) < 0$ for all $x > \hat{x}$. We thus conclude that y has at most one local maximum in $(0, \infty)$. □

5 A second-order problem with linear dispersion: pdf solutions

The homogeneous equation associated with equation (4.1) is

$$xy''(x) - ay'(x) - by(x) = 0,$$

which has a general solution of the form

$$y_h(x) = A_1 x^{\nu/2} I_\nu(2\sqrt{bx}) + A_2 x^{\nu/2} K_\nu(2\sqrt{bx}),$$

where A_1 and A_2 are constants, $\nu = a + 1 = g/D - 1$, and I_ν and K_ν denote the modified Bessel functions of order ν. Since g and D are positive,

$$-1 < \nu < \infty.$$

The function I_ν is not bounded as $x \to \infty$, but

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \tag{5.1}$$

as $x \to \infty$ for all ν. Moreover, for $\nu > 0$,

$$K_\nu(x) \sim \frac{\Gamma(\nu)}{2} \left(\frac{2}{x}\right)^\nu$$

as $x \to 0^+$, and since $K_\nu = K_{-\nu}$,

$$K_\nu(x) \sim \frac{\Gamma(|\nu|)}{2} \left(\frac{2}{x}\right)^{|\nu|} \tag{5.2}$$

as $x \to 0^+$ for all $\nu \neq 0$. In addition, it is known that

$$K_0(x) \sim -\left(\log\left(\frac{x}{2}\right) + \gamma\right), \tag{5.3}$$

where γ is Euler's constant, as $x \to 0^+$. Let

$$y_h(x) = x^{\nu/2} K_\nu(2\sqrt{bx}).$$

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

For any value of v, equation (5.1) implies that the homogeneous solution y_h decays exponentially as $x \to \infty$. The asymptotic relations (5.2) and (5.3), however, partition the classes of solutions according to their behaviour as $x \to 0^+$. Specifically, if $v > 0$, then

$$\lim_{x \to 0^+} y_h(x) = \frac{\Gamma(v)}{2b^{v/2}}; \tag{5.4}$$

if $v < 0$, then

$$y_h(x) \sim \frac{\Gamma(|v|)}{2b^{|v|/2}x^{|v|}}, \tag{5.5}$$

as $x \to 0^+$, so that $y_h(x) \to \infty$. For $v = 0$, equation (5.3) implies

$$y_h(x) \sim \frac{1}{2}\log x$$

as $x \to 0^+$.

We anticipate a pdf solution to equation (4.1) in the form of an infinite series in $y_h(\alpha^k x)$, and the behaviour of y_h as $x \to 0^+$ signals a partition in the solution classes based on the value of g/D. We use the term *fast growth rate* for the case $g/D > 1$ ($v > 0$) and *slow growth rate* for the case $g/D < 1$ ($v < 0$). The term *neutral growth rate* will be used for the case $g/D = 1$ ($v = 0$). We begin with the fast growth rate case.

Suppose $v > 0$. Applying the Mellin transform to equation (4.1) gives

$$\begin{aligned} &x^s y'(x)|_0^\infty - (s-a)x^{s-1}y(x)|_0^\infty \\ &+(s+a)(s-1)M(s-1) - M(s) = -\frac{1}{\alpha^{s-1}}M(s). \end{aligned}$$

We anticipate a solution in the form of a series of the modified Bessel functions K_v, which decay exponentially. The values of xy' and y as $x \to 0^+$ are not known at this stage, but the asymptotic behaviour of y_h as $x \to 0^+$ for $v > 1$ indicates that these terms vanish at $x = 0$. The boundary terms are set to zero. If the resulting solution from the Mellin transform is a pdf, then we can appeal to Theorem 4.3 to assert uniqueness. We thus look for solutions to

$$(s+a)(s-1)M(s-1) = b\left(1 - \frac{1}{\alpha^{s-1}}\right)M(s) \tag{5.6}$$

defined in some fundamental strip that includes $s = 1$ such that condition (3.3) is satisfied. Proceeding as in the first-order case, we seek solutions of the form (3.5), where

$$(s+a)(s-1)F(s-1) - bF(s) = 0. \tag{5.7}$$

We know that y_h solves the homogeneous problem; hence, a solution is

$$M[y_h(x); s] = \frac{\Gamma(s)\Gamma(s+a+1)}{2b^{(2s+a+1)/2}}.$$

We can use the normalised transform

$$F(s) = \frac{b^{(a+3)/2}}{\Gamma(a+2)}\frac{\Gamma(s)\Gamma(s+a+1)}{b^{(2s+a+1)/2}}. \tag{5.8}$$

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

Equations (5.6) and (5.7) imply that Q satisfies equation (3.6) and therefore $Q(s)$ is given by (3.7). In terms of the parameter, $v = g/D - 1$, a solution to equation (5.6) is therefore

$$M(s) = \frac{Cb^{(v+2)/2}}{\Gamma(v+1)} \frac{\Gamma(s)\Gamma(s+v)}{b^{(2s+v)/2}} \prod_{k=0}^{\infty} \left(1 - \frac{1}{\alpha^{k+s}}\right). \tag{5.9}$$

This transform can be inverted using the Euler identity to get

$$y(x) = \frac{2Cb^{(v+2)/2}}{\Gamma(v+1)} \sum_{k=0}^{\infty} p_k (\alpha^k x)^{v/2} K_v(2\sqrt{b\alpha^k x}). \tag{5.10}$$

The functions $y_h(\alpha^k x)$ are bounded in $[0, \infty)$ and it follows readily that the series defining y is uniformly convergent on $[0, \infty)$. It is known that the modified Bessel functions K_v are holomorphic in the half plane $\mathrm{Re} z > 0$ and it follows that y is holomorphic on the positive real axis. The conditions of Theorem 4.3 are satisfied and consequently y is the unique pdf solution to equation (4.1) for the fast growth rate case.

The solution is a generalised version of the K-distribution, which has found many applications over the last decades. Jakeman [15] gives a brief survey of applications of this distribution with some history. Tough [22] discusses the time evolution of the K-distribution in the context of the Fokker–Planck equation and also gives several references to applications.

It is clear from equation (5.1) that

$$y(x) \sim \frac{2Cb^{(v+2)/2}}{\Gamma(v+1)} x^{v/2} \sqrt{\frac{\pi}{4\sqrt{bx}}} e^{-2\sqrt{bx}},$$

as $x \to \infty$. The asymptotic behaviour of y as $x \to 0^+$ is less clear, but the Euler identity and relation (5.2) can be used to show that $y(x) \to 0$ as $x \to 0^+$. This relation also follows readily from well-known asymptotic relations for the inverse Mellin transform, and this approach is also useful for the neutral and slow growth cases.

Let f be a function with the Mellin transform $M[f(x); s]$ defined in a fundamental strip $A < \mathrm{Re} s < B$. Recall that the inversion formula is

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M[f(x); s] x^{-s}\, ds,$$

where $A < c < B$. If $f \in \mathbb{D}^2(0, \infty)$, then the Riemann–Lebesgue lemma and integration by parts show that $M[f(x); \sigma + it] = o(|t|^2)$ as $t \to \pm\infty$, where σ is a fixed real number. If $M[f(x); s]x^{-s}$ can be meromorphically continued to a strip $A_1 < \mathrm{Re} s < B$, where $A_1 < A$, and if $A_1 < -M < A$, then the inversion formula and the asymptotic relation can be used to show

$$f(x) = \sum_{\lambda_k \in \mathscr{S}} \mathrm{Res}(M[f(x); s]x^{-s}; s = \lambda_k) + O(x^M), \tag{5.11}$$

where $\mathscr{S}$ denotes the set of singularities in the strip $-M < \mathrm{Re} s < c$. Here, it is assumed that $M[f(x); s]x^{-s}$ has no singularities on the line $\mathrm{Re} s = -M$.

The function y satisfies the conditions for the asymptotic estimate (5.11). The gamma function has simple poles for $s = 0, -1, -2, \ldots$, but the infinite product in the Mellin

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

transform also vanishes at these values. In fact,

$$\lim_{s\to 0}\Gamma(s)\left(1-\frac{1}{\alpha^s}\right)=\log\alpha. \tag{5.12}$$

The function $\Gamma(s)$ thus does not contribute any singularities to the Mellin transform (5.9). The singularities come from $\Gamma(s+v)$, and the fundamental strip for $M(s)$ is $-v<\mathrm{Re}\, s<\infty$. After the simple pole at $s=-v$, the next singularity is the simple pole at $s=-v-1$. We can thus take M such that $v<M<v+1$. Since

$$\begin{aligned}\mathrm{Res}(M[f(x);s]x^{-s};s=-v)&=\frac{Cb^{v+1}}{\Gamma(v+1)}\prod_{k=0}^{\infty}\left(1-\frac{1}{\alpha^{k-v}}\right)x^v\\&=\theta_0 x^v,\end{aligned}$$

we have

$$y(x)=\theta_0 x^v+O(x^M),$$

where $0<v<M$. The relation shows that $y(x)\to 0$ as $x\to 0^+$. It also connects directly the order of the zero for y at $x=0$ and the order of the underlying Bessel function. A similar analysis can be used to study the first derivative. The Mellin transform can be obtained from the relation $M[y'(x);s]=-(s-1)M[y(x);s-1]$, *viz.*

$$M[y'(x);s]=-\frac{C}{\Gamma(v+1)}\frac{(s-1)\Gamma(s-1)\Gamma(s+v-1)}{b^{s-2}}\prod_{k=0}^{\infty}\left(1-\frac{1}{\alpha^{k+s-1}}\right),$$

which has a fundamental strip $1-v<\mathrm{Re}\, s<\infty$. Now,

$$\mathrm{Res}[M[y'(x);s];s=1-v]=\frac{C}{\Gamma(v+1)}vb^{1+v}\Gamma(-v)\prod_{k=0}^{\infty}\left(1-\frac{1}{\alpha^{k-v}}\right), \tag{5.13}$$

which is positive for $v>0$, and thus

$$y'(x)=\theta_1 x^{-(1-v)}+O(x^M)$$

as $x\to 0^+$, where θ_1 is a positive constant and $0<M<v$. If $v>1$, then the derivative also vanishes at $x=0$; if $v<1$, then the derivative is not bounded at this point. The interesting case is when $v=1$, which corresponds to $a=0$ in equation (4.1). In this case there is a simple pole at $s=0$ for the transform and, using equations (3.9) and (5.12),

$$\mathrm{Res}\left(M[f(x);s]x^{-s};s=0\right)=(\alpha-1)b^2\log\alpha.$$

Thus,

$$\lim_{x\to 0^+}y'(x)=(\alpha-1)b^2\log\alpha.$$

Higher-order derivatives can be studied in a similar way to connect the behaviour of derivatives at $x=0$ with the order of the Bessel function. This can, of course, also be done directly byusing the asymptotic relations for the Bessel functions, although one must appeal to the Euler identity in cases where $y_h^{(n)}(\alpha^k x)$ is finite and non-zero as $x\to 0^+$.

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

The derivative of the homogeneous solution is

$$y_h'(x) = -x^{(v-1)/2}K_{v-1}(2\sqrt{bx})\sqrt{b},$$

and therefore

$$y'(x) \sim -\frac{2Cb^{(v+1)/2}}{\Gamma(v)}x^{(v-1)/2}\sqrt{\frac{\pi}{4\sqrt{bx}}}e^{-2\sqrt{bx}},$$

as $x \to \infty$. This equation implies that $y'(x) < 0$ for all x sufficiently large. Since $y(0) = 0$, the pdf must have at least one local maximum. Theorem 4.4 can be used to show that the distribution must be unimodal.

The analysis for the slow and neutral growth rate cases is much the same as that for the fast growth rate case. The character of the solution as $x \to 0^+$, however, is markedly different and the solution is decreasing on $(0, \infty)$.

Suppose that $-1 < v < 0$ (slow growth rate case). We can still use the Mellin transform (5.9) to construct a solution of the form (5.10) and appeal to Theorem 4.3 to assert uniqueness. Since $-1 < v < 0$, the Mellin transform (5.9) has a simple pole at $s = -v > 0$. The fundamental strip is $-v < \mathrm{Re}s < \infty$, and the next singularity is at $s = -v - 1$. Relation (5.11) yields

$$y(x) = \theta_2 x^v + O(x^M)$$

as $x \to 0^+$, where we can take $0 < M < 1 + v$, and θ_2 is a positive constant. It is clear from this equation that $y(x) \to \infty$ as $x \to 0^+$. The solution decays exponentially as $x \to \infty$, and since $|v| < 1$, the above equation shows that the solution is integrable on $[0, \infty)$.

The asymptotic behaviour of y' as $x \to \infty$ is similar to the fast growth case, and it can be shown that $y'(x) < 0$ for all x sufficiently large. The Mellin transform for y' gives

$$y'(x) = \theta_3 x^{v-1} + O(x^M)$$

as $x \to 0^+$, where θ_3 is a negative constant. We thus have $y'(x) < 0$ for all x sufficiently small. The proof of Theorem 4.4 precludes the existence of a local minimum in $(0, \infty)$; hence y cannot have a local maximum in $(0, \infty)$. The function is therefore decreasing on $(0, \infty)$.

The neutral growth rate case is similar to the slow growth rate case. The main difference is that y is finite at $x = 0$. If $v = 0$, then

$$M(s) = \frac{C\Gamma^2(s)}{b^{s-1}}\prod_{k=0}^{\infty}\left(1 - \frac{1}{\alpha^{k+s}}\right).$$

The fundamental strip is $0 < \mathrm{Re}s < \infty$, and $M(s)$ has a simple pole at $s = 0$. Equation (5.12) can be used to show

$$\mathrm{Res}[M(s)x^{-s}; s = 0] = b\log\alpha,$$

so that

$$y(x) = b\log\alpha + O(x^M),$$

as $x \to 0^+$, where $M > 0$. We thus have $y(x) \to b\log\alpha$ as $x \to 0^+$. The arguments for the

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

slow growth rate case can be used to show that y is decreasing on $(0, \infty)$. In summary, we have the following results concerning these cases.

Theorem 5.1 *The unique pdf solution to equation (4.1) is given by (5.10) with* $\nu = g/D - 1$. *If* $g/D > 1$, *then* $y(0) = 0$ *and the pdf is unimodal. If* $g/D \leqslant 1$, *then the solution is decreasing on* $(0, \infty)$. *If* $g/D < 1$, *then* $y(x) \to \infty$, *as* $x \to 0^+$; *if* $g/D = 1$, *then* $y(x) \to b \log \alpha$, *as* $x \to 0^+$.

6 Conclusions

There are few explicit solutions available for the pantograph-type equations with non-constant coefficients. In this paper, we studied a second-order equation with a linear dispersion term. The equation comes from the Fokker–Planck equation that models cell growth, and the solution is required to be a pdf. It was shown that there is a unique pdf solution to this problem. The Mellin transform played a central role in the analysis and by use of this transform we were able to construct a solution in terms of a series of the modified Bessel functions. The transform was also useful in studying the behaviour of solutions as $x \to 0^+$. It was shown that the shape of the solutions differs markedly depending on the growth to dispersion ratio.

The above analysis provides some framework in which one might study more general cases. The Mellin transform technique is useful, but it is sensitive to coefficients (even monomials) and explicit solutions can be formidable to obtain unless the difference equation is simple. Future work certainly includes work on more general cases that are compatible with the cell growth model. Another analytical direction for work is on solutions to the problem that are not pdfs.

References

[1] Ambartsumyan, V. A. (1944) On the fluctuation of the brightness of the Milky Way. *Dokl. Akad. Nauk SSSR* **44**, 223–226.

[2] Basse, B., Baguley, B., Marshall, E., Joseph, W., van Brunt, B., Wake, G. & Wall, D. (2004) Modelling cell death in human tumour cell lines exposed to the anticancer drug paclitaxel. *J. Math. Bio.* **49**(4), 329–357.

[3] Basse, B., Wake, G., Wall, D. & van Brunt, B. (2004) On a cell-growth model for plankton. *IMA J. Math. Med. Bio.* **21**, 49–61.

[4] Bogachev, L., Derfel, G., Molchanov, S. & Ockendon, J. (2008) On bounded solutions of the balanced generalized pantograph equation. In: P. L. Chow, B. Mordukhovich & G. Yin (editors), *Topics in Stochastic Analysis and Nonparametric Estimation, IMA,* Vol. 145, Springer, New Mexico, pp. 29–49.

[5] Derfel, G. (1989) Probabilistic method for a class of functional differential equations. *Ukrain. Mat. Zh.* **41**, 1322–1327. (English translation: (1990) *Ukrainian Math. J.* **41**, 1137–1141.

[6] Derfel, G. & Iserles, A. (1997) The pantograph equation in the complex plane. *J. Math. Anal. Appl.* **213**, 117–132.

[7] Epstein, B. (1948) Some applications of the Mellin transform in statistics. *Ann. Math. Statist.* **19**, 370–379.

[8] Fox, L., Mayers, D. F., Ockendon, J. R. & Tayler, A. B. (1971) On a function differential equation. *J. Inst. Math. App.* **8**, 271–307.

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press

[9] Gardiner, C. W. (2004) *Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences*, Springer-Verlag, Berlin.
[10] Gaver, D. P. (1964) An absorption probablility problem. *J. Math. Anal. Appl.* **9**(N3), 384–393.
[11] Hall, A. J., Wake, G. C. & Gandar, P. W. (1991) Steady size distributions for cells in one dimensional plant tissues. *J. Math. Bio.* **30**(2), 101–123.
[12] Hall, A. J. & Wake, G. C. (1989) A functional differential equation arising in the modelling of cell-growth. *J. Aust. Math. Soc. Ser.* B **30**, 424–435.
[13] Hall, A. J. & Wake, G. C. (1990) A functional differential equation determining steady size distributions for populations of cells growing exponentially. *J. Aust. Math. Soc. Ser.* B, **31**, 344–353.
[14] Iserles, A. (1993) On the generalized pantograph functional differential equation. *Eu. J. Appl. Math.* **4**, 1–38.
[15] Jakeman, E. (1999) K-distributed noise. *J. Opt.* A*: Pure Appl. Opt.* **1**, 784–789.
[16] Kato T. & McLeod, J. B. (1971) The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$. *Bull. Am. Math. Soc.* **77**, 891–937.
[17] Kim, H. K. (1998) *Advanced Second Order Functional Differential Equations*, Ph.D. thesis, Massey University, New Zealand.
[18] Marshall, J., van-Brunt, B. & Wake, G. (2002) Natural boundaries for solutions to a certain class of functional differential equations. *J. Math. Anal. Appl.* **268**, 157–170.
[19] Morgan, D. (2000) A remarkable sequence derived from Euler products. *J. Math. Phys.* **41**(10), 7109–7121.
[20] Ockendon, J. & Tayler, A. (1971) The dynamics of a current collection system for an electric locomotive. *Proc. R. Soc. Lond. Ser.* A **322**, 447–468.
[21] Springer, M. & Thompson, W. (1966) The distribution of products of independent random variables. *SIAM J. Appl. Math.* **14**(3), 511–526.
[22] Tough, R. (1987) A Fokker-Planck description of K-distributed noise. *J. Phys.* A*: Math. Gen.* **20**, 551–567.
[23] van-Brunt, B., Wake, G. C. & Kim, H. K. (2001) On a singular Sturm–Liouville problem involving an advanced functional differential equation. *Euro. J. Appl. Math.* **12**, 625–644.
[24] Wake, G. C., Cooper, S., Kim, H. K. & van-Brunt, B. (2000) Functional differential equations for cell-growth models with dispersion. *Commun. Appl. Anal.* **4**, 561–574.

https://doi.org/10.1017/S0956792510000367 Published online by Cambridge University Press