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The maximal subgroups of the Lyons group

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1. *Introduction*

In this paper we complete the work begun in [2] on the subgroup structure of the Lyons simple group Ly of order

$$51\,765\,179\,004\,000\,000 = 2^8.3^7.5^6.7.11.31.37.67.$$

In [2] we classified the local subgroups and did some preliminary investigations concerning the non-local subgroups. To determine the maximal subgroups completely, it now suffices to classify the normalizers of subgroups isomorphic to A_6, $L_2(7)$, $L_2(8)$, $L_2(11)$, $U_3(3)$ or M_{11}. Each such group can be found by extending a smaller group by a suitable element. We quote the possibilities for the smaller groups from [2], and then consider all possible extending elements either by hand or, more often, by heavy use of a computer. The computations were done using the 111-dimensional representation over $\mathbb{F}_5$ recently constructed by R. A. Parker (see [1]). It was proved by Meyer & Neutsch in [1] that Parker's matrices do in fact generate the Lyons group.

Our main result is that there are no new subgroups of Ly. In other words, we prove:

THEOREM. *The Lyons group has exactly* 9 *conjugacy classes of maximal subgroups, as follows:*

$G_2(5)$	$3\cdot M^cL\colon 2$	$5^3\cdot L_3(5)$
$2\cdot A_{11}$	$5^{1+4}_+\colon 4S_6$	$3^5\colon (2\times M_{11})$
$3^{2+4}\colon 2A_5.D_8$	$67\colon 22$	$37\colon 18$

Note. In [2] it was asserted, incorrectly, that there is no $L_3(4)$ in $3\cdot M^cL\colon 2$. The argument given there which deduces from this the (false) conclusion that $L_3(4)$ is not a subgroup of Ly in fact shows only that any $L_3(4)$ in Ly has centralizer of order 3, and its normalizer is therefore contained in $3\cdot M^cL\colon 2$. This, however, is sufficient for our purposes, since Theorem 2 of [2] remains valid.

2. *Subgroups isomorphic to* A_6

Any A_6 may be constructed by taking a group A_5 and extending a subgroup A_4 to S_4. Now it was shown in [2] that there are 5 classes of A_5 in the Lyons group, three of which are centralized by a $3A$-element. If we start with an A_5 from one of the latter three classes, then the corresponding four-group normalizer $(2^{2+4}\times 3)\cdot(S_3\times S_3)$ is also contained in $N(3A)\cong 3\cdot M^cL\colon 2$, and hence so is the normalizer of any A_6 constructed in this way.

The next class of A_5 is in $2\cdot A_{11}$ and has orbits $1+5+5$ on the 11 points on which

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$2 \cdot A_{11}$ acts. Its normalizer is $(2 \times A_5) \cdot 2$, and the normalizer of the A_4 is $3 \times (2^2 \times A_4)$: 2, so any A_6 containing this A_5 has centralizer of order 2, and normalizer contained in $2 \cdot A_{11}$.

The final class of A_5 is also in $2 \cdot A_{11}$ and has orbits $5 + 6$ on the 11 points. Its normalizer is $2 \times S_5$, and the normalizer of the A_4 is $S_3 \times S_4$. But these groups together generate a group 3^4: $(2 \times S_5)$, contained in 3^5: $(2 \times M_{11})$, so this class of A_5 does not extend to A_6.

3. *Subgroups isomorphic to* $L_2(11)$ *or* M_{11}

Any $L_2(11)$ can be constructed by taking a group 11: 5 and adjoining an involution inverting a given element of order 5 therein. Now $N_{Ly}(11) \cong 11\colon 5 \times S_3$, in which the elements of order 5 are of class $5B$, and $N_{Ly}(5B) \cong (5 \times 5_+^{1+2}\colon S_3)\colon 4$. It turns out to be quite easy to find generators for these groups in terms of Parker's generators for the Lyons group. For Parker has generators for $2 \cdot A_{11}$ and its subgroup $2 \times M_{11}$, and an extra element extending the latter group to $S_3 \times M_{11}$. We first find $S_3 \times 11\colon 5$ in $S_3 \times M_{11}$, and then extend $5 \times S_3$ to $(5 \times 5_+^{1+2}\colon S_3)\colon 4$ by adjoining $N_{2 \cdot A_{11}}(5) \cong 2 \times 5^2\colon 4$.

Now each $5B$-element extends to just 100 groups D_{10}. In each of these 100 cases we multiplied each of the 5 involutions in the D_{10} by the original 11-element, and checked whether the product had order 3, since the following is a presentation for $L_2(11)$:

$$L_2(11) \cong \langle x, y, z | x^{11} = y^5 = z^2 = (yz)^2 = (xz)^3 = x^y x^8 = 1 \rangle.$$

It turns out that 11: 5 extends to exactly four groups $L_2(11)$, falling into orbits of sizes 1 and 3 under the action of the S_3 centralizing our 11: 5. In other words, there are exactly two classes of $L_2(11)$ in Ly, with normalizers

$$S_3 \times L_2(11) < 3 \cdot M^cL\colon 2$$

and

$$2 \times L_2(11) < 2 \cdot A_{11},$$

respectively.

Now any M_{11} can be constructed by taking a group $L_2(11)$ and extending a subgroup A_5 to S_5. We have seen that any $L_2(11)$ in the Lyons group is contained in $2 \cdot A_{11}$. Furthermore, one of the two classes of A_5 in any $L_2(11)$ has orbits $1 + 10$ on the 11 points on which $2 \cdot A_{11}$ acts, and has normalizer $S_3 \times A_5$ in Ly, so does not extend to S_5. We must therefore extend the other class of A_5, which in each case has orbits $5 + 6$ on the 11 points. If we start with the $L_2(11)$ whose normalizer is $S_3 \times L_2(11)$, then the corresponding A_5 has normalizer $S_3 \times S_5$, and these two groups together generate $S_3 \times M_{11}$. Similarly, if we start with the $L_2(11)$ whose normalizer is $2 \times L_2(11)$, then the corresponding A_5 has normalizer $2 \times S_5$, and these two groups together generate $2 \times M_{11}$. It follows that there are exactly two classes of M_{11} in the Lyons group, with normalizers

$$S_3 \times M_{11} < 3 \cdot M^cL\colon 2,$$

and

$$2 \times M_{11} < 2 \cdot A_{11},$$

respectively.

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4. *Subgroups isomorphic to* $L_2(8)$

Any $L_2(8)$ can be constructed by taking a Frobenius group 2^3: 7 and adjoining an involution inverting a given element of order 7 therein. In [2], section 2, it was shown that there are exactly two classes of elementary Abelian 2^3-groups in Ly, and their normalizers are

$$N(2^3)_A \cong 3 \times 2^4 : L_3(2)$$

and

$$N(2^3)_B \cong 2^3 \cdot L_3(2).$$

Thus there are just two classes of Frobenius group 2^3: 7, with normalizers

$$N(2^3 : 7)_A \cong 6 \times 2^3 : 7 : 3$$

and

$$N(2^3 : 7)_B \cong 2^3 : 7 : 3.$$

If we try to extend the first type of 2^3: 7 to $L_2(8)$, we find that the central involution of $N(2^3{:}7)$ is also central in $N(7) \cong (7 : 3 \times 2A_4) : 2$, and this involution therefore centralizes any such $L_2(8)$. Hence there is a unique class of such $L_2(8)$, and its normalizer is $2 \times L_2(8)$: 3, contained in $2 \cdot A_{11}$.

Consider now the case of an $L_2(8)$ containing a 2^3: 7 of the second type, for which we find it necessary to use the computer. The first problem is to construct the subgroup 2^3: 7. Now the 2^3-group can be found in $2 \cdot A_{11}$ (see [2]), in which its normalizer is $2^3 \cdot S_4 \cong 2^{1+4}.S_3$. Next we find an element of Ly which conjugates the central element of $2 \cdot A_{11}$ to one of the other elements of the 2^3-group. This is easy, given a few random involutions in Ly, since if x and y have order 2 and xy has odd order $2n-1$, then $(xy)^n$ conjugates x to y. Then the inverse of this transformation conjugates our 2^3-group to another 2^3-group in the same $2 \cdot A_{11}$. We can then conjugate one of these to the other by an element of $2 \cdot A_{11}$, and hence obtain an element normalizing the original 2^3-group, and extending $2^3 \cdot S_4$ to $2^3 \cdot L_3(2)$.

We now choose an element of order 7 in $2^3 \cdot L_3(2)$, and find its normalizer 7: 3 therein. The next problem is to find the full normalizer $(7 : 3 \times 2A_4) : 2$ of this 7-element, but this is too hard to do directly, so we begin by finding the centralizer of the 3-element which normalizes it. This has the shape $3^{2+4} : 2S_5$, and a method for finding it is described in the next section. We can then search through the 621 involutions in $3^{2+4} : 2S_5$ to find the one which centralizes the given 7-element. We conjugate this involution to the central involution of $2 \cdot A_{11}$ by the method described above, and identify the image of the 7-element under this transformation, as an element of our $2 \cdot A_{11}$. We can then write down generators for the normalizer of the 7-element.

It is now a simple matter to take each of the 84 involutions inverting the 7-element and check whether its product with some fixed involution in the original 2^3-group has order 3. It turns out that in no case does this occur, and hence there is no $L_2(8)$ containing a 2^3-group of the second type.

5. *Subgroups isomorphic to* $L_2(7)$ *or* $U_3(3)$

Any $L_2(7)$ can be constructed from a Frobenius group 7: 3 by adjoining an involution inverting a given element of order 3 therein. Since $N_{Ly}(7) \cong (7 : 3 \times 2A_4) : 2$, it follows that there are just two classes of 7: 3 in the Lyons group, with respective normalizers

$$N(7 : 3)_A \cong (7 : 3 \times 2A_4) : 2$$

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and $$N(7:3)_B \cong 7:3 \times 6.$$

In both cases the group is contained in $2 \cdot A_{11}$, and so can easily be found from Parker's matrices. Also, in each class of $7:3$ the 3-elements are of class $3B$, and so have normalizer $3^{2+4}:4S_5$.

In the first case, we can already see a subgroup $3^2:4S_5 \cong (3^2 \times 2A_5).2^2$ of the $3B$-normalizer inside $2 \cdot A_{11}$. To extend this to $3^{2+4}:4S_5$, we seek an extra element centralizing this 3^2-group, and we begin by looking for the centralizer of one of the $3A$-elements therein. Since the group generated by two elements of class $3A$ is one of a very restricted class of groups (such as 3^2 or $2A_4$ or $2A_5$), it is not hard to find the $3A$-centralizer computationally, and then to find the centralizer of the $3B$-element therein.

Having found the normalizer of the $3B$-element, we look at the 2970 involutions inverting it. (The number of cases can be reduced to 240 by considering orbits under the centralizer $2A_4$ of our $7:3$.) In each case, then, we multiply the involution by the original 7-element, and see if the product has order 3, thereby checking whether the following presentation for $L_2(7)$ is satisfied:

$$L_2(7) \cong \langle x, y, z | x^7 = y^3 = z^2 = (yz)^2 = (xz)^3 = x^y x^5 = 1 \rangle.$$

It turns out that just three of the involutions satisfy this condition, and so there are just two classes of $L_2(7)$ containing the first type of $7:3$, with normalizers as follows:

$$2 \times (3 \times L_2(7)):2 < 2 \cdot A_{11} \quad \text{and} \quad 3 \cdot M^cL:2,$$
$$(3 \times L_2(7)):2 < 3 \cdot M^cL:2.$$

Consider now the case of an $L_2(7)$ containing a group $7:3$ of the second type. In order to find the normalizer of a 3-element in this $7:3$, we try to conjugate it to the $3B$-element whose normalizer we have already found. We use the group $2^3 \cdot L_3(2)$ found above to conjugate the central involution of $(2A_4 \times 7:3):2$ to a non-central involution of $2 \cdot A_{11}$ commuting with our 3-element. We then identify the image of our 3-element under the inverse transformation, as an element of $2 \cdot A_{11}$, and conjugate this by an element of $2 \cdot A_{11}$ to our standard 3-element.

Again, we have 2970 cases to check, which can be reduced to 480 by considering orbits under the centralizer of the group $7:3$. It turns out that just two of these involutions extend $7:3$ to $L_2(7)$, and since such an $L_2(7)$ does not extend to $L_2(7):2$, it follows that both these involutions give rise to the same $L_2(7)$. It is now easy to see that this $L_2(7)$ has normalizer $3 \times L_2(7)$, contained in $3 \cdot M^cL:2$.

We have thus shown that any $L_2(7)$ in Ly is contained in $3 \cdot M^cL:2$. Now $U_3(3)$ can be constructed from $L_2(7)$ by extending S_4 to $4^2:S_3$. But the $3A$-element centralizing the $L_2(7)$ is characteristic in the entire four-group normalizer, so the normalizer of any group constructed in this way is contained in $3 \cdot M^cL:2$.

This concludes the proof of the Theorem.

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