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# Weakly 3-Connected Graphs

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A 3-connected graph $G$ is *weakly* 3-connected if, for every edge $e$ of $G$, at most one of $G\backslash e$ and $G/e$ is 3-connected. The main result of this paper is that any weakly 3-connected graph can be reduced to $K_4$ by a sequence of simple operations. This extends a result of Dawes [5] on minimally 3-connected graphs.

## 1. Introduction

A 3-connected graph $G$ is *weakly* 3-connected if, for every edge $e$ of $G$, at most one of $G\backslash e$ and $G/e$ is 3-connected. The main result of this paper is the following theorem; the necessary definitions follow its statement.

**Main Theorem.** *Let $G$ be a weakly 3-connected graph on at least four nodes. Then, either*

(i) *$G$ has a triad $T$ and $e \in T$ such that $\lfloor G\backslash e \rfloor$ is weakly 3-connected, or*
(ii) *$G$ has a triangle $T$ and $e \in T$ such that $\lceil G/e \rceil$ is weakly 3-connected, or*
(iii) *$G$ has a degree-three node $u$ such that $G\backslash u$ is weakly 3-connected, or*
(iv) *$G$ has a triangle $T$ such that $G/T$ is weakly 3-connected, or*
(v) *$G = K_4$.*

Thus, by repeated application of the reductions described in (i)–(iv) of the Main Theorem, any weakly 3-connected graph $G$ can be reduced to $K_4$ in at most $|V(G)|$ steps.

Graphs in this paper are finite. Cycles and paths are simple (*i.e.*, no repeated nodes), and are equated with their edge sets. A *triangle* of a graph $G$ is a cycle of length three. A *triad* of $G$ is the set of edges incident to a degree-three node; this node is the *centre* of the triad. A *triangle-triad* of a graph $G$ is a subset of four edges, three of which form a triangle of $G$, three of which form a triad of $G$, and no two of which are parallel. A *parallel class* of a graph is a subset of edges, every pair of which are parallel, and is

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maximal with respect to this property. To *simplify* a graph is to delete all but one edge from each parallel class; the simplification of a graph $G$ is denoted $\lceil G \rceil$. An *arc* of a graph is a path, each internal node of which has degree two in the graph, and is maximal with respect to this property. To *cosimplify* a graph is to contract all but one edge from each arc; the cosimplification of a graph $G$ is denoted $\lfloor G \rfloor$.

The notion of 3-connectivity used here is that of Tutte [12]. A *k-separation*, for a positive integer $k$, of a connected graph $G$ is a partition $\{E_1, E_2\}$ of the edge set of $G$ such that $|E_1| \geqslant k \leqslant |E_2|$ and the edge-induced subgraphs $G[E_1]$ and $G[E_2]$ have at most $k$ nodes in common. A graph possessing a $k$-separation is said to be *k-separable*. A connected graph $G$ is *n-connected*, for a positive integer $n$, if it is not $k$-separable for any $k < n$. An $n$-connected graph $G$ is *weakly* $n$-connected if, for every edge $e$ of $G$, at most one member of $\{G\backslash e, G/e\}$ is $n$-connected. An $n$-connected graph $G$ is *minimally* $n$-connected if, for every edge $e$ of $G$, the graph $G\backslash e$ is not $n$-connected. Evidently, every minimally $n$-connected graph is weakly $n$-connected.

Minimal 3-connectivity has been studied by Dawes [5], Coullard, Gardner and Wagner [4], Halin [6, 7, 8], Mader [10], among others. Weak $n$-connectivity, under the name *minor-minimal n-connectivity*, has been studied by Oxley [11] and Leo [9], primarily in the context of matroids. Of particular interest to this paper is the work of Dawes; the Main Theorem above can be seen as an extension of the following result in [5].

**Theorem.** *Let $G$ be a minimally 3-connected graph on at least four nodes. Then, either*

(i) *$G$ has a triad $T$ and $e \in T$ such that $\lfloor G\backslash e \rfloor$ is minimally 3-connected, or*
(ii) *$G$ has a degree-three node $u$ such that $G\backslash u$ is minimally 3-connected, or*
(iii) $G = K_4$.

It is also of interest to note that a slight modification of the definition of weakly 3-connected graphs yields the class of wheel graphs, which were characterized in the following famous theorem due to Tutte [12].

**Theorem.** *A 3-connected graph $G$ is a wheel if and only if, for every edge $e$ of $G$, neither $G\backslash e$ nor $G/e$ is 3-connected.*

The remainder of the paper is outlined as follows. The next section contains preliminary results on 3-connected graphs. The third section contains the proof of the Main Theorem. The fourth and final section contains some additional results and comments on weak 3-connectivity.

## 2. 3-connected graphs

The first two results of this section recall known facts about 3-connected graphs. The first one is implicit in Tutte [12]; an explicit matroid version is given in Bixby [2].

**Lemma 2.1.** *Let $G$ be a 3-connected graph, and let $e$ be an edge of $G$. Then, either $\lfloor G\backslash e \rfloor$ or $\lceil G/e \rceil$ is 3-connected.*

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The following lemma collects several well-known facts about operations on 3-connected graphs. The validity of operations (i)–(iii) can be found in Tutte [12]. The validity of operation (iv) follows easily from the definition of 3-connectivity. Two definitions are required. Let $v$ be a node a of a graph $G$. To *expand* $v$ *into a cycle* means to construct a graph $G'$ from $G$ by replacing $v$ by a cycle $C$ made up of new edges and nodes, and such that an edge is incident to $v$ in $G$ if and only if it is incident to a node of $C$ in $G'$. To *split* a node $v$ of $G$ is to construct a graph $G''$ from $G$ by replacing $v$ by two new adjacent nodes, say $v_1$ and $v_2$, such that a node is a neighbour of $v$ in $G$ if and only if it is a neighbour of at least one of $v_1$ and $v_2$ in $G''$. Note that a neighbour of $v$ in $G$ might be a neighbour of both $v_1$ and $v_2$ in $G''$.

**Lemma 2.2.** *Let $G$ be a 3-connected graph on at least four nodes, and let $G'$ a graph obtained from $G$ by applying one of the following operations. Then, $G'$ is 3-connected if and only if it does not have any loops, parallel edges, degree-one or degree-two nodes.*

(i) *Split a node of $G$.*
(ii) *Expand a node of $G$ into a cycle.*
(iii) *Add a new edge to $G$, each end of which is either a node of $G$ or a new node that results from subdividing an edge of $G$.*
(iv) *Add a new node $v$ of degree at least three to $G$, each neighbour of which is either a node of $G$ or a new node that results from subdividing an edge of $G$, and such that the neighbours of $v$ in $G'$ do not all lie on a common arc of $G'\backslash v$.*

Theorem 2.3 below proves that every 3-connected graph $G$ on at least two nodes either has a triangle-triad or a node $v$ such that $\lfloor G\backslash v\rfloor$ is 3-connected. This is a slight generalization of a nice result due to Barnette [1]. Specifically, Barnette's theorem contains the additional condition that $G$ must be different from $K_4$ and $K_{3,3}$. It is not clear why Barnette chose to include $K_4$ in this condition, since $K_4$ clearly contains a triangle-triad. By using Tutte's definition of 3-connectivity (as opposed to the standard 'disjoint-paths' definition), one also avoids the need to include $K_{3,3}$. For completeness, a proof of Theorem 2.3 is provided below. This proof is conceptually similar to that of Barnette, but is simpler and shorter.

Theorem 2.3 also generalizes a theorem of Chartrand, Kaugars and Lick [3], who proved that if $G$ is a 3-connected graph of minimum degree four, then $G$ has a node $v$ such that $G\backslash v$ is 3-connected.

The proof of Theorem 2.3 requires a few definitions. To this end, let $G$ be a connected graph. Then, $G$ is said to be *cyclically* 3*-connected* if $\lfloor G\rfloor$ is 3-connected. A node having degree at least three in $G$ is a *principal* node of $G$. Let $v$ be a principal node of $G$, and let $S_1,\ldots,S_t$ denote the arcs of $G$ that have $v$ as an end node. To *remove* $v$ from $G$ is to delete $v$ together with all nodes of $G$ that are internal to one of $S_1,\ldots,S_t$; the graph so obtained is denoted $G\neg v$. If $G$ and $G\neg v$ are both cyclically 3-connected, then $v$ is said to be a *removable* node of $G$.

Let $H$ be a subgraph of $G$, and let $v$ be a node of $G$ not in $H$. Let $P_1,\ldots,P_t$ be paths of $G$ that start at $v$, end in $V(H)$, are pairwise node-disjoint other than $v$, and are internally node-disjoint from $H$. The subgraph $H'$ of $G$ induced by the subset of edges

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$E(H) \cup P_1 \cup \cdots \cup P_t$ is said to be obtained from $H$ by *adjoining* $v$ *using* $P_1, \ldots, P_t$. Now suppose additionally that $H$ is cyclically 3-connected and has at least four principal nodes. Then, the proof of Theorem 2.3 makes repeated use of the following observation, which follows from Lemma 2.2: $H'$ is cyclically 3-connected if and only if $t \geqslant 2$ and $v_1, \ldots, v_t$ do not all lie on a common arc of $H$, where $v_i$ denotes the end of $P_i$ different from $v$.

**Theorem 2.3.** *Let $G$ be a 3-connected graph on at least two nodes. Then, either $G$ has a node $v$ such that $G\backslash v$ is cyclically 3-connected or $G$ has a triangle-triad.*

**Proof.** If $G$ has fewer than four nodes, the result is easily verified. Thus, assume $G$ has at least four nodes. Also, assume $G$ does not have a triangle-triad. It is well known that every 3-connected graph on at least four nodes has a subgraph homeomorphic to $K_4$. Thus, $G$ has at least one subgraph that is cyclically 3-connected and has a removable node; among all such subgraphs, let $H$ denote one that has the maximum number of principal nodes.

**Claim 1.** *$H$ is a spanning subgraph of $G$.*

**Proof of Claim 1.** Suppose that $H$ is not spanning. Then, there exists a node of $G$ that is not in $H$. If $u$ is a node of $G$ not in $H$ that can be adjoined to $H$ using at least three paths to produce a cyclically 3-connected subgraph $K$ of $G$, then $K$ is a contradiction to the choice of $H$ since it has more principal nodes than $H$ and has $u$ as a removable node. Thus, no such node $u$ exists. Now, let $v$ be any node of $G$ that is not in $H$. Since $G$ is 3-connected, there do exist three paths, say $P_1$, $P_2$, and $P_3$, such that $v$ can be adjoined to $H$ using $P_1$, $P_2$, and $P_3$ to produce a subgraph $H'$ of $G$. But, by the above, $H'$ is not cyclically 3-connected. Thus, $v_1$, $v_2$, and $v_3$, the respective ends of $P_1$, $P_2$, and $P_3$ different from $v$, lie on a common arc, say $S$, of $H$.

Let $s_1$ and $s_2$ denote the end nodes of $S$, and assume the ordering of the above nodes of $S$ is given by $s_1$, $v_1$, $v_3$, $v_2$, and $s_2$. Let $d_i$, for $i \in \{1, 2\}$, denote the distance from $s_i$ to $v_i$ on $S$. Consistent with the above choice of $H$, assume that $H$, $v$, $P_1$, $P_2$, $P_3$ (and thus $S$) are chosen so that $d_1 + d_2$ is minimized. Let $T$ denote the $v_1v_2$-subpath of $S$. Note that $\{v_1, v_2\}$ constitutes a two-node cut of $H'$. Thus, by the 3-connectivity of $G$, there exists a path $Z$ of $G$ that satisfies the following properties: (i) no internal node of $Z$ is a node of $H'$; (ii) one end node of $Z$, say $z_1$, is either an internal node of $T$, $P_1$, $P_2$, or $P_3$, or is equal to $v$; and (iii) the other end node of $Z$, say $z_2$, is a node of $H$ that is not a node $S$, or is a node $S$ that is not a node of $T$.

Suppose that $z_2$ is a node of $H$ that is not a node of $S$. Assume that $z_1$ is either an internal node of $T$, $P_1$ or $P_2$, or is equal to $v$. (The case where $z_1$ is internal to $P_3$ is similar.) Construct a graph $H''$ from $H$ by adding $Z$, $P_1$, and $P_2$. Then, $H''\neg z_1$ is cyclically 3-connected since it is homeomorphic to $H$. Moreover, $H''$ is cyclically 3-connected. Thus, $z_1$ is a removable node of $H''$. This, however, is a contradiction since $H''$ has more principal nodes than $H$.

Now, suppose that $z_2$ is a node of $S$ that is not a node of $T$. Suppose that $z_2$ is on the $s_1v_1$-subpath of $S$. (The case where $z_2$ is on the $s_2v_2$-subpath of $S$ is similar.) Also, assume

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that $z_1$ is either an internal node of $T$, $P_1$, or $P_2$, or is equal to $v$. (Again, the case where $z_1$ is internal to $P_3$ is similar.) Construct a graph $H''$ from $H$ by adding $Z$, $P_1$, and $P_2$. Then, a contradiction to the choice of $H'$ is obtained since $H''\neg z_1$ is homeomorphic to $H$, which implies it is cyclically 3-connected, has the same number of principal nodes as $H$, and has a removable node. Moreover, the node $z_1$ can be adjoined to $H''\neg z_1$ to produce a graph that has a smaller value of $d_1 + d_2$. □

The next claim is aimed at showing that $H$ has a removable node such that each arc of $H$ that ends at this node consists of a single edge. To this end, let $v$ be a removable node of $H$. Then, Claim 2 below shows that if some arc of $H$ that ends at $v$ has more than one edge, then this arc can be 'shortened' by adding an appropriate edge $e$ to $H$ and then deleting an appropriate edge $f$. Moreover, the resulting graph satisfies all of the conditions imposed on $H$. Namely, it is cyclically 3-connected, has a removable node (namely, $v$), and has the maximum number of principal nodes. The claim uses the following notation: $P_0, P_1, \ldots, P_t$ denote the arcs of $H$ that end at $v$ with $P_0$ having at least two edges.

**Claim 2.** *There exist an edge $e$ of $G$ not in $H$ and an edge $f$ of $H\neg v$, such that adding $e$ to $H$ and deleting $f$ results in a graph $H'$ that is cyclically 3-connected, has the same number of principal nodes as $H$, and has $v$ as a removable node. Moreover, the arcs of $H'$ that end at $v$ are $P_0', P_1, \ldots, P_t$, where $P_0'$ is a proper subpath of $P_0$.*

**Proof of Claim 2.** Since $G$ is 3-connected, there exists an edge $e = xy$ of $G$ not in $H$ such that $x$ is an internal node of $P_0$ and $y$ is not a node of $P_0$. Let $K$ be the graph obtained from $H$ by adding $e$. Then, by Lemma 2.2, $K$ is cyclically 3-connected and has more principal nodes than $H$. Thus, by the choice of $H$, $K$ does not have a removable node. Note this implies that $H$ has at least five principal nodes, since otherwise it is homeomorphic to $K_4$, in which case it easy to check that, in fact, $v$ is a removable node of $K$.

Denote by $v_i$ the end node of $P_i$ different from $v$. Since $K$ does not have a removable node, $K\neg x$ is not cyclically 3-connected. Since $v$ is a removable node of $H$, it must be that $v_1, \ldots, v_t$ lie on a common arc $S$ of $H\neg v$. Moreover, since $H$ has at least five principal nodes, $v_0$ is not on $S$, for otherwise $H$ is not cyclically 3-connected.

Since $v$ is not a removable node of $K$, $K\neg v$ is not cyclically 3-connected. Thus, it must be the case that $v_0$ and $y$ lie on the same arc $T$ of $H\neg v$. Note, $T$ is different from $S$. Let $R$ be the $v_0y$-subpath of $T$. Let $K'$ be the graph obtained from $K$ by deleting the edges and internal nodes of $R$. Observe that $K'\neg v$ is homeomorphic to $H\neg v$, and is thus cyclically 3-connected. Let $P_0'$ be the $vx$-subpath of $P_0$. Then, $K'$ is obtained from $K\neg v$ by adjoining $v$ using $P_0', P_1, \ldots, P_t$. Since $x$ is not on $S$, $K'$ is cyclically 3-connected. Therefore, $v$ is a removable node of $K'$. Moreover, observe that the number of principal nodes of $K'$ is at least as great as (and therefore equal to) that of $H$. Therefore, by Claim 1, $K'$ is spanning, which implies that $R$ consists of a single edge, say $f$. □

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The shortening technique given by Claim 2 (*i.e.*, adding $e$ and deleting $f$) can be applied repeatedly until each arc incident to $v$ consists of a single edge. Thus, it can be assumed that $H$ and $v$ are chosen consistent with the previous conditions imposed on them, and such that each arc incident to $v$ consists of a single edge. It can further be assumed that $H$ is chosen consistent with these conditions, and such that it has the maximum number of edges. A simple, but key, observation about the shortening technique is that it does not change the number of edges in the graph; this is used in Claim 3 below.

If $H$ contains all of the edges of $G$, then the theorem follows. Thus, there exists an edge of $G$ not in $H$. Define an edge $e$ of $G$ not in $H$ to be a *chord* of $H$ if the ends of $e$ lie on a common arc of $H\backslash v$. From the choice of $H$ and Lemma 2.2, each edge of $G$ not in $H$ must be a chord of $H$. Note also that each arc of $H\backslash v$ having at least two edges has an internal node adjacent to $v$ in $H$. The theorem will now follow from Claim 3 below.

**Claim 3.** *$H$ has no chords.*

Suppose that some arc $S$ of $H\backslash v$ has a chord. First, it is claimed that some internal node of the subpath of $S$ defined by the ends of any chord of $S$ is adjacent to $v$. To see this, let $e = st$ be a chord of $S$, and suppose that no internal node of the $st$-subpath of $S$ is adjacent to $v$. Now, construct a graph $H'$ by adding the edge $e$ to $H$ and deleting all internal nodes of the $st$-subpath of $S$. Note that this $st$-subpath of $S$ must contain at least one internal node, for otherwise $e$ is parallel to an edge of $S$ contradicting the 3-connectivity of $G$. Then, $H'$ is homeomorphic to $H$, but is not spanning, contradicting Claim 1.

Let $e = st$ again be a chord of $S$, and let $x$ be an internal node of the $st$-subpath of $S$ that is adjacent to $v$. Let $H''$ be the graph obtained from $H$ by adding the edge $e$. By Lemma 2.2, $H''$ is cyclically 3-connected, and has at least as many principal nodes as $H$. If $x$ is a removable node of $H''$, then a contradiction is easily obtained as follows. From Claim 2 (and the key observation that followed it), it can be assumed that every arc of $H''$ incident to $x$ consists of a single edge. This, however, contradicts the choice of $H$ since $H''$ has one more edge than $H$. Thus, it can be assumed that $x$ is not a removable node of $H''$.

Since $x$ is not a removable node of $H''$, $H''\neg x$ is not cyclically 3-connected. Note, the graph $H'''$ obtained by removing $v$ from $H''\neg x$ is homeomorphic to $H\backslash v$, and is thus cyclically 3-connected. Since $H''\neg x$ is not cyclically 3-connected, it must be that all of the neighbours of $v$ in $H''\neg x$ lie on a common arc $T$ of $H'''$. Note, every arc of $H'''$, except $S$, is an arc of $H\backslash v$. Thus, $T$ is an arc of $H\backslash v$ different from $S$. Therefore, $x$ is the unique internal node of $S$ that is incident to $v$. If $S$ has exactly one internal node, then $G$ has a triangle-triad consisting of $e$, $S$, and the edge $xv$, a contradiction. Thus, $S$ has at least two internal nodes. From the 3-connectivity of $G$, every internal node of $S$, except for $x$, is incident to a chord of $S$. Thus, it can be assumed that $s$ is an internal node of $S$ that is adjacent to $x$. If $t$ is also adjacent to $x$, then $G$ has a triangle-triad, a contradiction. Thus, the $xt$-subpath of $S$ contains at least one internal node. Now, consider the graph obtained from $H''$ by deleting all of the internal nodes of this $xt$-subpath of $S$; the resulting graph is homeomorphic to $H$, but is not spanning, a contradiction to Claim 1. □

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The next pair of lemmas of this section are dual to each other.

**Lemma 2.4.** *Let $G$ be a 3-connected graph on at least five nodes, and suppose that $G\backslash f$ is 2-separable for some edge $f$. Let $g$ be an edge different from $f$ and such that no triad of $G$ contains both $f$ and $g$, and such that $f$ is an edge of $\lceil G/g\rceil$. Then, $G' := \lfloor G\backslash g\rfloor\backslash f$ is 2-separable and $G'' := \lceil G/g\rceil\backslash f$ is 1- or 2-separable.*

**Proof.** Let $\{E_1, E_2\}$ be a 2-separation of $G\backslash f$. Then, $G[E_1]$ and $G[E_2]$ have exactly two nodes in common. Moreover, $f$ has one end in $V(G[E_1]) - V(G[E_2])$ and the other in $V(G[E_2]) - V(G[E_1])$. If $|V(G[E_1]) - V(G[E_2])| = 1$, then $E_1 \cup \{f\}$ is a triad of $G$. Since $g$ is not in this triad, it follows that $\lfloor G\backslash g\rfloor$ contains a triad containing $f$, and $\lceil G/g\rceil$ contains either a triad containing $f$ or a degree-two node to which $f$ is incident. It follows that $G'$ is 2-separable and $G''$ is 1- or 2-separable.

It can now be assumed that $G[E_1]$ and $G[E_2]$ each contain at least two nodes that are not in the other. Since each node of $G$ has degree at least three, it follows that $|E_1| \geqslant 5 \leqslant |E_2|$.

Consider $G'$, and let $E_1'$ and $E_2'$ be the restrictions of $E_1$ and $E_2$, respectively, to $E(G')$. Then, $G'[E_1']$ and $G'[E_2']$ have at most two nodes in common. Note that $|E(G\backslash f) - E(G')| \leqslant 3$, from which it follows that $|E_1'| \geqslant 2 \leqslant |E_2'|$. Thus, $\{E_1', E_2'\}$ is a 2-separation of $G'$.

Now, consider $G''$, and let $E_1''$ and $E_2''$ be the restrictions of $E_1$ and $E_2$, respectively, to $E(G'')$. Then, $G''[E_1'']$ and $G''[E_2'']$ have at most two nodes in common. Without loss of generality, assume $g \in E_1$. If the ends of $g$ in $G$ coincide with the nodes common to $G[E_1]$ and $G[E_2]$, then it is easy to verify that $G''$ is 1-separable. Thus, assume at least one end of $g$ is a node of $G[E_1]$ that is not in $G[E_2]$. It follows that at most two edges of $E_2$ are in a triangle of $G$ with $g$. Since $|E_2| \geqslant 5$, at least three edges of $E_2$ are not in a triangle of $G$ with $g$, and therefore $|E_2''| \geqslant 3$. It also follows that it can be assumed that any triangle of $G$ that contains $g$ and meets $E_1$ contributes an edge to $E_1''$. Since $|E_1| \geqslant 5$, it follows that $|E_1''| \geqslant 2$. Thus, $\{E_1'', E_2''\}$ is a 2-separation of $G''$. □

**Lemma 2.5.** *Let $G$ be a 3-connected graph on at least five nodes, and suppose that $G/f$ is 2-separable for some edge $f$. Let $g$ be an edge different from $f$ and such that no triangle of $G$ contains both $f$ and $g$, and such that $f$ is an edge of $\lfloor G\backslash g\rfloor$. Then, $G' := \lceil G/g\rceil/f$ is 2-separable and $G'' := \lfloor G\backslash g\rfloor/f$ is 1- or 2-separable.*

**Proof.** Let $\{E_1, E_2\}$ be a 2-separation of $G/f$. Then, $G[E_1]$ and $G[E_2]$ have exactly three nodes in common, with $f$ joining two of these nodes. If $|V(G[E_1]) - V(G[E_2])| \leqslant 1$, then $E_1 \cup \{f\}$ contains a triangle of $G$ that contains $f$. Since $g$ is not in this triangle, it follows that $\lceil G/g\rceil$ contains a triangle containing $f$, and $\lfloor G\backslash g\rfloor$ either contains a triangle containing $f$ or a pair of parallel edges, one of which is $f$. Thus, $G'$ is 2-separable and $G''$ is 1- or 2-separable, as required.

It can now be assumed that $G[E_1]$ and $G[E_2]$ each contain at least two nodes that are not in the other. Since each node of $G$ has degree at least three, it follows that $|E_1| \geqslant 5 \leqslant |E_2|$.

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Consider $G'$, and let $E_1'$ and $E_2'$ be the restrictions of $E_1$ and $E_2$, respectively, to $E(G')$. Then, $G'[E_1']$ and $G[E_2']$ have at most two nodes in common. Without loss of generality, assume $g \in E_1$. Since $G[E_2]$ has at least two nodes that are not in $G[E_2]$, $E_2$ has at least two edges that are not in a triangle of $G$ with $g$, from which follows that $|E_2'| \geqslant 2$. It also follows that it can be assumed that any triangle of $G$ that contains $g$ and meets $E_1$ contributes an edge to $E_1'$. Since $|E_1| \geqslant 5$, it follows that $|E_1'| \geqslant 2$. Thus, $\{E_1', E_2'\}$ is a 2-separation of $G'$.

Now, consider $G''$, and let $E_1''$ and $E_2''$ be the restrictions of $E_1$ and $E_2$, respectively, to $E(G'')$. Then, $G''[E_1'']$ and $G''[E_2'']$ have at most two nodes in common. Note that $|E(G/f) - E(G'')| \leqslant 3$, from which it follows that $|E_1''| \geqslant 2 \leqslant |E_2''|$. Thus, $\{E_1'', E_2''\}$ is a 2-separation of $G''$. □

## 3. Proof of Main Theorem

The Main Theorem is proved through a sequence of six lemmas.

**Lemma 3.1.** *Let $G$ be a weakly 3-connected graph. Then, either*

(i) *$G$ has a triad $T$ and $e \in T$ such that $\lfloor G\backslash e \rfloor$ is 3-connected, or*
(ii) *$G$ has a triangle $T$ and $e \in T$ such that $\lceil G/e \rceil$ is 3-connected.*

**Proof.** If $G$ has fewer than five nodes, the lemma is easy to verify. Thus, assume $G$ has at least five nodes. If $G$ has a triangle-triad, then Lemma 2.1 implies that either $\lceil G/e \rceil$ or $\lfloor G\backslash e \rfloor$ is 3-connected, where $e$ is an edge common to both the triangle and triad. Thus, it can be assumed that $G$ does not have a triangle-triad. By Theorem 2.3, $G$ has a node $v$ such that $\lfloor G\backslash v \rfloor$ is 3-connected.

First, consider the case that $v$ has degree three in $G$. Let $e$ be an edge incident to $v$. Observe that the graph $\lfloor G\backslash e \rfloor$ can be obtained from $\lfloor G\backslash v \rfloor$ by adding a single edge, say $f$. Moreover, the ends of $f$ lie in different arcs of $\lfloor G\backslash v \rfloor$, for otherwise $G$ has a triangle-triad. Thus, by Lemma 2.2, the graph $\lfloor G\backslash e \rfloor$ is 3-connected, as required. Therefore, it can be assumed that $v$ has degree at least four in $G$.

Second, consider the case that a neighbour of $v$, say $u$, has degree three in $G$, and let $e = uv$. Observe that $\lfloor G\backslash e \rfloor$ can be obtained from $\lfloor G\backslash v \rfloor$ by adding a node of degree at least three. Thus, by Lemma 2.2, $\lfloor G\backslash e \rfloor$ is 3-connected so long as all of the neighbours of $v$ in $\lfloor G\backslash e \rfloor$ do not all lie on a common arc of $G\backslash v$. But this must be the case, for otherwise $G$ has a triangle-triad.

Finally, consider the case where $v$ and all of its neighbours in $G$ have degree at least four. It follows that $G\backslash v$ is 3-connected. Let $e$ be an edge of $G$ incident to $v$. Then, by Lemma 2.2, $G\backslash e$ is also 3-connected. By the weak 3-connectivity of $G$, $G/e$ is 2-separable. Observe that $G/e$ can be obtained from $G\backslash v$ by adding edges. Thus, by Lemma 2.2, $G/e$ must contain parallel edges, which, in turn, implies that $e$ is in a triangle of $G$ and $\lceil G/e \rceil$ is 3-connected, as required. □

Let $G$ be a graph that is 3-connected, but not weakly 3-connected. Thus, $G$ has an edge $e$ such that both $G\backslash e$ and $G/e$ are 3-connected. Call such an edge a *violator* of $G$.

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**Lemma 3.2.** *Let $G$ be a weakly 3-connected graph on at least four nodes, and suppose that $G$ has an edge $e$ that is in a triad of $G$ and for which $\lfloor G\backslash e\rfloor$ is 3-connected. Then, either $G$ has a triangle-triad, or degree-three node $u$ such that $G\backslash u$ is weakly 3-connected, or an edge $f$ that is in a triad of $G$ and for which $\lfloor G\backslash f\rfloor$ is weakly 3-connected.*

**Proof.** If $G$ has five or fewer nodes, then the lemma is easy to verify. Thus, assume it has at least six nodes. If $\lfloor G\backslash e\rfloor$ is weakly 3-connected, the lemma is proved. So, suppose that it is not weakly 3-connected, and let $f$ be a violator of $\lfloor G\backslash e\rfloor$. By the weak 3-connectivity of $G$, at least one of $G\backslash f$ or $G/f$ is 2-separable.

If $G/f$ is 2-separable, then, by Lemma 2.5, $e$ and $f$ are in a triangle of $G$, which implies that $G$ has a triangle-triad of $G$, as required. Thus, it can be assumed that $G\backslash f$ is 2-separable. Since $G\backslash f$ is 2-separable, Lemma 2.4 implies that $e$ and $f$ are in a triad, say $T$, of $G$. Let $u$ denote the centre of $T$, and let $G' := G\backslash u$.

Suppose that $G'$ is 3-connected; if it is weakly 3-connected, then the proof is complete. So, suppose otherwise, and let $g$ be a violator of $G'$. If the ends of $g$ are both neighbours of $u$ in $G$, then $G$ has a triangle-triad. Thus, suppose this is not the case. It follows that the neighbours of $u$ in $G$ are three distinct nodes in $G'/g$. Evidently, $G/g$ is obtained from $G'/g$ by adding the node $u$. Thus, by Lemma 2.2, $G/g$ is 3-connected. Similarly, $G\backslash g$ is 3-connected. This, however, contradicts the weak 3-connectivity of $G$.

Now, suppose that $G'$ is 2-separable. By the 3-connectivity of $\lfloor G\backslash e\rfloor\backslash f$, the edge $f$ is in exactly one triad of $G$, namely $T$. Now, consider $\lfloor G\backslash f\rfloor$. If it has parallel edges, then $G$ contains a triangle-triad, as required. Thus, suppose this is not the case. Then, Lemma 2.2 implies that $\lfloor G\backslash f\rfloor$ is 3-connected since it can be obtained from $\lfloor G\backslash e\rfloor\backslash f$ by adding an edge. If $\lfloor G\backslash f\rfloor$ is weakly 3-connected, then the lemma is proved. Thus, it must have a violator, say $h$. By the weak 3-connectivity of $G$, either $G/h$ or $G\backslash h$ is 2-separable.

If $G/h$ is 2-separable, then, by Lemma 2.5, $f$ and $h$ are in a triangle, which implies that $G$ has a triangle-triad, as required.

Suppose $G\backslash h$ is 2-separable. By Lemma 2.4 and the fact that $T$ is the unique triad containing $f$, it follows $h \in T$. Since $h$ is a violator of $\lfloor G\backslash f\rfloor$, the graph $\lfloor G\backslash f\rfloor\backslash h$ is 3-connected. But, observe that $\lfloor G\backslash f\rfloor\backslash h$ is the same as $G'$, which contradicts the 2-separability of $G'$. □

Lemma 3.3 below is the dual of Lemma 3.2.

**Lemma 3.3.** *Let $G$ be a weakly 3-connected graph on at least four nodes, and suppose that $G$ has an edge $e$ that is in a triangle of $G$ and for which $\lceil G/e\rceil$ is 3-connected. Then, either $G$ has a triangle-triad, a triangle $T$ such that $G/T$ is weakly 3-connected, or an edge $f$ that is in a triangle of $G$ and for which $\lceil G/f\rceil$ is weakly 3-connected.*

**Proof.** If $G$ has five or fewer nodes, the lemma is easy to verify. Thus, assume that $G$ has at least six nodes. If $\lceil G/e\rceil$ is weakly 3-connected, the lemma proved. So, suppose that it is not weakly 3-connected, and let $f$ be a violator of $\lceil G/e\rceil$. By the weak 3-connectivity of $G$, at least one of $G\backslash f$ or $G/f$ is 2-separable.

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If $G\backslash f$ is 2-separable, then, by Lemma 2.4, $e$ and $f$ are in a triad of $G$, which implies that $G$ has a triangle-triad, as required. Thus, it can be assumed that $G/f$ is 2-separable. Since $G/f$ is 2-separable, Lemma 2.5 implies that $e$ and $f$ are in a triangle, say $T'$, of $G$. Let $G' := G/T'$.

Suppose that $G'$ is 3-connected; if it is weakly 3-connected, then the proof is complete. So, suppose otherwise, and let $g$ be a violator of $G'$. Evidently, $G\backslash g$ can be obtained from $G'\backslash g$ by expanding a node into the triangle $T'$. If some node $v$ of $T'$ has degree two in $G\backslash g$, then it must be that $g$ is incident to $v$ in $G$, from which it follows that $G$ has a triangle-triad. Thus, by Lemma 2.2, $G\backslash g$ is 3-connected. Similarly, $G/g$ is 3-connected. This, however, contradicts the weak 3-connectivity of $G$.

Now, suppose that $G'$ is 2-separable. By the 3-connectivity of $\lceil G/e\rceil/f$, the edge $f$ is in exactly one triangle of $G$, namely $T'$. Now, consider $\lceil G/f\rceil$. If it has a degree-two node, then $G$ contains a triangle-triad. Thus, suppose this is not the case. Then, Lemma 2.2 implies that $\lceil G/f\rceil$ is 3-connected since it can be obtained from $\lceil G/e\rceil/f$ by splitting a node. If $\lceil G/f\rceil$ is weakly 3-connected, then the lemma is proved. Thus, it must have a violator, say $h$. By the weak 3-connectivity of $G$, either $G/h$ or $G\backslash h$ is 2-separable.

If $G\backslash h$ is 2-separable, then, by Lemma 2.4, $f$ and $h$ are in a triad, which implies that $G$ has a triangle-triad, as required.

Suppose $G/h$ is 2-separable. By Lemma 2.5 and the fact that $T'$ is the unique triangle containing $f$, it follows that $h \in T'$. Since $h$ is a violator of $\lceil G/f\rceil$, the graph $\lceil G/f\rceil/h$ is 3-connected. But, observe that $\lceil G/f\rceil/h$ is the same as $G'$, which contradicts the 2-separability of $G'$. □

In the absence of a triangle-triad, the Main Theorem follows from Lemmas 3.1–3.3 above. The case when a triangle-triad is present is surprisingly complicated, and is addressed by next three lemmas.

**Lemma 3.4.** *Let $G$ be a weakly 3-connected graph, and suppose that $G$ has two triangles that have exactly one edge $g$ in common. Further, suppose that one end of $g$ has degree three in $G$. Then, either $\lfloor G\backslash g\rfloor$ or $\lceil G/g\rceil$ is weakly 3-connected.*

**Proof.** If $G$ has exactly four nodes, then $G = K_4$ and the lemma is easy to verify; so assume $G$ has at least five nodes. Observe that $g$ is in exactly two triangles and exactly one triad of $G$; denote by $T$ the union of the two triangles.

Assume first that $\lfloor G\backslash g\rfloor$ is 3-connected, but is not weakly 3-connected. Let $f$ be a violator of $\lfloor G\backslash g\rfloor$. By the weak 3-connectivity of $G$, either $G\backslash f$ or $G/f$ is 2-separable. If $G\backslash f$ is 2-separable, then Lemma 2.4 implies that $f$ and $g$ are in a triad of $G$. If $G/f$ is 2-separable, then Lemma 2.5 implies that $f$ and $g$ are in a triangle of $G$. In either case, it must be that $f \in T$, which implies that $f$ is in a triangle of $\lfloor G\backslash g\rfloor$, contradicting the 3-connectivity of $\lfloor G\backslash g\rfloor/f$. Thus, it can be assumed that $\lfloor G\backslash g\rfloor$ is 2-separable.

Now assume, by Lemma 2.1, that $\lceil G/g\rceil$ is 3-connected, but is not weakly 3-connected. Let $f$ be a violator of $\lceil G/g\rceil$. By the weak 3-connectivity of $G$, either $G\backslash f$ or $G/f$ is 2-separable. If $G\backslash f$ is 2-separable, then Lemma 2.4 implies that $f$ and $g$ are in a triad of

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$G$. If $G/f$ is 2-separable, then Lemma 2.5 implies that $f$ and $g$ are in a triangle of $G$. In either case, it must be that $f \in T$.

Observe that it can be assumed that $\lceil G/g \rceil$ is obtained from $\lfloor G\backslash g \rfloor$ by deleting an edge, say $h$. Since $\lceil G/g \rceil$ is 3-connected and $\lfloor G\backslash g \rfloor$ is 2-separable, Lemma 2.2 implies that, in $\lfloor G\backslash g \rfloor$, $h$ is parallel to an edge of $E(G) - T$. It follows that in $\lceil G/g \rceil$, $f$ is in a triangle, which contradicts the 3-connectivity of $\lceil G/g \rceil / f$. □

Lemma 3.5 below is the dual of Lemma 3.4. Moreover, a proof of Lemma 3.5 can be obtained by a direct dualization of the above proof of Lemma 3.4, and is thus omitted.

**Lemma 3.5.** *Let $G$ be a weakly 3-connected graph, and suppose that $G$ has two triads that have exactly one edge $g$ in common. Further, suppose that $g$ is in a triangle of $G$. Then, either $\lfloor G\backslash g \rfloor$ or $\lceil G/g \rceil$ is weakly 3-connected.* □

The final lemma can now be proved.

**Lemma 3.6.** *Let $G$ be a weakly 3-connected graph, and let $T = \{e, f, g, h\}$ be a triangle-triad of $G$, with the triad consisting of edges $\{e, f, g\}$ and centre node $u$, and the triangle consisting of edges $\{f, g, h\}$. Then, at least one of $\lfloor G\backslash f \rfloor$, $\lfloor G\backslash g \rfloor$, $\lceil G/f \rceil$, $\lceil G/g \rceil$, $G\backslash u$, or $G/\{f, g, h\}$ is weakly 3-connected.*

**Proof.** By Lemma 3.4, it can be assumed that each of $f$ and $g$ is in exactly one triangle of $G$, and by Lemma 3.5, it can be assumed that each of them is in exactly one triad of $G$. It is easily checked that $G$ must have at least six nodes. By Lemma 2.1, either $\lfloor G\backslash f \rfloor$ or $\lceil G/f \rceil$ is 3-connected. Observe that $\lceil G/f \rceil$ is isomorphic to $\lfloor G\backslash g \rfloor$, and $\lfloor G\backslash f \rfloor$ is isomorphic to $\lceil G/g \rceil$. Because of this symmetry between $f$ and $g$, it can be assumed that $\lfloor G\backslash f \rfloor$ is 3-connected. If $\lfloor G\backslash f \rfloor$ is weakly 3-connected, the lemma is proved. Thus, let $m$ be a violator of $\lfloor G\backslash f \rfloor$. By the weak 3-connectivity of $G$, either $G\backslash m$ or $G/m$ is 2-separable.

First consider that case that $G\backslash m$ is 2-separable. By Lemma 2.4, $m$ is in a triad of $G$ with $f$. Since $f$ is in a unique triad, it follows that $m \in \{e, g\}$. Moreover, it follows that $m$ is the unique violator of $\lfloor G\backslash f \rfloor$. Since $m$ is a violator of $\lfloor G\backslash f \rfloor$, it also follows that $G\backslash u$ is 3-connected since it is the same as $\lfloor G\backslash f \rfloor \backslash m$. Thus, if $G\backslash u$ is weakly 3-connected, the lemma is proved. Therefore, it can be assumed that $G\backslash u$ has a violator, say $p$. Note that Lemma 2.2 implies that $G\backslash p$ is 3-connected since it can be obtained from $G\backslash u\backslash p$ by adding the node $u$. Similarly, if $p \neq h$, then Lemma 2.2 implies that $G/p$ is 3-connected. Thus, by the weak 3-connectivity of $G$, it must be that $p = h$. Note, $h$ is not a violator of $\lfloor G\backslash f \rfloor$. Thus, at least one of $\lfloor G\backslash f \rfloor \backslash h$ or $\lfloor G\backslash f \rfloor / h$ is 2-separable. This leads to two subcases.

Assume first that $\lfloor G\backslash f \rfloor \backslash h$ is 2-separable. Observe that $\lfloor G\backslash f \rfloor \backslash h$ can be obtained from $G\backslash u\backslash h$ by adding the edge $m$. Since $G\backslash u\backslash h$ is 3-connected, Lemma 2.2 implies that $m$ is parallel to an edge, say $q$, of $\lfloor G\backslash f \rfloor \backslash h$, which, in turn, implies that $\{e, g, q\}$ constitutes a triangle of $G$, contradicting the assumption that $g$ is in a unique triangle of $G$.

The case when $\lfloor G\backslash f \rfloor / h$ is 2-separable is similar. Namely, $\lfloor G\backslash f \rfloor / h$ can be obtained from $G\backslash u/h$ by adding the edge $m$. Since $G\backslash u/h$ is 3-connected, Lemma 2.2 implies that $m$ is parallel to an edge, say $q$, of $\lfloor G\backslash f \rfloor / h$, which, in turn, implies that either $\{e, g, q\}$ or

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$\{e, f, q\}$ constitutes a triangle of $G$, contradicting the assumption that both $f$ and $g$ are in unique triangles of $G$.

Now, suppose that $G/m$ is 2-separable. As observed, $\lceil G/g \rceil$ is isomorphic to $\lfloor G \backslash f \rfloor$. Thus, $m$ (or more precisely, its image under this isomorphism) is a violator of $\lceil G/g \rceil$. By Lemma 2.5, $m$ is in a triangle of $G$ with $g$. Since $g$ is in a unique triangle of $G$, it follows that $m \in \{f, h\}$. Moreover, it must be that $m$ is the unique violator of $\lceil G/g \rceil$. Since $m$ is a violator of $\lceil G/g \rceil$, it also follows that $G/\{f, g, h\}$ is 3-connected. If $G/\{f, g, h\}$ is weakly 3-connected, the lemma is proved. Therefore, it can be assumed that $G/\{f, g, h\}$ has a violator, say $q$. Note that Lemma 2.2 implies that $G/q$ is 3-connected since it can be obtained from $G/\{f, g, h, q\}$ by expanding a node into a triangle. Similarly, if $q \neq e$, then Lemma 2.2 implies that $G \backslash q$ is 3-connected. Thus, by the weak 3-connectivity of $G$, it must be that $q = e$. Note, $e$ is not a violator of $\lceil G/g \rceil$. Thus, at least one of $\lceil G/g \rceil / e$ or $\lceil G/g \rceil \backslash e$ is 2-separable. This leads to two subcases.

Assume first that $\lceil G/g \rceil / e$ is 2-separable. Observe that $\lceil G/g \rceil / e$ can be obtained from $G/\{e, f, g, h\}$ by a node splitting. Suppose that $G/\{e, f, g, h\}$ has at least four nodes. Then, the 3-connectivity of $G/\{e, f, g, h\}$ together with Lemma 2.2 imply that $m$ is incident to a degree-two node of $\lceil G/g \rceil / e$. Letting $q$ denote the other edge incident to this degree-two node, it follows that $\{f, h, q\}$ constitutes a triad of $G$, contradicting the assumption that $f$ is in a unique triad of $G$. Now, suppose $G/\{e, f, g, h\}$ has fewer than four nodes. Since $G$ has at least six nodes, $G/\{e, f, g, h\}$ must have exactly three nodes, and is thus a triangle. Therefore, $G$ has six nodes, seven edges, and a triangle-triad. It is easy to check that no such 3-connected graph exists.

The case when $\lceil G/g \rceil \backslash e$ is 2-separable is similar. Namely, $\lceil G/g \rceil \backslash e$ can be obtained from $G/\{f, g, h\} \backslash e$ by a node splitting. Since $G/\{f, g, h\} \backslash e$ is 3-connected, Lemma 2.2 implies that $m$ is incident to a degree-two node of $\lceil G/g \rceil \backslash e$. Letting $q$ denote the other edge incident to this degree-two node, it follows that either $\{f, h, q\}$ or $\{g, h, q\}$ constitutes a triad of $G$, contradicting the assumption that both $f$ and $g$ are in unique triads of $G$. □

The Main Theorem can now be easily proved.

**Proof of Main Theorem.** By Lemma 3.1, $G$ either has a triad $T$ and an edge $e \in T$ such that $\lfloor G \backslash e \rfloor$ is 3-connected, or a triangle $T$ and an edge $e \in T$ such that $\lceil G/e \rceil$ is 3-connected. Thus, by Lemmas 3.2 and 3.3, either $G$ has a triangle-triad or one of (i)–(iv) of the Main Theorem is true. In the former case, Lemma 3.6 implies that one of (i)–(iv) is true. □

## 4. Additional results and comments

Lemma 3.1, which implies that every weakly 3-connected graph either has a triangle or triad, can be seen as an extension of a result of Halin [7], who proved that every minimally 3-connected graph has a triad. (Halin actually proved the more general statement that every minimally $n$-connected graph has a node of degree $n$.) The following result extends a second result of Halin [8], who proved that if $e$ is an edge of a minimally 3-connected graph $G$ that is not in a triad, then $G/e$ is minimally 3-connected.

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**Proposition 4.1.** *Let $G$ be a weakly 3-connected graph, and let $e$ be an edge that is neither in a triangle or triad of $G$. Then, either $G\backslash e$ or $G/e$ is weakly 3-connected.*

**Proof.** By Lemma 2.1, either $G\backslash e$ or $G/e$ is 3-connected. Let $f$ be an edge of $G$ different from $e$. By the weak 3-connectivity of $G$, either $G\backslash f$ or $G/f$ is 2-separable.

First, assume $G\backslash e$ is 3-connected. If $G\backslash f$ is 2-separable, then Lemma 2.4 implies $G\backslash\{e,f\}$ is 2-separable. If $G/f$ is 2-separable, then Lemma 2.5 implies that $G\backslash e/f$ is 1- or 2-separable. Thus, $G\backslash e$ is weakly 3-connected.

Now, assume that $G/e$ is 3-connected. If $G\backslash f$ is 2-separable, then Lemma 2.4 implies $G/e\backslash f$ is 1- or 2-separable. If $G/f$ is 2-separable, then Lemma 2.5 implies that $G/\{e,f\}$ is 2-separable. Thus, $G/e$ is weakly 3-connected. □

As a final comment, it is noted that the all of the results presented here have natural matroid generalizations. Indeed, it is conjectured that the Main Theorem can be extended to matroids.

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