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Summability methods which include the Riesz typical means. I

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A number of special results exist for summability methods B which include Riesz summability (R, λ, κ)—for example, when B is generalized Abel summability (A, λ, ρ) [Kuttner (5)], or Riemann summability $(\mathfrak{R}, \lambda, \mu)$ [Russell (14)], or Riemann–Cesàro summability $(\mathfrak{R}, \lambda, p, \alpha)$ [Rangachari (12)], or generalized Cesàro summability (C, λ, κ) [Meir (9); Borwein and Russell (1)]. The question of necessary and sufficient conditions to be satisfied by an arbitrary method B in order that $B \supseteq (R, \lambda, \kappa)$ has received an answer only for limited values of λ and κ—for example, by Lorentz [(6), Theorem 10] for $\kappa = 1$; the restrictions on λ in this case were removed by Maddox [(8), Theorem 1]. Thus (apart from the well-known case $\kappa = 0$) the case $\kappa = 1$ is the only one for which a complete solution exists, though application of a theorem of Russell [(13), Theorem 1 *A*] yields one form of a result for $0 < \kappa \leqslant 1$. Maddox's results, however, suggest an alternative form capable of generalization to all $\kappa \geqslant 0$, and in this paper we obtain a complete solution for $0 < \kappa \leqslant 1$ in that form, without restriction on λ. We first recall the following definitions.

For an arbitrary $A = (a_{\rho\nu})$ (ρ may be a continuous or a discrete parameter) we denote by c_A the set of A-limitable sequences; $\chi(A) = \lim_{\rho} \sum_{\nu} a_{\rho\nu} - \sum_{\nu} \lim_{\rho} a_{\rho\nu}$ whenever this exists finitely; and $\|A\| = \sup_{\rho} \sum_{\nu} |a_{\rho\nu}|$, finite or not. Given two matrices A, B, we say that B is not weaker than A if $c_A \subseteq c_B$; if, in addition, A and B are consistent (every A-limitable sequence is B-limitable to the same value) we write $A \subseteq B$. Limits of summation are assumed throughout to be 0, ∞ unless otherwise specified, and $\Delta x_n = x_n - x_{n+1}$.

When matrices A and $\bar{A}$ are normal (lower triangular with non-zero diagonal elements) they possess unique two-sided normal inverses A', A'' respectively, and if A and $\bar{A}$ are related by $a_{n\nu} = \Delta_\nu \bar{a}_{n\nu}$, it is easy to show that

$$a''_{\mu k} = a'_{\mu k} - a'_{\mu-1,\,k} (a'_{-1,\,k} = 0), \quad a'_{\mu k} = \sum_{\nu=0}^{\mu} a''_{\nu k}; \quad \mu, k = 0, 1, 2, \ldots. \tag{1}$$

Also A is a regular sequence-to-sequence matrix (a T-matrix) if and only if $\bar{A}$ is a regular series-to-sequence matrix (a γ-matrix).

We shall make use of results connected with the set $\mathscr{M}^*$ of matrices A having the following property:

$$\mathscr{M}^* \begin{cases} A \text{ is normal and non-negative and, for } 0 \leqslant \nu \leqslant n,\ n = 0, 1, 2, \ldots, \\ a_{n+1,\,\nu} = d_{n\nu} a_{n\nu}, \quad \text{where} \quad 0 \leqslant d_{nn} \leqslant \ldots \leqslant d_{n1} \leqslant d_{n0} \leqslant 1. \end{cases}$$

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For $\kappa \geqslant 0$ and $\lambda = \{\lambda_n\}$ a strictly increasing unbounded real sequence with $\lambda_0 \geqslant 0$, the Riesz sum of an arbitrary real or complex series Σc_ν, with partial sums s_ν, is

$$C^\kappa(\omega) = \sum_{\lambda_\nu < \omega} (\omega - \lambda_\nu)^\kappa c_\nu = \sum_{\lambda_\nu < \omega} s_\nu \Delta_\nu (\omega - \lambda_\nu)^\kappa \ (\omega \geqslant 0), C(\omega) = C^0(\omega); \tag{2}$$

here, and throughout, we make the convention that $(\omega - \lambda_n)^\kappa = 0$ if $\omega \leqslant \lambda_n$; and we shall also denote $\Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n)$ $(n = 0, 1, \ldots)$.

Σc_n, or $\{s_n\}$, is summable (R, λ, κ) to s when $\omega^{-\kappa} C^\kappa(\omega) \to s$ as $\omega \to \infty$; and summable (R^*, λ, κ) to s when $\lambda_n^{-\kappa} C^\kappa(\lambda_n) \to s$ as $n \to \infty$. Jurkat (4) has shown that, when $0 \leqslant \kappa \leqslant 1$, the (R^*, λ, κ) T-matrix has property $\mathscr{M}^*$; and also that

$$(R, \lambda, \kappa) \sim (R^*, \lambda, \kappa) \quad \text{for } \lambda \text{ arbitrary}, \quad 0 \leqslant \kappa \leqslant 1. \tag{3}$$

The main results to be proved here are as follows:

THEOREM 1. *Let $\kappa \geqslant 0$. In order that a sequence-to-sequence or sequence-to-function method $B = (b_{\rho\mu})$ shall be not weaker than (R, λ, κ) it is necessary that*

$$\text{(i)} \ \exists \lim_\rho b_{\rho\mu} = \beta_\mu \ (\mu = 0, 1, \ldots), \quad \text{(ii)} \ \exists \lim_\rho \sum_\mu b_{\rho\mu} = \beta, \tag{4}$$

and that a family of functions $\{g_\rho\}$ exists, defined in $[\lambda_0, \infty)$, such that

$$b_{\rho\mu} = \Delta_\mu \int_{\lambda_\mu}^\infty (\omega - \lambda_\mu)^\kappa \, dg_\rho(\omega), \quad \int_{\lambda_0}^\infty \omega^\kappa \, |dg_\rho(\omega)| = M_\rho \leqslant M. \tag{5}$$

If $0 \leqslant \kappa \leqslant 1$ (without restriction on λ) then (4) and (5) are also sufficient. If we require B to be consistent with (R, λ, κ), then the theorem holds provided that $\beta_\mu = 0$, $\beta = 1$ in (4).

THEOREM 2. *Let $\kappa > 0$ and $\Lambda_n \neq O(1)$, and when $\kappa > 1$ assume $\Lambda_{n-1} = O(\Lambda_n)$. In order that a series-to-sequence or series-to-function method $\bar{B} = (\bar{b}_{\rho\mu})$ shall be not weaker than (R, λ, κ) it is necessary that*

$$\exists \lim_\rho \bar{b}_{\rho\mu} = \bar{\beta}_\mu \quad (\mu = 0, 1, \ldots), \tag{6}$$

$$|\bar{b}_{\rho\mu}| \leqslant H_\rho \Lambda_\mu^{-\kappa}, \tag{7}$$

and that a family of functions $\{g_\rho\}$ exists, defined in $[\lambda_0, \infty)$, such that

$$\bar{b}_{\rho\mu} = \int_{\lambda_\mu}^\infty (\omega - \lambda_\mu)^\kappa \, dg_\rho(\omega), \quad \int_{\lambda_0}^\infty \omega^\kappa \, |dg_\rho(\omega)| = M_\rho \leqslant M. \tag{8}$$

When $0 < \kappa \leqslant 1$, (6)–(8) are also sufficient. If we require $\bar{B}$ to be consistent with (R, λ, κ), then the theorem holds provided that $\bar{\beta}_\mu = 1$ in (6).

The proofs of the sufficiency parts of these theorems are an almost immediate consequence of the four lemmas which follow. The necessity parts are already essentially known.

LEMMA 1. *Let $A \in \mathscr{M}^*$, $\chi(A) \neq 0$. In order that $c_A \subseteq c_B$ it is necessary and sufficient that $\chi(B)$ exists, and that $T \equiv BA'$ exists and satisfies $B = TA$, $\|T\| < \infty$. If A and B are required also to be consistent, then we must add to the necessary and sufficient conditions*

$$\lim_n a_{n\nu} = \lim_\rho b_{\rho\nu} \ (\nu = 0, 1, \ldots), \quad \lim_n \sum_\nu a_{n\nu} = \lim_\rho \sum_\nu b_{\rho\nu}.$$

Proof. This lemma is given by Russell [(16), Theorem 3 and Remark 6].

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LEMMA 2. *If $A \in \mathscr{M}^*$ then its two-sided normal inverse $A' = (a'_{nv})$ satisfies*

$$\sum_{\nu=\mu}^{n} a'_{n\nu} \geqslant 0 \quad (0 \leqslant \mu \leqslant n,\ n = 0, 1, \ldots), \tag{9}$$

$$a'_{n\nu} \leqslant 0 \quad (\nu < n), \quad a'_{nn} = 1/a_{nn} > 0, \tag{10}$$

$$\sum_{\nu=0}^{n} |a'_{n\nu}| \leqslant 2a'_{nn}; \tag{11}$$

if, in addition, $\sum_{\nu=0}^{n} a_{n\nu}$ is non-decreasing as n increases, then

$$|a'_{n\nu}| \leqslant |a'_{\nu+1,\nu}| \quad (n \geqslant \nu+1). \tag{12}$$

Proof. Peyerimhoff [(11), Satz 4] has proved (9) under the weaker assumption that A is normal, non-negative, and $a_{n+1,\nu} \leqslant a_{n\nu}$ $(0 \leqslant \nu \leqslant n)$. Wilansky and Zeller [(17), p. 261, Bemerkung 2] state (10), with an indication that it can be proved by induction; and (11) is a trivial deduction from (9) and (10). The proof of (12) begins in the same way as (10); thus by eliminating the diagonal elements of A' between the two equations expressing the nth and $(n+1)$th rows of AA' we find, for $0 \leqslant \nu < n$,

$$a_{n+1,n+1} a'_{n+1,\nu} = \sum_{k=\nu+1}^{n} a_{nk}(d_{n\nu} - d_{nk})\, a'_{k\nu}$$

(then since $a_{\nu+1,\nu+1} a'_{\nu+1,\nu} = -a_{\nu+1,\nu}/a_{\nu\nu} \leqslant 0$, (10) follows by induction on n). Now write $\bar{a}_{n\nu} = \sum_{i=\nu}^{n} a_{ni} = \bar{a}_{n0} - \sum_{i=0}^{\nu-1} a_{ni}$; then, for fixed i and ν, and $n\uparrow$, we have, by hypothesis, $\bar{a}_{n0}\uparrow$ and $a_{ni}\downarrow$, whence $a_{n\nu}\uparrow$. Now suppose that, for a given fixed ν and for some $n \geqslant \nu+1$, we have $|a'_{k\nu}| \leqslant K$ when $\nu+1 \leqslant k \leqslant n$; then

$$\begin{aligned} a_{n+1,n+1} |a'_{n+1,\nu}| &\leqslant K \sum_{k=\nu+1}^{n} a_{nk}(1 - d_{nk}) \\ &= K(\bar{a}_{n,\nu+1} - \bar{a}_{n+1,\nu+1} + a_{n+1,n+1}) \\ &\leqslant K a_{n+1,n+1}. \end{aligned}$$

Since we may take $K = |a'_{\nu+1,\nu}|$ when $n = \nu+1$, (12) follows by induction.

LEMMA 3. *If $0 < \kappa \leqslant 1$, $\lambda_n \leqslant \omega$ and $C^\kappa(\omega)$ is defined as in (2), then*

$$\left| \int_0^{\lambda_n} \kappa(\omega - \tau)^{\kappa-1} C(\tau)\, d\tau \right| \leqslant |C^\kappa(\lambda_\eta)| \quad \textit{for some} \quad \eta \textit{ in } 0 \leqslant \eta \leqslant n.$$

Proof. This result is due to Jurkat [(4), Satz 1].

LEMMA 4. *Let $\kappa \geqslant 0$ and*

$$b_\mu = \Delta_\mu \int_{\lambda_\mu}^{\infty} (\omega - \lambda_\mu)^\kappa\, dg(\omega), \quad \int_{\lambda_0}^{\infty} \omega^\kappa\, |dg(\omega)| = M < \infty. \tag{13}$$

Then

$$\Sigma |b_\mu| \leqslant M. \tag{14}$$

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Moreover, if $0 \leqslant \kappa \leqslant 1$ *and* A *is the* (R^*, λ, κ) *T-matrix, with inverse* A', *then*

$$t_\nu \equiv \sum_{\mu=\nu}^{\infty} b_\mu a'_{\mu\nu} \tag{15}$$

exists and satisfies

$$\Sigma |t_\nu| \leqslant 3M \tag{16}$$

and

$$b_m = \sum_{\nu=m}^{\infty} t_\nu a_{\nu m}. \tag{17}$$

Proof. It is easy to deduce from (13) that

$$|b_\mu| \leqslant \Delta_\mu \int_{\lambda_\mu}^{\infty} (\omega - \lambda_\mu)^\kappa \, |dg(\omega)|,$$

and (14) follows. When $\kappa = 0$ we have $t_\nu = b_\nu$ and the lemma is trivial. When $0 < \kappa \leqslant 1$ the existence of t_ν as defined in (15) follows from (14) and (12).

The analysis will be slightly simpler if we can dispense with the differences occurring in the definitions of A and b_μ. Accordingly, let $\bar{A}$ denote the (R^*, λ, κ) γ-matrix, with inverse C, so that, by (1),

$$\bar{a}_{n\mu} = \lambda_{n+1}^{-\kappa} (\lambda_{n+1} - \lambda_\mu)^\kappa \quad (0 \leqslant \mu \leqslant n), \qquad c_{\mu\nu} = a'_{\mu\nu} - a'_{\mu-1,\nu} \quad (a'_{-1,\nu} = 0);$$

and define

$$\bar{b}_\mu \equiv \sum_{\nu=\mu}^{\infty} b_\nu = \int_{\lambda_\mu}^{\infty} (\omega - \lambda_\mu)^\kappa \, dg(\omega). \tag{18}$$

Thus, for fixed ν, $\bar{b}_{\mu+1} a'_{\mu\nu} = o(1) \cdot O(1)$, by (18) and (12), and a partial summation on (15) therefore gives

$$t_\nu = \sum_{\mu=\nu}^{\infty} \bar{b}_\mu c_{\mu\nu} = \sum_{\mu=\nu}^{\infty} c_{\mu\nu} \int_{\lambda_\mu}^{\infty} (\omega - \lambda_\mu)^\kappa \, dg(\omega).$$

We shall show that the order of summation and integration can be interchanged here, but since there may not be absolute convergence, the procedure is a little more involved. Write

$$C_\nu^\rho(\omega) = \sum_{\lambda_\mu < \omega} (\omega - \lambda_\mu)^\rho c_{\mu\nu} \quad (\rho \geqslant 0),$$

so that, in particular, $C_\nu^\kappa(\lambda_{n+1}) = \lambda_{n+1}^\kappa \sum_{\mu=\nu}^{n} \bar{a}_{n\mu} c_{\mu\nu} = \lambda_{n+1}^\kappa \delta_{n\nu}$,

$$C_\nu(\omega) = C_\nu^0(\omega) = \sum_{\mu=0}^{n} (a'_{\mu\nu} - a'_{\mu-1,\nu}) = a'_{n\nu} \quad (\lambda_n < \omega \leqslant \lambda_{n+1}).$$

Thus, for $\lambda_n < \omega \leqslant \lambda_{n+1}$ and $0 < \kappa \leqslant 1$,

$$\begin{aligned} C_\nu^\kappa(\omega) &= \int_0^\omega \kappa(\omega - \tau)^{\kappa-1} C_\nu(\tau) \, d\tau \\ &= \int_0^{\lambda_n} \kappa(\omega - \tau)^{\kappa-1} C_\nu(\tau) \, d\tau + a'_{n\nu} \int_{\lambda_n}^{\omega} \kappa(\omega - \tau)^{\kappa-1} \, d\tau \end{aligned}$$

and hence, by Lemma 3, there exists η such that

$$|C_\nu^\kappa(\omega)| \leqslant \lambda_\eta^\kappa \delta_{\eta-1,\nu} + (\omega - \lambda_n)^\kappa |a'_{n\nu}| \quad (\lambda_n < \omega \leqslant \lambda_{n+1}, \ 0 \leqslant \eta \leqslant n). \tag{19}$$

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A similar inequality also holds for any section of the sum defining $C_\nu^\kappa(\omega)$; thus denote

$$S_q(\omega) = \sum_{\lambda_\mu < \min(\omega, \lambda_q)} (\omega - \lambda_\mu)^\kappa c_{\mu\nu},$$

so that $S_q(\omega) = C_\nu^\kappa(\omega)$ for $\lambda_q \geqslant \omega$, while for $\lambda_q < \omega$ we have

$$\begin{aligned} |S_q(\omega)| &= \left| \int_0^{\lambda_q} (\omega - \tau)^\kappa \, dC_\nu(\tau) \right| \\ &= \left| (\omega - \lambda_q)^\kappa C_\nu(\lambda_q) + \int_0^{\lambda_q} \kappa(\omega - \tau)^{\kappa-1} C_\nu(\tau) \, d\tau \right| \\ &\leqslant (\omega - \lambda_q)^\kappa |a'_{q-1,\nu}| + \lambda_\xi^\kappa \delta_{\xi-1,\nu} \quad (\lambda_q < \omega,\ 0 \leqslant \xi \leqslant q). \end{aligned}$$

In view of (12) it now follows that

$$|S_q(\omega)| \leqslant K_\nu \omega^\kappa \quad \text{for any} \quad \omega \geqslant 0 \quad \text{and any} \quad q \geqslant 0. \tag{20}$$

Now
$$\sum_{\mu=0}^{q-1} \bar{b}_\mu c_{\mu\nu} = \sum_{\mu=0}^{q-1} c_{\mu\nu} \int_{\lambda_\mu}^{\infty} (\omega - \lambda_\mu)^\kappa \, dg(\omega) = \int_{\lambda_\nu}^{\infty} S_q(\omega) \, dg(\omega); \tag{21}$$

but this last integral converges uniformly in q, by (20) and the convergence of $\int \omega^\kappa |dg(\omega)|$. Consequently we may now let $q \to \infty$ in (21) to give

$$t_\nu = \int_{\lambda_\nu}^{\infty} C_\nu^\kappa(\omega) \, dg(\omega). \tag{22}$$

Hence
$$\sum_{\nu=0}^{N} |t_\nu| \leqslant \int_{\lambda_0}^{\infty} |dg(\omega)| \sum_{\lambda_\nu < \min(\omega, \lambda_{N+1})} |C_\nu^\kappa(\omega)| \leqslant \int_{\lambda_0}^{\infty} |dg(\omega)| \sum_{\lambda_\nu < \omega} |C_\nu^\kappa(\omega)|.$$

But noting that $a'_{nn} = \Lambda_n^\kappa$ and that $\omega - \lambda_n \leqslant \omega \Lambda_n^{-1}$ for $\omega \leqslant \lambda_{n+1}$, we see from (19) and (11) that if $\lambda_n < \omega \leqslant \lambda_{n+1}$ then

$$\begin{aligned} \sum_{\lambda_\nu < \omega} |C_\nu^\kappa(\omega)| &\leqslant \sum_{\nu=0}^{n} \{\lambda_\eta^\kappa \delta_{\eta-1,\nu} + (\omega - \lambda_n)^\kappa |a'_{n\nu}|\} \\ &\leqslant \lambda_n^\kappa + (\omega - \lambda_n)^\kappa \cdot 2\Lambda_n^\kappa \\ &< \omega^\kappa + \omega^\kappa \Lambda_n^{-\kappa} \cdot 2\Lambda_n^\kappa = 3\omega^\kappa. \end{aligned} \tag{23}$$

Thus (16) follows, since we now have

$$\sum_{\nu=0}^{N} |t_\nu| \leqslant \int_{\lambda_0}^{\infty} 3\omega^\kappa |dg(\omega)| = 3M, \quad \text{for any} \quad N.$$

For the final part of the lemma we first have, for $q > m$, by (22),

$$\sum_{\nu=m}^{q-1} t_\nu a_{\nu m} = \sum_{\nu=m}^{q-1} a_{\nu m} \left(\int_{\lambda_\nu}^{\lambda_q} + \int_{\lambda_q}^{\infty} \right) C_\nu^\kappa(\omega) \, dg(\omega) = I_1(q) + I_2(q), \tag{24}$$

where
$$I_1(q) = \int_{\lambda_m}^{\lambda_q} dg(\omega) \sum_{\lambda_\nu < \omega} C_\nu^\kappa(\omega) a_{\nu m}, \quad I_2(q) = \int_{\lambda_q}^{\infty} dg(\omega) \sum_{\nu=m}^{q-1} C_\nu^\kappa(\omega) a_{\nu m}.$$

Now we easily have
$$\sum_{\lambda_\nu < \omega} C_\nu^\kappa(\omega) a_{\nu m} = \Delta_m (\omega - \lambda_m)^\kappa,$$

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where $(\omega-\lambda_n)^\kappa$ is interpreted as zero if $\omega \leqslant \lambda_n$; so it follows from the definition of b_μ in (13) that

$$I_1(q) \to b_m \quad \text{as} \quad q \to \infty. \tag{25}$$

Since $0 \leqslant a_{\nu m} \leqslant 1$ for every ν and m, (23) shows that

$$|I_2(q)| \leqslant \int_{\lambda_q}^{\infty} |dg(\omega)| \sum_{\nu=m}^{q-1} |C_\nu^\kappa(\omega)|\, a_{\nu m} \leqslant \int_{\lambda_q}^{\infty} 3\omega^\kappa |dg(\omega)| = o(1).$$

Using this result together with (25), and letting $q \to \infty$ in (24), we obtain (17), and this completes the proof of the lemma.

Proof of Theorem 1.

Necessity. The necessity of (4) (with $\beta_\mu = 0$, $\beta = 1$ if consistency is also required) is immediate, since (R, λ, κ) is regular. The necessity of (5) is due essentially to Peyerimhoff (10) and Maddox (8). Briefly [see (10), § 8, and the argument in (8), p. 166, with minor modifications] the general continuous linear functional f in the space $R_{\lambda\kappa}(F, c)$ of (R, λ, κ)-limitable sequences is of the form

$$f(\{s_\nu\}) = \alpha \lim_\omega \omega^{-\kappa} C^\kappa(\omega) + \int_{\lambda_0}^{\infty} C^\kappa(\omega)\, dg(\omega), \quad \int_{\lambda_0}^{\infty} \omega^\kappa |dg(\omega)| < \infty.$$

Now if $\sum_\nu b_{\rho\nu} s_\nu$ converges for each ρ whenever $\omega^{-\kappa} C^\kappa(\omega) \to s$, then $f_\rho(\{s_\nu\}) = \sum_\nu b_{\rho\nu} s_\nu$ is a linear functional in $R_{\lambda\kappa}(F, c)$ and hence

$$\sum_\nu b_{\rho\nu} s_\nu = \alpha_\rho s + \int_{\lambda_0}^{\infty} C^\kappa(\omega)\, dg_\rho(\omega), \quad \int_{\lambda_0}^{\infty} \omega^\kappa |dg_\rho(\omega)| = M_\rho < \infty,$$

$$\|f_\rho\| = |\alpha_\rho| + M_\rho.$$

In particular, the choice $s_\mu = 1$, $s_\nu = 0$ $(\nu \neq \mu)$ leads to the expression for $b_{\rho\mu}$ in (5). Now if B is not weaker than (R, λ, κ), then $|\sum_\nu b_{\rho\nu} s_\nu|$ is bounded whenever $\{s_\nu\}$ is (R, λ, κ)-limitable, and the uniform boundedness principle shows that $|\alpha_\rho| + M_\rho \leqslant M$ (so that $M_\rho \leqslant M$).

Sufficiency. For $0 \leqslant \kappa \leqslant 1$ we may, by (3), replace (R, λ, κ) by the (R^*, λ, κ) T-matrix A, with inverse A'. Inserting an extra parameter ρ in Lemma 4, we see that (5) implies that $T \equiv BA'$ exists and satisfies $B = TA$, $\|T\| < \infty$. The required result now follows from Lemma 1.

Proof of Theorem 2.

Necessity. Since $\bar{B}$ is not weaker than the regular method (R, λ, κ), $\bar{B}$ must be a conservative series transformation, so that (6) is necessary (and if consistency is required, then $\bar{B}$ must be regular, whence $\bar{\beta}_\mu = 1$). No restrictions on λ are of course needed for this.

The necessity of (7), under the assumption that $\Lambda_{n-1} = O(\Lambda_n)$ if $\kappa > 1$, and without restriction for $0 < \kappa \leqslant 1$, follows from Russell [(15), Theorem 2]; this had earlier been proved by Jurkat [(3), Satz 4], with a heavier restriction on λ when $\kappa > 1$.

The necessity of (8) is given by Maddox [(8), Lemma 1]; however, no restrictions on λ are needed (compare the similar proof in Theorem 1 above) provided that we

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insert an additional term α_ρ on the right-hand side of the expression for $\bar{b}_{\rho\mu}$ in (8), which then gives $\bar{b}_{\rho\mu} = \alpha_\rho + o(1)$ as $\mu \to \infty$ (ρ fixed). α_ρ will vanish if we can independently show that $\bar{b}_{\rho\mu}$ has 0 as a limit point when $\mu \to \infty$, and this holds if we have (7) and $\Lambda_n \neq O(1)$. Since we impose the restriction $\Lambda_{n-1} = O(\Lambda_n)$ when $\kappa > 1$ in order to obtain (7), this restriction is also used in order to obtain (8) in the form given. No real generality is lost by the assumption $\Lambda_n \neq O(1)$, since otherwise (R, λ, κ) would be equivalent to convergence for all $\kappa \geqslant 0$.

Sufficiency. Suppose that (6), (7) hold, and define $b_{\rho\mu} = \Delta_\mu \bar{b}_{\rho\mu}$. Then (4) (i) holds with $\beta_\mu \equiv \lim_\rho b_{\rho\mu} = \Delta\bar{\beta}_\mu$; and, by (7), $\bar{b}_{\rho\mu} \to 0$ as $\mu \to \infty$ (ρ fixed), so that $\sum_{\mu=0}^{\infty} b_{\rho\mu} = \bar{b}_{\rho 0}$ and (4) (ii) holds with $\beta \equiv \lim_\rho \sum_\mu b_{\rho\mu} = \bar{\beta}_0$. Also (8) implies (5). Thus the conditions of Theorem 1 hold for the method B.

Now given any series Σc_ν, with partial sums s_ν, we have

$$\sum_{\nu=0}^{N} \bar{b}_{\rho\nu} c_\nu = \sum_{\nu=0}^{N-1} b_{\rho\nu} s_\nu + \bar{b}_{\rho N} s_N. \tag{26}$$

If Σc_ν is summable (R, λ, κ) to s, then we may assume without loss of generality that $s = 0$ (since (R, λ, κ) is regular and $\lim_\rho \sum_\mu b_{\rho\mu}$ exists) and then, by the limitation theorem for Riesz means [(2), Theorem 21], $s_N = o(\Lambda_N^\kappa)$. Hence, by (7), we have $\bar{b}_{\rho N} s_N \to 0$ as $N \to \infty$, so that, by (26), $\sum_\nu \bar{b}_{\rho\nu} c_\nu = \sum_\nu b_{\rho\nu} s_\nu$ whenever either side exists. Since B satisfies the conditions of Theorem 1, it follows that B, and hence $\bar{B}$, is not weaker than (R, λ, κ). If $\bar{\beta}_\mu = 1$ then $\beta_\mu = 0$, $\beta \neq 1$ and hence B and $\bar{B}$ are consistent with (R, λ, κ).

Remarks. (i) In Theorem 1, an order condition on $b_{\rho\mu}$ similar to (7) of Theorem 2, namely

$$|b_{\rho\mu}| \leqslant H_\rho \Lambda_\mu^{-\kappa}, \tag{27}$$

is in fact necessary for all $\kappa \geqslant 0$, if it is assumed that $\Lambda_{n-1} = O(\Lambda_n)$ when $\kappa > 1$ [see Russell (15), Theorem 2]. However, when $0 \leqslant \kappa \leqslant 1$ it is easy to show that (5) implies (27), so that (27) does not appear explicitly in Theorem 1. But any extension of the sufficiency part of Theorem 1 to $\kappa > 1$ would no doubt have to add (27) to hypotheses (4) and (5).

(ii) Counter-examples have been given by Maddox [(7), § 6] for $\kappa = 2$ which show that, unless some restriction is imposed on λ (here we have assumed $\Lambda_{n-1} = O(\Lambda_n)$), (7) may not be necessary in Theorem 2; and (7) and (8) may not be sufficient even to imply the existence of $\sum_\nu \bar{b}_{\rho\nu} c_\nu$ when Σc_ν is summable $(R, \lambda, 2)$.

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