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Zeros of generalized hypergeometric functions

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Introduction. In this paper we derive the conditions which have to be satisfied in order to obtain some classes of zeros of the generalized hypergeometric series of the type

$$ {}_nF_{n-1}\left[\begin{matrix} a_1, a_2, \ldots, a_{n-1}, a_n \\ c_1, c_2, \ldots, c_{n-1} \end{matrix} ; 1\right]. $$

These conditions read:

$$ \left.\begin{aligned} a_s - c_r + 1 &= n_1 \\ a_k - c_l + 1 &= n_2 \end{aligned}\right\} \quad \begin{matrix} l \neq r, \quad k \neq s \\ n_i \text{ positive integer.} \end{matrix} $$

To establish this result we use the connexion existing between the ${}_nF_{n-1}$-series and a class of functions known as dual amplitudes or 'multiveneziano amplitudes' which have been recently introduced in strong-interaction physics.‡

For completeness, we give in section 1 a short account of the main properties of these functions, and we derive the conditions determining some classes of zeros of the dual amplitudes. In section 2 we give the connexion with the generalized hypergeometric series and derive the above conditions giving classes of zeros of

$$ {}_nF_{n-1}\left[\begin{matrix} (a_i) \\ (c_i); \end{matrix} 1\right]. $$

1. Let $Z \equiv \{z_0, z_1, \ldots, z_N, z_{N+1}\}$ be a set of $N+2$ points ordered as in Fig. 1 on a circle in the complex plane, and $P[i-1, i; j, j+1]$ a partition of the set into two sets of consecutive points (Fig. 1). We adjoin the partition to the anharmonic ratio:

$$ u_{i,\, i+1,\, \ldots,\, j} \equiv u_{j+1,\, \ldots,\, 0,\, N+1,\, \ldots,\, i-1} = (z_i, z_{i-1}, z_j, z_{j+1}) = \frac{(z_i - z_j)(z_{i-1} - z_{j+1})}{(z_{i-1} - z_j)(z_i - z_{j+1})} \qquad (1) $$

which is a real positive number between 0 and 1, since the two couples (z_j, z_{i-1}) and (z_i, z_{j+1}) separate.

Consider now the quantity

$$ \prod_P u_{i,\ldots,j}^{\alpha_i \ldots j-1} \qquad (2) $$

where the product is extended to all the partitions, in which, with reference to Fig. 1, $i \neq j$ and $i-1 \neq j+1$. The parameters $\alpha_{i \ldots j}$ like the anharmonic ratios $u_{i \ldots j}$ are in

‡ See references from (1) to (7).

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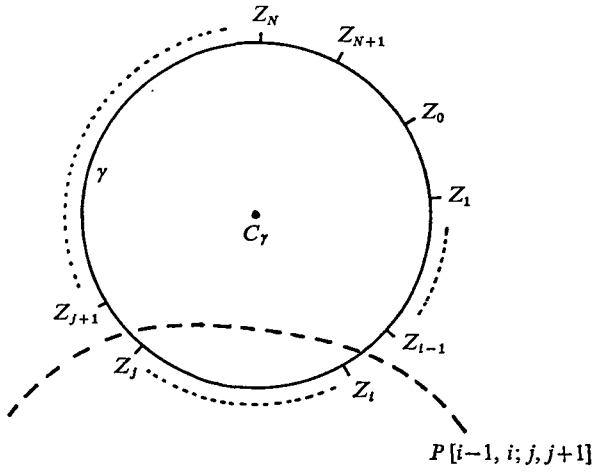


Fig. 1

a one-to-one correspondence with the above-mentioned partitions of Z; in writing them, the following convention is understood:

$$\alpha_{i\dots j} \equiv \alpha_{j+1,\dots,N+1,0,\dots,i-1}. \tag{3}$$

The number of the $\alpha_{i\dots j}$'s is therefore $\frac{1}{2}(N+2)(N-1)$. Since the anharmonic ratios (1) are invariant under a Möbius transformation:

$$z_i \to z_i' = \frac{\alpha z_i + \beta}{\gamma z_i + \delta} \quad (\alpha\delta - \beta\gamma \neq 0) \tag{4}$$

the variables $u_{i\dots j}$ actually depend on $N-1$ z's only, because three of them, say z_a, z_b and z_c, may be mapped by means of equation (4) into three arbitrarily given points of the complex plane. Notice that transformation (4) maps the circle γ into a new one γ', which may coincide with a straight-line if it contains the infinite point; moreover, it does not affect the ordering of the z's although it may transform a clock-wise ordering into a counter-clockwise one, or vice versa.

If now we integrate the expression (2) with a volume element invariant under transformation (4), in a domain $D(0, 1, \dots, N+1)$ defined by the constraints:

$$\theta_0 < \theta_1 < \dots < \theta_{N+1} < \theta_0 + 2\pi, \tag{5}$$

where θ_i is the argument of z_i on γ: $z_i = c_\gamma + \rho_\gamma e^{i\theta_i}$, we get: (6) (7)

$$B(\alpha_{i\dots j}) = \int\dots\int_{D(0,1,\dots,N+1)} dV^{(N-1)} \prod_P u^{a\,i\dots j-1}_{i,\dots,j}. \tag{6}$$

It may be easily verified that a possible invariant volume element is:

$$dV^{(N-1)} = \frac{\prod_{i\neq a,b,c} dz_i (z_a - z_b)(z_b - z_c)(z_c - z_a)}{\prod_{i=0}^{N+1} (z_i - z_{i+2})}. \tag{7}$$

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Since in equation (6) both the integrand and the integration domain $D(0, 1, \ldots, N+1)$ are invariant under projective transformations (4), the integral depends neither on the choice of γ nor on the position of z_a, z_b and z_c on it. Moreover, it is independent of the choice of indices a, b and c of the variables kept fixed in the integration. In fact, if we take $z_a = \infty$, $z_b = 0$ and $z_c = 1$ which is not restrictive as we have just seen, and we perform the change of variables:

$$x_i = (z_{a'}, z_{b'}, z_{c'}, z_i) \tag{8}$$

we get $x_{a'} = \infty$, $x_{b'} = 0$ and $x_{c'} = 1$ where a', b' and c' are any three indices in $\{0, 1, \ldots, N+1\}$.

As the transformation (8) is of the same type of (4)†, it leaves the form of the integral invariant except that the variables kept fixed become $x_{a'}$, $x_{b'}$ and $x_{c'}$ (6)(7).

Finally, we want to stress that in the definition of $B(\alpha_{i \ldots j})$ for real values of the α's, the real determination of the multi-valued functions $u_{i,\ldots,j}^{\alpha_{i \ldots j}-1}$ is understood so that $B(\alpha_{i \ldots j})$ is also real.

In order to cast equation (6) in a more convenient form, it is useful to introduce a new set of $\frac{1}{2}(N+2)(N+1)$ variables defined as follows:

$$\left.\begin{aligned} \beta_{ij} \equiv \beta_{ji} &= \alpha_{i \ldots j} + \alpha_{i+1, \ldots, j-1} - \alpha_{i \ldots j-1} - \alpha_{i+1 \ldots j} \quad (j \neq i+1, i+2), \\ \beta_{i, i+2} \equiv \beta_{i+2, i} &= \alpha_{i, i+1, i+2} - \alpha_{i, i+1} - \alpha_{i, i+2}, \\ \beta_{i, i+1} \equiv \beta_{i+1, i} &= \alpha_{i, i+1}, \end{aligned}\right\} \tag{9a}$$

$$\alpha_{i \ldots j} = -\binom{j-i}{2} + \sum_{\substack{h>k \\ h,k=i}}^{j} \beta_{hk}. \tag{9b}$$

Because of their definition (9), these new variables satisfy the $N+2$ linear relations:

$$\sum_{\substack{i=0 \\ i \neq j}}^{N+1} \beta_{ij} = N - 1. \tag{10}$$

In terms of these variables we may rewrite equation (6) in the form (6)(7):

$$B_{[0, 1, \ldots, N+1]}(\beta_{ij}) = \int \cdots \int_{D(0, 1, \ldots, N+1)} (z_a - z_b)(z_b - z_c)(z_c - z_a) \prod_{i>j} (z_i - z_j)^{\beta_{ij}-1} \prod_{i \neq a, b, c} dz_i. \tag{11}$$

In conclusion, given the set of variables β_{ij} satisfying equation (10) and an arbitrary permutation $(\lambda_0, \lambda_1, \ldots, \lambda_{N+1})$ of $(0, 1, \ldots, N+1)$ we may associate it with the real function:

$$\begin{aligned} B_{[\lambda_0, \lambda_1, \ldots, \lambda_{N+1}]}(\beta_{ij}) = \int \cdots \int_{D(\lambda_0, \lambda_1, \ldots, \lambda_{N+1})} & (z_{\lambda_\pi} - z_{\lambda_s})(z_{\lambda_s} - z_{\lambda_t})(z_{\lambda_t} - z_{\lambda_\pi}) \\ & \times \prod_{i>j} (z_{\lambda_i} - z_{\lambda_j})^{\beta_{\lambda_i \lambda_j}-1} \prod_{i \neq \pi, s, t} dz_i. \end{aligned} \tag{12}$$

† Actually the coefficients of the transformation (8) are not constant as in (4) but depend on $z_{a'}$, $z_{b'}$ and $z_{c'}$; it can, however, be checked by a direct calculation that our conclusions are not affected by this fact.

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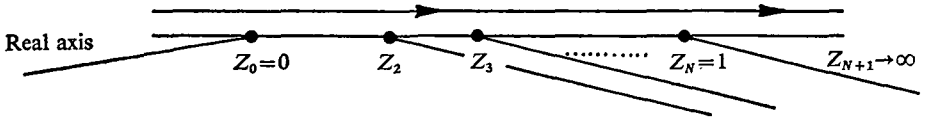


Fig. 2. Integration path in the cut Z_1 complex plane.

As the domain of integration D is defined by means of a projective invariant, it is unaffected by a cyclic or anticyclic permutation of the indices $\lambda_0, \lambda_1, \ldots, \lambda_{N+1}$. Since the integrand also is left unchanged by such permutations, it follows that the B-functions are defined within cyclic or anticyclic permutations of the indices, so that we get only $\frac{1}{2}(N+1)!$ different functions.

These functions, however, are not all linearly independent. To prove this (9) let us come back to equation (11) and take $a = 0$, $b = N$ and $c = N+1$. We get

$$B_{[0,1,\ldots,N+1]}(\beta_{ij}) = \int_0^1 dz_{N-1} \int_0^{z_{N-1}} dz_{N-2} \cdots \int_0^{z_3} dz_2 \int_0^{z_2} dz_1 \prod_{\substack{i=1 \\ i>j}}^{N} (z_i - z_j)^{\beta_{ij}-1}. \tag{13}$$

Let us consider now an integral identical to the right-hand side of equation (13) except that z_1 is now integrated on the real axis from $-\infty$ to $+\infty$ as shown in Fig. 2. This integral may be decomposed into the sum of $N+1$ integrals which correspond to the following intervals of integration:

$$(z_0 = 0, z_2); (z_2, z_3); \ldots; (z_{N-1}, z_N = 1); (1, \infty); (-\infty, 0).$$

The integration in each interval gives respectively:

$$\begin{aligned} &B[0,1,\ldots,N+1];\ B[0,2,1,\ldots,N+1]\, e^{-i\pi(\beta_{12}+1)}; \\ &B[0,2,3,1,\ldots,N+1]\, e^{-i\pi(\beta_{12}+\beta_{13}+2)}\ldots, \\ &B[0,2,3,\ldots,N+1,1]\, e^{-i\pi(\beta_{12}+\beta_{13}+\cdots+\beta_{1,N+1}+N)}, \end{aligned} \tag{14}$$

where the exponentials are due to the sign change of some of the differences $z_i - z_j$. Now the integral from $-\infty$ to $+\infty$ in z_1 is zero because of the Cauchy theorem so that we get:

$$\begin{aligned} &B[0,1,\ldots,N+1] + e^{-i\pi(\beta_{12}+1)} B[0,2,1,\ldots,N+1] \\ &\quad + e^{-i\pi(\beta_{12}+\beta_{13}+2)} B[0,2,3,1,\ldots,N+1] + \ldots \\ &\quad + e^{-i\pi(\beta_{12}+\beta_{13}+\ldots+\beta_{1,N+1}+N)} B[0,2,3,\ldots,N+1,1] = 0. \end{aligned}$$

Notice that, because of the reality of the B-functions, equation (14) is equivalent to two linear relations with real coefficients. A large number of relations of this kind may be written down; actually we may start from a function $B[\lambda_0, \lambda_1, \ldots, \lambda_{N+1}]$ corresponding to an arbitrary permutation of $(0, 1, \ldots, N+1)$ and integrate from $-\infty$ to $+\infty$ in the variable we like.

All those relations, first studied by Plathe (9), are not linearly independent, and the problem of finding, for a given N, the number of independent B-functions is not trivial and up to now has not been solved for $N \geqslant 4$.

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For $N = 2$, the B-functions are simply Euler Beta-functions and we have a single complex linear relation:

$$B(\beta_{01}, \beta_{12}) + e^{-i\pi(\beta_{12}+1)} B(\beta_{02}, \beta_{03}) + e^{-i\pi(\beta_{12}+\beta_{13})} B(\beta_{02}, \beta_{23}) = 0 \tag{15}$$

$$(\beta_{02} = \beta_{13};\ \beta_{01} = \beta_{23};\ \beta_{03} = \beta_{12};\ \beta_{12} + \beta_{23} + \beta_{13} = 1),$$

(i.e. two real linear relations among the 3 different Beta functions). For $N = 3$, the B-functions are proportional, as we shall see in the next section, to the ${}_3F_2$-hypergeometric series so that the four-term complex relations among them are equivalent to the well-known three-term relations among ${}_3F_2$ series.

Let us now focus our attention on the zeros of the B-functions (8), in view of the connexion with the generalized hypergeometric series that we shall establish in the next section.

Consider equation (14) and put:

$$\left.\begin{aligned} \beta_{1,3} &= n_3 \\ \beta_{1,4} &= n_4 \\ \dots &= \dots \\ \beta_{1,N+1} &= n_{N+1} \end{aligned}\right\} \quad (n_3, n_4, \dots, n_{N+1} \text{ positive integers}). \tag{16}$$

Equation (14) becomes:

$$B[0, 1, \dots, N+1] + e^{-i\pi(\beta_{12}+1)}(B[0, 2, 1, \dots, N+1] + \dots + (-1)^{n_3+n_4+\dots+n_{N+1}+N-1} B[0, 2, \dots, N+1, 1]) = 0, \tag{17}$$

which, since β_{12} is arbitrary, implies for real values of the β's:

$$B[0, 1, \dots, N+1] = 0. \tag{18}$$

The condition of positiveness on the n_i's in equations (16) is necessary in order to avoid singularities in the B-functions involved in equation (17) which would invalidate the argument leading to equation (18). Actually for non-positive integer values of β_{ij} all the B-functions in which i and j are contiguous indices share a pole.†

For a function $B[\lambda_0, \lambda_1, \dots, \lambda_{N+1}]$ corresponding to a generic permutation of the indices the conditions giving a class of zeros of the function read:

$$\left.\begin{aligned} \beta_{\lambda_\kappa, \lambda_{\kappa+2}} &= n_2 \\ \beta_{\lambda_\kappa, \lambda_{\kappa+3}} &= n_3 \\ \dots\dots\dots\dots & \\ \beta_{\lambda_\kappa, \lambda_{\kappa+N}} &= n_N \end{aligned}\right\} \quad (n_2, n_3, \dots, n_N \text{ positive integers}), \tag{19}$$

where $\kappa \in \{0, 1, \dots, N+1\}$.

2. In order to connect the B-functions with the hypergeometric series ${}_NF_{N-1}$ let us perform on the right-hand side of equation (13) the following substitution:

$$y_i = \frac{z_i}{z_{i+1}}.$$

† A detailed account of the singularity structure of the B-functions may be found in the literature (see refs. from (1) to (7)).

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We get (1)–(5),

$$B[0,1,\ldots,N+1]=\int_0^1\ldots\int_0^1\prod_{i=1}^{N-1}\left\{dy_i\,y_i^{\alpha_{0,1,\ldots,i}-1}\prod_{\kappa=0}^{N-i-1}(1-y_iy_{i+1}\ldots y_{i+\kappa})^{\beta_{i,i+\kappa+1}-1}\right\}. \quad (20)$$

Let us consider first the particular case $N=3$. We have ((10), p. 108)

$$B[0,1,\ldots,N+1]=\int_0^1 dy_1\int_0^1 dy_2\,y_1^{\beta_{01}-1}y_2^{\beta_{34}-1}(1-y_1)^{\beta_{12}-1}(1-y_2)^{\beta_{23}-1}(1-y_1y_2)^{\beta_{13}-1}$$

$$=\frac{\Gamma(\beta_{01})\,\Gamma(\beta_{12})\,\Gamma(\beta_{23})\,\Gamma(\beta_{34})}{\Gamma(\beta_{01}+\beta_{12})\,\Gamma(\beta_{23}+\beta_{34})}\,{}_3F_2\left[\begin{matrix}\beta_{01},\beta_{34},1-\beta_{13}\\ \beta_{01}+\beta_{12},\beta_{23}+\beta_{34}\end{matrix};1\right]. \quad (21)$$

Without going into details, since the properties of the ${}_3F_2$ series and of the integral (21) have been widely studied, we only stress that:

(*a*) The invariance of the B-functions under cyclic and anticyclic permutations of the indices gives, through equation (21), the well known transformation properties of the ${}_3F_2$ series ((10), p. 52).

(*b*) The twelve essentially different functions

$$\frac{B[\lambda_0,\lambda_1,\ldots,\lambda_4]}{\prod_{i=0}^{4}\Gamma(\beta_{\lambda_i,\lambda_{i+1}})}\quad(\lambda_5=\lambda_0)$$

are just the twelve functions $F_p(i)$ and $F_n(i)$ if a suitable identification is made between our β_{ij} variables and the α_{ijk} in the Whipple notation ((10), pp. 115–121).

(*c*) With the same identification, one easily recognizes that the imaginary parts of the complex relations

$$B[0,1,2,3,4]+e^{-i\pi(\beta_{01}+1)}B[1,0,2,3,4]+e^{-i\pi(\beta_{01}+\beta_{02})}B[1,2,0,3,4]$$
$$+e^{-i\pi(\beta_{01}+\beta_{02}+\beta_{03}-1)}B[1,2,3,0,4]=0$$

are the three-term relations among the F_p and F_n functions. In the general case $N>3$, the B-functions are proportional to a hypergeometric series only if we impose the $\frac{1}{2}\{N(N-3)\}$ constraints:

$$\beta_{i,i+\kappa+1}=1\quad\begin{cases}\kappa=1,2,\ldots,N-i-1 & (i\neq 1)\\ \kappa=1,2,\ldots,N-3 & (i=1)\end{cases}. \quad (22)$$

Then the right-hand side of equation (20) becomes ((10), p. 108):

$$\int_0^1\ldots\int_0^1\prod_{i=1}^{N-1}\{dy_i\,y_i^{\alpha_{01}\ldots i-1}(1-y_i)^{\beta_{i,i+1}-1}\}\,(1-y_1\ldots y_{N-1})^{\beta_{1N}-1}$$

$$=\prod_{i=1}^{N-1}\frac{\Gamma(\alpha_{0,1,\ldots,i})\,\Gamma(\beta_{i,i+1})}{\Gamma(\alpha_{0,1,\ldots,i}+\beta_{i,i+1})}\,{}_NF_{N-1}\left[\begin{matrix}\alpha_{01},\alpha_{012},\ldots,\alpha_{01\ldots N-1},1-\beta_{1,N}\\ \alpha_{01}+\beta_{12},\alpha_{012}+\beta_{23},\ldots,\alpha_{01\ldots N-1}+\beta_{N-1,N}\end{matrix};1\right]. \quad (23)$$

Notice that conditions (22) are not invariant under cyclic and anticyclic permutations of the indices as they assign a privileged role to the indices 0 and $N+1$. Therefore, contrary to what happens in the case $N=3$ (where no condition (22) is needed to get a connection between the B-functions and the ${}_3F_2$ series) for a general N, the cyclic invariance of the B-functions cannot be directly interpreted in terms of hypergeometric series. For the same reason we cannot interpret the linear relations (14) from

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the point of view of hypergeometric series. However, and here we come to the main point of our work, if we add to equations (22) the two constraints

$$\beta_{i0} = n_0; \quad \beta_{i,N+1} = n_{N+1}, \tag{24}$$

where i is any index in $\{2, 3, \ldots, N-1\}$ and n_0 and n_1 are positive integers, the set of conditions (19) giving a zero of $B[0, 1, \ldots, N+1]$ is fulfilled. So equations (24) give some classes of zeros of both sides of equation (23) and, unless some Γ-function in the denominator of the right-hand side of equation (23) becomes infinite, this implies that the ${}_NF_{N-1}$ vanishes.†

To summarize the result in the usual notation for the hypergeometric series, we put

$$\left.\begin{aligned} \alpha_{0,1,\ldots,i} &= a_i \\ 1-\beta_{1N} &= a_N \\ \alpha_{01\ldots i}+\beta_{i,i+1} &= c_i \end{aligned}\right\} \quad (i = 1, 2, \ldots, N-1), \tag{25}$$

which, by means of equations (9), may be written in the form

$$\left.\begin{aligned} \beta_{0,i} &= a_i - c_{i-1} + 1, \\ \beta_{i,i+1} &= c_i - a_i, \\ \beta_{N+1,i} &= a_{i-1} - c_i + 1, \\ \beta_{0,N+1} &= \sum_{i=1}^{N} c_i - \sum_{i=1}^{N} a_i - 1, \end{aligned}\right\} \tag{26}$$

where we have defined $c_N = 1$ and the index i has to be understood modulo N.

In terms of the new parameters defined in equation (25) the statement about the zeros of the hypergeometric series reads: if c_i is not a non-positive integer the series:

$${}_NF_{N-1}\left[\begin{matrix} a_1, a_2, \ldots, a_N \\ c_1, c_2, \ldots, c_{N-1} \end{matrix}; 1\right]$$

vanishes whenever

$$\left.\begin{aligned} a_i - c_{i-1} + 1 &= n_1, \\ a_{i-1} - c_i + 1 &= n_2, \end{aligned}\right\} \tag{27}$$

where n_1 and n_2 are positive integers.

Owing to the symmetry of the hypergeometric series with respect to any permutation of its parameters a_i's and c_i's, the condition (27) can be generalized to

$$\left.\begin{aligned} a_s - c_r + 1 &= n_1 \\ a_k - c_l + 1 &= n_2 \end{aligned}\right\} \quad \text{where } l \neq r \quad \text{and} \quad k \neq s. \tag{28}$$

† In the case $N = 3$ the two conditions (24) alone give a zero of $B(0, 1, \ldots, 4)$ and therefore also of the ${}_3F_2$.

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