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Homotopical dynamics: suspension and duality

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Abstract. Flow-type suspension and homotopy suspension agree for attractor–repellor homotopy data. The connection maps associated in Conley index theory to an attractor–repellor decomposition with respect to the direct flow and its inverse are Spanier–Whitehead duals in the stably parallelizable context and are duals modulo a certain Thom construction in general.

1. *Introduction*

The Morse inequalities as well as their degenerate counterpart, the Lusternik–Schnirelmann inequality [**17**], are early examples of applications of homotopy theory to the study of smooth flows. More recently, the subject has developed along three directions:

- Various numerical homotopical invariants related to the Lusternik–Schnirelmann (LS) category have been investigated intensively.
- Particular attention has been given to Morse–Smale flows where the use of homotopical methods has had much success.
- The index theory of Conley [**5**] has introduced homotopical methods in the analysis of general flows.

The last few years have seen a certain convergence of these topics having at the center the notion of stabilization. For example, given a two-connected manifold M, for k large enough, one can construct functions on the manifold $M \times D^k$ pointing inwards on the boundary and with no more than $\mathrm{cat}(M) + 2$ critical points [**7**]. Attaining the best general lower bound, $\mathrm{cat}(M) + 1$, depends, for now, on showing that the LS-category and another homotopical invariant, the cone-length of M agree [**6**]. This seems likely for most closed manifolds [**10**] but for arbitrary CW-complexes it is not true and recently new homotopical examples have been constructed [**11**] for which the two invariants differ.

Conversely, one can look for critical point estimates for functions that are defined on $M \times D^k$ with M closed and which are pointing out on $M \times D^{k-t-1} \times S^t$ and pointing in on the complement of this set in $\partial(M \times D^k)$. The lower bound in this case is $\mathrm{cat}(M, M \times S^t)$ [**9**]. For any CW-complex, this invariant can be shown [**20**] to be strictly bigger than a

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more stable version of the LS-category $\sigma^{(t+1)} - \mathrm{cat}$ (defined as the least n for which the $(t+1)$-suspension of the nth filtration in Milnor's classifying construction applied to ΩM admits a homotopy section [**25**]).

This class of functions contains the functions quadratic at infinity studied in relation to the Arnold conjecture in symplectic geometry. They are at the center of the 'stable' Morse and LS theory of Eliashberg and Gromov [**12**]. Because of the Arnold conjecture one does then expect that $\mathrm{cat}(M, M \times S^t) \geq \mathrm{cat}(M) + 1$. This inequality is known to hold in many cases and it implies [**9**] $\mathrm{cat}(M \times S^t) = \mathrm{cat}(M) + 1$. In its turn, the validity of this equality for all CW-complexes is known as the Ganea conjecture. The conjecture has been recently disproved [**16**]. However, for symplectic, closed manifolds of symplectic form ω and with $\omega|_{\pi_2(M)} = 0$ by [**21**] one has $\sigma^{(t+1)} - \mathrm{cat}(M) = \mathrm{cat}(M) = \mathrm{cat}(M, M \times S^t) - 1 = \dim(M)$.

Finally, again in relation to the Arnold conjecture and Floer homology, Cohen *et al* [**4**] have introduced a stable homotopy point of view in studying Morse–Smale flows (see also [**15**]).

All of this suggests that in the flow context suspending should be understood to mean taking the product of the given flow with the gradient flow of a quadratic form on $\mathbf{R}^n$.

The present paper is concerned with the next step in the study of the homotopical properties of flows: attractor–repellor data. We address two main issues in this context.

- Understand the behaviour of attractor–repellor data under suspension.
- Determine in what measure and in what sense the homotopical information provided by a smooth flow and its inverse are redundant.

The key point is that suspension is central in dealing with the second matter.

Attractor–repellor data is best understood for Morse–Smale functions. Passage through a critical point of a Morse–Smale function corresponds to the attachment of a cell. Hence, for two consecutive critical points one has a relative attaching map $d : S^{q-1} \longrightarrow S^k$ where k is the index of the first point and q that of the second. This map has been described by Franks in [**13**]. The negative of the original function being also Morse–Smale there is also a relative attaching map $d' : S^{n-k-1} \longrightarrow S^{n-q}$ with n being the dimension of the supporting manifold M. Franks has also shown that, if M is stably parallelizable, then d and d' are stably the same up to sign.

Conley index theory allows one to define analogues of these maps for general attractor–repellor pairs of some general flow γ (which will be supposed here to be defined on a smooth manifold and to be itself smooth). Assume S is an isolated invariant set of γ and (A, A^*) is an attractor–repellor decomposition of S. Then one has the map $\delta : c_\gamma(A^*) \longrightarrow \Sigma c_\gamma(A)$ called the connection map of the attractor–repellor pair (A, A^*). Let $-\gamma$ be the inverse flow of γ, $-\gamma_t(x) = \gamma_{-t}(x)$. As (A^*, A) is an attractor–repellor decomposition of S with respect to $-\gamma$ we also have the inverse connection map $\delta' : c_{-\gamma}(A) \longrightarrow \Sigma c_{-\gamma}(A^*)$.

Here $c_\gamma(-)$ is the Conley index of the respective invariant set. Specialized to the Morse–Smale case and consecutive critical points, $\delta = \Sigma d$ and $\delta' = \Sigma d'$.

The results of the paper imply two main things:

- Flow-type suspension and homotopy suspension agree for attractor–repellor homotopy data.
- The maps δ and δ' are Spanier–Whitehead duals in the stably parallelizable context and are duals modulo a certain Thom construction in general.

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The duality result implies the (co)-homological results of McCord [**19**]. In the stably parallelizable case the duality together with the suspension result imply that, up to flow suspension, the direct and inverse attractor–repellor homotopy data are redundant. Conley index theory works quite well for continuous flows on metrizable spaces, hence in singular or stratified contexts, and parts of the theory extend even to semiflows. Our suspension result is also likely to extend to the most general setting. Not so the duality. This result seems to be specific to flows on manifolds and shows that Spanier–Whitehead duality is a key ingredient in understanding the related homotopical data. Possibly, the smoothness restriction might be relaxed.

Of course, the result of Franks on the stable equality up to sign of d and d' is also extended (as the Spanier–Whitehead duality of maps between spheres is stable equality up to sign). His paper [**13**] is closest in spirit to ours even if, due to the fact that for general flows one has no control on stable and unstable manifolds, our methods are forced to be quite different.

In trying to make the paper more easily accessible we have included, in the second section, a relatively detailed recall of Spanier–Whitehead duality and of the main properties of the Conley index.

The third section contains the precise statements and proofs and is organized as follows. We first discuss the Thom construction in the setting of the Conley index. This is needed to prove the suspension result and it also reduces to proving the duality to the study of a flow defined on a sphere. For such a flow the duality of the two connection maps is obtained by identifying each of them to a connectant in a certain cofibration sequence and then showing that these two cofibration sequences are dual.

The paper is written in the language of connected simple systems for two reasons. Firstly, the constructions related to the Conley index contain many choices and in this way one is sure of the invariance of the result. Secondly, connected simple systems reflect precisely the homotopical information that can be extracted from the flow.

2. *Recall of some well-known facts*

We recall a number of well-known facts, first on Spanier–Whitehead duality and then concerning the Conley index.

2.1. *Spanier–Whitehead duality*

2.1.1. *Definitions.* Two pointed, connected CW-complexes X, X' are Spanier–Whitehead dual [**24, 23**] if there is a map $\mu : X \wedge X' \longrightarrow S^m$ such that the slant product $\mu^*(i_m)^*/ : H_q(X') \longrightarrow H^{m-q}(X)$ gives an isomorphism. The map μ is called an m-duality. For completeness recall (see, e.g. [**14**]) that the slant product α/γ with $\alpha \in H^{p+q}(A \times B)$ and $\gamma \in H_p(A)$ is a cohomology class in $H^q(B)$ that satisfies $\langle \alpha/\gamma, \beta \rangle = \langle \alpha, \gamma \times \beta \rangle$ for all $\beta \in H_q(B)$, where $\times$ is the exterior cross product and $\langle , \rangle$ is the usual cohomology–homology pairing. The natural map $A \times B \longrightarrow A \wedge B$ allows one to use in the slant product a class $\alpha \in H^{p+q}(A \wedge B)$ by pulling it back to the product $A \times B$.

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Spanier–Whitehead duality behaves well under suspension: if X and X' are m-duals with duality map μ, then $\Sigma^k X$ and $\Sigma^q X'$ are $(m+k+q)$-duals with respect to the obvious suspension of μ. Assume X, X' are duals by an m-duality, μ, and Y, Y' are duals by an m-duality ν. Two pointed maps $f : X \longrightarrow Y$, $g : Y' \longrightarrow X'$ are duals if for some k, k' the following diagram homotopy commutes:

$$\begin{array}{ccc} \Sigma^k X \wedge \Sigma^{k'} Y' & \xrightarrow{f\wedge 1} & \Sigma^k Y \wedge \Sigma^{k'} Y' \\ {\scriptstyle 1\wedge g}\downarrow & & \downarrow{\scriptstyle \nu} \\ \Sigma^k X \wedge \Sigma^{k'} X' & \xrightarrow[\mu]{} & S^{m+k+k'} \end{array}$$

In general, only by allowing k and k' to be sufficiently large does this notion of duality give a duality isomorphism between (stable) classes of maps $D_m(\mu, \nu) : \{X, Y\} \approx \{Y', X'\}$.

Spanier–Whitehead duality implies Poincaré duality via the Thom isomorphism. For example, let $(W^n; V_0, V_1)$ be a smooth manifold triad with $V_0 \coprod V_1 = \partial W$. Let ν be the stable normal bundle of W and assume it is of rank $m \gg n$. Then W/V_0 is $(m+n)$-Spanier–Whitehead dual to $T^\nu(W)/T^\nu(V_1)$ where $T^\nu(-)$ is the respective Thom space [**2**] (recall that the Thom space of an orthogonal bundle η over a space X is obtained from the total space of the unit disk bundle associated with η by identifying to a point the total space of the unit sphere bundle). We have the Thom isomorphism $H^{m+k}(T^\nu(W)/T^\nu(V_1)) \simeq H^k(W/V_1)$. On the other hand we also have the Spanier–Whitehead duality isomorphism $H^{m+k}(T^\nu(W)/T^\nu(V_1)) \simeq H_{n-k}(W/V_0)$.

2.1.2. *Construction of duality maps.* Here is one way to obtain duality maps [**24**]. Assume $A \subset S^{m+1}$, $B \subset S^{m+1}$ such that the inclusion induces an isomorphism $H_*(B) \longrightarrow H_*(S^{m+1} - A)$, and A, B are connected and disjoint. Define a map in the following way: fix a point $p \in S^{m+1} - \left(A \bigcup B\right)$ and identify $S^{m+1} - \{p\}$ with $\mathbf{R}^{m+1}$. We then define $v : A \times B \longrightarrow S^m$ by $v(x, y) = (x - y)/\|x - y\|$. Connectivity implies that this map is null-homotopic when restricted to $A \vee B$. It defines a map $A \wedge B \longrightarrow S^m$ which is a duality map. One can identify dual maps in this setting: if A', B' is another pair of subspaces of S^{m+1} that satisfies the conditions above and if $A \xrightarrow{i} A'$ and $B' \xrightarrow{i'} B$ are inclusions, then i and i' are dual with respect to the duality maps constructed as above.

2.2. *The Conley index.* We will use here and in the rest of the paper the conventions of [**22**].

2.2.1. *Connected simple systems.* Recall first that the pointed homotopy category of pointed topological spaces $Ho - \mathcal{T}$ has as objects pointed topological spaces and as morphisms homotopy classes of pointed (continuous) maps. A connected simple system is a subcategory $\mathcal{C}$ of $Ho-\mathcal{T}$ such that for any two objects X_1, X_2 of $\mathcal{C}$ the set $\mathrm{mor}_{\mathcal{C}}(X_1, X_2)$ consists of exactly one element. Any map $X_1 \longrightarrow X_2$ whose homotopy class belongs to $\mathrm{mor}_{\mathcal{C}}(X_1, X_2)$ is called a comparison map. Clearly, all comparison maps are homotopy equivalences. A morphism between two connected simple systems X and Y associates

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with each pair $X_0 \in Ob(X)$ and $Y_0 \in Ob(Y)$ a unique homotopy class of maps $X_0 \longrightarrow Y_0$ with the obvious compatibility properties. We will sometimes call such a morphism a *map* of connected simple systems. Connected simple systems also form a category $\mathcal{CS}$ (see also [**18**]). Let $X, Y \in Ob(\mathcal{CS})$.

Assume that we have a (non-void) set $\mathcal{P}$ of pairs (A, B) with $A \in Ob(X)$ and $B \in Ob(Y)$ and maps $f_{A,B} : A \longrightarrow B$ that are defined for each such pair. We say that these maps *induce* a morphism $f : X \longrightarrow Y$ if for any two pairs (A, B) and (A', B') in $\mathcal{P}$ the maps $f_{A,B}$ and $f_{A',B'}$ commute, up to homotopy, with the obvious comparison maps and f is the unique morphism of connected simple systems represented by the maps $f_{A,B}$. Of course, defining f is possible even if $\mathcal{P}$ has a single element. However, in many of the constructions below the maps $f_{A,B}$ are naturally defined for many pairs (A, B) and it is useful to know that the induced morphism does not depend on the particular choice of the pair. Hence, in the following we will speak of an *induced map* at the level of connected simple systems by always considering $\mathcal{P}$ to be as large as is allowed by the definition of the maps $f_{A,B}$.

2.2.2. *Cofibration sequences.* An obvious functor $\Sigma : \mathcal{CS} \longrightarrow \mathcal{CS}$ is defined by taking the supension of spaces and maps. By definition, a cofibration sequence of connected simple systems is a triple of maps $X \xrightarrow{i} Y \xrightarrow{p} Z \xrightarrow{\delta} \Sigma X$ with $X, Y, Z \in Ob(\mathcal{CS})$ with the property that there are $X_0 \in Ob(X)$, $Y_0 \in Ob(Y)$, $Z_0 \in Ob(Z)$, a cofibration sequence $X_0 \xrightarrow{i_0} Y_0 \xrightarrow{p_0} Z_0$ with $[i_0] = i$, $[p_0] = p$ and if $\delta_1 : Z_0 \longrightarrow \Sigma X_0$ is the obvious connectant, then $\delta = [\delta_1]$. The map δ is called the connection map of the cofibration sequence. The cofibration sequence $X_0 \longrightarrow Y_0 \longrightarrow Z_0$ is called a representing cofibration sequence. The cofibration sequence is determined by the choice of i_0 (however, because push outs in $Ho - \mathcal{T}$ do not coincide, in general, with homotopy push outs, the knowledge of i only does not suffice). For completeness we recall the definition of a cofibration sequence. An inclusion $j : A \longrightarrow B$ is a cofibration if it satifies the homotopy extension property. This means that given a homotopy $h : A \times [0, 1] \longrightarrow X$ and a map $F : B \longrightarrow X$ such that $F \circ j = h_0$ then there is a homotopy $H : B \times [0, 1] \longrightarrow X$ that extends h and with $H_0 = F$. This concept is useful here because if j is a cofibration then the homotopy type of the quotient B/A is the same as that of $B \bigcup_j CA$ (CA is the cone over A). In view of this, we will sometimes identify C and B/A. The pair of maps $A \xrightarrow{j} B \xrightarrow{r} C$ form a cofibration sequence if j is a cofibration, $C = B/A$ and r is the obvious collapsing map. A cofibration sequence extends to the right giving a sequence of maps: $A \xrightarrow{j} B \xrightarrow{r} C \xrightarrow{\delta_1} \Sigma A \xrightarrow{\Sigma j} \Sigma B \xrightarrow{\Sigma r} \cdots$ where δ_1 is the collapsing $C \simeq B \bigcup CA \longrightarrow \left(B \bigcup CA\right)/B = \Sigma A$.

2.2.3. *The Conley index.* We recall now a few elements of Conley index theory [**5, 22**] for the case of a continuous flow $\gamma : N \times \mathbf{R} \longrightarrow N$, N being a compact metric space. Let $S \subset N$ be an isolated invariant set. A pair (N_1, N_0) of compact sets in N is an index pair for S in N if $N_0 \subset N_1$, $N_1 - N_0$ is a neighborhood of S, S is the maximal invariant set in the closure of $N_1 - N_0$, N_0 is positively invariant in N_1 and, if for $x \in N_1$ there is some

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$t \geq 0$ such that $\gamma_t(x) \notin N_1$, then there exists a $\tau > 0$ with $\gamma_t(x) \in N_1$ for $0 \leq t \leq \tau$ and $\gamma_\tau(x) \in N_0$, [**22**].

The basic result in this theory is the existence of index pairs inside any isolating neighborhood. Moreover, for any two index pairs of S, (N_1, N_0) and (N_1', N_0) there are flow induced comparison maps $N_1/N_0 \longrightarrow N_1'/N_0'$. In the following we need a uniform choice for the comparison maps. We will refer to the maps defined by the formula of Lemma 4.7 of [**22**] as the *standard comparison maps*. This gives the set of quotients N_1/N_0 the structure of a connected simple system denoted by $c_\gamma(S)$ and called the Conley index of S with respect to γ.

Assume that S admits an attractor–repellor decomposition, denoted by $(S; A, A^*)$, of attractor A and repellor A^* (this means that if $x \in S - \left(A \bigcup A^*\right)$ then $\lim_{t\to\infty} \gamma_t(x) \in A$ and $\lim_{t\to-\infty} -\gamma_t(x) \in A^*$). We have a cofibration sequence of connected simple systems denoted by $c_\gamma(S; A, A^*)$: $c_\gamma(A) \xrightarrow{i} c_\gamma(S) \xrightarrow{p} c_\gamma(A^*) \xrightarrow{\delta} \Sigma c_\gamma(A)$ such that δ as well as i and p are flow defined [**22**]. In this case δ is called the connection map of the given attractor–repellor decomposition.

Moreover, for any triple $N_0 \subset N_1 \subset N_2$ with the inclusions being inclusions of NDR's and such that (N_2, N_1) is an index pair of A^*, (N_1, N_0) is an index pair of A and (N_2, N_0) an index pair of S the cofibration sequence $N_1/N_0 \longrightarrow N_2/N_0 \longrightarrow N_2/N_1$ represents $c_\gamma(S; A, A^*)$.

For $M \subset N$ we use the notation $I(M)$ for the maximal invariant set inside M.

2.2.4. *Duality and connected simple systems.* The definition of Spanier–Whitehead duality induces one for simple connected systems. Consider the connected simple system formed by a single space S^m and let $X, X' \in Ob(\mathcal{CS})$. We have $X \wedge X' \in Ob(\mathcal{CS})$. The two connected simple systems are Spanier–Whitehead duals if there is a morphism $X \wedge X' \longrightarrow S^m$ giving a duality map whenever restricted to $X_0 \wedge X_0'$, $X_0 \in Ob(X)$, $X_0' \in Ob(X')$. A similar definition applies to maps.

Let $X \xrightarrow{i} Y \xrightarrow{p} Z \xrightarrow{\delta} \Sigma X$ and $Z' \xrightarrow{p'} Y' \xrightarrow{i'} X' \xrightarrow{\delta'} \Sigma Z'$ be two cofibration sequences of connected simple systems. They are called dual cofibration sequences if the pairs of maps (i, i'), (p, p') and (δ, δ') are respectively dual. To ensure this duality it is enough to find cofibration sequences of spaces: $X_0 \xrightarrow{i_0} Y_0 \xrightarrow{p_0} Z_0$ and $Z_0' \xrightarrow{p_0'} Y_0' \xrightarrow{i_0'} X_0'$ representing, respectively, the two above and with (i_0, i_0') and (p_0, p_0') respectively duals. Indeed, in this case, by the basic properties of Spanier–Whitehead duality [**23**], we have that the associated connectant maps $\delta_0 : Z_0 \longrightarrow \Sigma X_0$ and $\delta_0' : X_0' \longrightarrow \Sigma Z_0'$ are Spanier–Whitehead duals. Of course, under these circumstances, one wants also to show that the duality relation obtained does not depend on the choice of representing cofibration sequences.

3. *Bundles, suspension and duality*

Let $\psi : F \longrightarrow E \xrightarrow{p} B$ be a locally trivial fibration of manifolds (possibly with boundary) with F and B compact, and let γ be a flow on B. A lift of γ to E is a flow γ' on E such that $p(\gamma_t'(x)) = \gamma_t(p(x))$ for all $x \in E$ and $t \in \mathbf{R}$.

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3.1. *The Thom index.* We fix a first construction with the next immediate fact.

LEMMA 3.1. *Let S be an isolated invariant set of the flow $\gamma : B \times \mathbf{R} \longrightarrow B$ and let γ' be a lift of γ to E. Then, $S' = p^{-1}(S)$ is an isolated, invariant set of γ' and there is an induced morphism $c_{\gamma'}(S') \longrightarrow c_\gamma(S)$ depending only on ψ and γ. This morphism is natural with respect to attractor–repellor pairs.*

Proof. Let (N_1, N_0) be an index pair of S. We first notice that $S' = I(f^{-1}(\overline{N_1 - N_0}))$. Moreover, the pair $(f^{-1}(N_1), f^{-1}(N_0))$ is an index pair for S' (with respect to the flow γ', of course). Denote $N_1' = f^{-1}(N_1)$ and $N_0' = f^{-1}(N_0)$. There is a map of pairs $(N_1', N_0') \longrightarrow (N_1, N_0)$ which induces a map of quotients: $p_N : N_1'/N_0' \longrightarrow N_1/N_0$. Notice that this map induces one of connected simple systems in the following way.

First, let (K_1, K_0) be another index pair of S and let, as above, (K_1', K_0') be the index pair of S' with $K_i' = p^{-1}(K_i)$, $i = 0, 1$. There are standard, flow defined [**22**], comparison maps, $N_1/N_0 \longrightarrow K_1/K_0$ and $N_1'/N_0' \longrightarrow K_1'/K_0'$. By construction, these maps commute with the projections p_N and p_K.

Now assume that (N_1'', N_0'') is another index pair of S' (and not necessarily one obtained as a preimage of one of S) and consider also an index pair of S, (T_1, T_0). We define a map $N_1''/N_0'' \longrightarrow T_1/T_0$ as the composite $N_1''/N_0'' \longrightarrow N_1'/N_0' \xrightarrow{p_N} N_1/N_0 \longrightarrow T_1/T_0$ where the first and last maps in this composition are the standard comparison maps. The homotopy class of this map does not depend on the choice of the pair (N_1, N_0).

To prove naturality with respect to attractor–repellor sequences consider an attractor–repellor decomposition $(S; A, A^*)$ for an invariant set S and a triple $N_0 \subset N_1 \subset N_2$ such that the pair (N_1, N_0) is an index pair for A, the pair (N_2, N_1) is an index pair for A^* and (N_2, N_0) is an index pair for S. Following the notations above we denote by $'$ the respective preimages by p of these sets. It is clear that A' and $(A^*)'$ form an attractor–repellor pair. The result now follows from the commutativity of the diagram:

$$\begin{array}{ccccc} N_1'/N_0' & \longrightarrow & N_2'/N_0' & \longrightarrow & N_2'/N_1' \\ \downarrow & & \downarrow & & \downarrow \\ N_1/N_0 & \longrightarrow & N_2/N_0 & \longrightarrow & N_2/N_1 \end{array}$$

The vertical arrows are given, as above, by projection. The rows are the cofibration sequences that represent the attractor–repellor cofibration sequences of connected-simple systems. Moreover, note that the right vertical map up is induced on cofibres by the left and middle projections and this implies that, by extending the cofibrations to the right, we will continue to get commutative squares. □

For a pointed space X let, as before, CX (or $C(X)$) be the (reduced) cone on X.

Definition. With the notations in the proof of Lemma 3.1, the Thom index of S with respect to ψ and γ is the connected simple system $\overline{c}_\gamma^\psi(S)$ having as objects the spaces $(N_1/N_0) \bigcup_{p_N} C(N_1'/N_0')$ and with the morphisms given by the homotopy classes of the maps $(N_1/N_0) \bigcup_{p_N} C(N_1'/N_0') \longrightarrow (K_1/K_0) \bigcup_{p_K} C(K_1'/K_0')$ induced by the standard comparison maps.

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LEMMA 3.2. *In the context of the lemma above and assuming that* $(S; A, A^*)$ *is an attractor–repellor decomposition there is a cofibration sequence of connected simple systems:* $\overline{c}_\gamma^\psi(A) \longrightarrow \overline{c}_\gamma^\psi(S) \longrightarrow \overline{c}_\gamma^\psi(A^*) \longrightarrow \Sigma\overline{c}_\gamma^\psi(A)$.

Proof. The naturality given by the previous lemma implies the existence of the maps in the statement. These maps give a cofibration sequence because of the general fact that if in the diagram:

$$\begin{array}{ccccc} A' & \longrightarrow & B' & \longrightarrow & C' \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ A'' & \longrightarrow & B'' & \longrightarrow & C'' \end{array}$$

the top two horizontal rows as well as the columns are cofibration sequences, then the third horizontal row is also a cofibration sequence. In our case, the top row contains the spaces representing the Conley index of γ'. The spaces in the middle row are obtained from the spaces giving the Conley indices with respect to γ by pasting the reduced cylinders of the projections. The vertical maps are inclusions in the free ends of these cylinders. □

Assume that $(S; A, A^*)$ is an attractor–repellor decomposition for the flow γ. Recall that we denote by $c_\gamma(S; A, A^*)$ the attractor–repellor cofibration sequence associated with this decomposition. We will denote by $\overline{c}_\gamma^\psi(S; A, A^*)$ the cofibration sequence given by Lemma 3.2.

Remark 1. Notice that this also applies to the limit case when the total space of ψ is void. In this case, $\overline{c}_\gamma^\psi(S; A, A^*)$ is isomorphic to $c_\gamma(S; A, A^*)$ (recall that $X/\emptyset$ is the disjoint union of X and a point).

Remark 2. Parts of the lemmas above can also be deduced from results in [**18**]. For a related construction see also [**3**].

We discuss now the following particular case. Consider a Riemannian fibre bundle $\eta : \mathbf{R}^n \longrightarrow T \longrightarrow B$. As before let γ be a flow on B and let γ'' be a lift of it to T (of course, such lifts always exist). Assume that there is a fixed quadratic form $q : \mathbf{R}^n \longrightarrow \mathbf{R}$ and that there exist local charts $U_i \subset B$ such that the restriction of η to each U_i is trivial and, moreover, with respect to these trivializations the vector field associated with γ'' splits as a direct sum of the vector field associated with γ and $-\nabla q$; also assume that q is compatible with the metric.

There is a particular locally trivial fibration that is associated with this context. Consider the associated sphere bundle of η: $S(\eta) \xrightarrow{p} B$. Denote by $e(q)$ the locally trivial fibration $e(q) \longrightarrow B$ whose total space is the subset of $S(\eta)$ where q is negative or null and whose projection is the restriction of p. On $e(q)$ there is an obvious lift of γ obtained by projecting first γ'' on $S(\eta)$ and then by restriction.

PROPOSITION 3.3. *Let* S *be an isolated invariant set of* γ. *The Thom index* $\overline{c}_\gamma^{e(q)}(S)$ *is isomorphic to* $c_{\gamma''}(S)$ *and this identification is natural with respect to attractor–*

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repellor pairs. More precisely, if $(S; A, A^*)$ *is an attractor–repellor decomposition, then* $\overline{c}_\gamma^{e(q)}(S; A, A^*)$ *is isomorphic to* $c_{\gamma''}(S; A, A^*)$.

Proof. Suppose (N_1, N_0) is an index pair for S with respect to the flow γ. Let N_i' be the total space of the restriction of the unit disk bundle associated with η restricted to N_i, $i = 0, 1$. Let $e(N_i)$ be the total space of the restriction of $e(q)$ to N_i. It is easy to see that the pair $\left(N_1', N_0' \bigcup e(N_1)\right)$ is an index pair for S with respect to the flow γ''. Notice that if D^n is the unit disk in $\mathbf{R}^n$ and S^{n-1} its boundary, then the region where q is negative or null in D^n has the structure of a cone over the boundary. This implies that the region $L \subset N_1'$ where q is negative or null is homeomorphic to $N_1 \bigcup (e(N_1) \times [0, 1])$ the identification used making $x \times 0$ correspond to the point $p(x)$ for each $x \in e(N_1)$. On the other hand we have a map of pairs $\left(L, \left(N_0' \bigcap L\right) \bigcup e(N_1)\right) \subset \left(N_1', N_0' \bigcup e(N_1)\right)$. This induces a map between the respective quotients: $t : L/\left(\left(N_0' \bigcap L\right) \bigcup e(N_1)\right) \longrightarrow N_1'/\left(N_0' \bigcup e(N_1)\right)$. Because of the description of L above, the domain of this map is a space in $\overline{c}_\gamma^{e(q)}$, the target is a space in $c_{\gamma''}(S)$. This map is a homotopy equivalence as both inclusions $L \subset N_1'$ and $\left(N_0' \bigcap L\right) \bigcup e(N_1) \subset N_0' \bigcup e(N_1)$ are homotopy equivalences (the first because the domain and the target are homotopy equivalent to N_1 and the second for a similar reason).

It is easy to verify, as in the lemma above, that the map t induces an isomorphism of connected simple systems and that this morphism is natural with respect to attractor–repellor pairs. □

Remark 3. (a) The proposition implies that the Thom index and the corresponding cofibration sequence are completely determined by the homotopical data extracted from the flow γ''.

(b) It is easy to see that the statement and proof above can be extended in many ways. For example, to the case when q is a more complicated germ of an isolated but still 'reasonable' singularity (see [**7**] for a definition of such reasonable critical points).

COROLLARY 3.4. *In the context of the proposition above, if* η *is trivial in a neighborhood of* S, *then we have* $c_{\gamma''}(S) \simeq \overline{c}_\gamma^{e(q)}(S) \simeq \Sigma^k c_\gamma(S)$ *and if* $(S; A, A^*)$ *is an attractor–repellor decomposition* $c_{\gamma''}(S; A, A^*) \simeq \overline{c}_\gamma^{e(q)}(S; A, A^*) \simeq \Sigma^k c_\gamma(S; A, A^*)$ *where* k *is the index of the quadratic form* q.

Proof. Let $K = \{x \in D^n : q(x) \leq 0\}$ and let $H = K \bigcap \partial D^n$. Notice that $H \simeq S^{k-1}$ and K is contractible. With the notations in the proof of the proposition, we have $L = N_1 \times K$, $N_0' \bigcap L = N_0 \times K$, $e(N_i) = N_i \times H$. The statement is an immediate consequence of the construction of the Thom index and of the fact that the cofibre of the projection $A \times S^{k-1} \longrightarrow A$ is, in a natural way, homotopy equivalent to $\Sigma^k A \vee S^k$. With this identification we have the commutative diagram below:

$$\begin{array}{ccc} N_0 \times H & \longrightarrow & N_1 \times H \\ \downarrow & & \downarrow \\ N_0 \times K & \longrightarrow & N_1 \times K \\ \downarrow & & \downarrow \\ \Sigma^k N_0 \vee S^k & \longrightarrow & \Sigma^k N_1 \vee S^k \end{array}$$

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the columns being (homotopy) cofibration sequences. By definition the Thom index of S is represented by the cofibre of the bottom horizontal map which is $\Sigma^k(N_1/N_0)$. □

Remark 4. The above corollary is well known at the space level (in that case being a consequence of the product formula for the Conley index [**5**]). For our purposes the key consequence of this corollary is the description of the connection map of γ'' in terms of that of γ. In particular, when the fibration is trivial they are related by suspension.

3.2. *Duality.* Assume now that B is a smooth, compact manifold of dimension n that is embedded in a sphere S^{n+k}. Let $\eta : \mathbf{R}^k \longrightarrow T \longrightarrow B$ be its normal bundle. Assume a Riemannian metric fixed on the total space of η. Let $S(\eta)$ be the induced unit sphere bundle. We recall that we denote by $-\gamma$ the inverse flow of γ: $-\gamma_t(x) = \gamma_{-t}(x)$.

THEOREM 3.5. *Let $(S; A, A^*)$ be an attractor–repellor pair for γ. The cofibration sequences $\overline{c}_\gamma^{S(\eta)}(S; A, A^*)$ and $c_{-\gamma}(S; A^*, A)$ are Spanier–Whitehead duals by a duality map that depends only on the embedding of B in S^{n+k}.*

Proof. We will construct a duality between two cofibration sequences representing those of the statement. Afterwards we show that the induced duality of cofibrations of connected simple systems does not depend on the choice of the representing sequences. The proof has four steps.

Reduction to a flow on S^{n+k}. Let U be a tubular neighborhood of B. In the following we identify it to T. Let $q : B \longrightarrow \mathbf{R}$ be a smooth function which locally has the form $q(x, y) = -||x||^2$ where x is the coordinate along the fibre of η and y the coordinate along the base. Consider on U the flow γ'' induced by the sum of the canonical horizontal lift of γ and of the flow induced by $-\nabla q$. Notice that $e(q) = S(\eta)$ and $e(-q) = \emptyset$. Applying Proposition 3.3 to γ'' we obtain isomorphisms $\overline{c}_\gamma^{S(\eta)}(S; A, A^*) \simeq c_{\gamma''}(S; A, A^*)$, $c_{-\gamma''}(S; A^*, A) \simeq \overline{c}_{-\gamma}(S; A^*, A) \simeq c_{-\gamma}(S; A^*, A)$. This reduces the proof to showing that $c_{\gamma''}(S; A, A^*)$ and $c_{-\gamma''}(S; A^*, A)$ are Spanier–Whitehead duals.

Construction of some special index pairs. Extend γ'' to a flow on S^{n+k} which is constant outside a neighborhood of U. We continue to denote this flow by γ''. Let S_1 be the maximal attractor of γ'' that does not contain S, let R_1 be the complementary repellor. Let S_2 be the attractor of γ'' obtained as the union of S_1 with the set of the points of A or situated on flow lines originating in A and let R_2 be the complementary repellor. Similarly, let S_3 be given by the union of S_2 with the set of the points of A^* or situated on flow lines originating at A^*, let R_3 be the complementary repellor. For each pair (S_i, R_i) for $1 \leq i \leq 3$ we may construct by classical techniques (see, for example, [**1**]) smooth Lyapounov functions, $f_i : S^{n+k} \longrightarrow \mathbf{R}$ such that $f_i(S_i) = 1$ and $f_i(R_i) = 0$. By Sard's theorem we may find regular values a and b of respectively f_1 and f_3 such that $f_1^{-1}(a)$ and $f_3^{-1}(b)$ intersect transversely. Notice that S is the maximal invariant set inside $N_2 = \{x : f_1(x) \geq a, f_3(x) \leq b\}$. Moreover, let $U_0 = f_1^{-1}(a) \bigcap N_2$ and $V_0 = f_3^{-1}(b) \bigcap N_2$. Then $\partial N_2 = U_0 \bigcup V_0$ and the pairs (N_2, U_0) and (N_2, V_0) are index pairs for S for, respectively, the flows γ'' and $-\gamma''$. We now choose a regular value c of f_2 such that $f_2^{-1}(c)$ cuts transversely $f_3^{-1}(b)$ and such that the inequality $f_1(x) \leq a$ implies $f_2(x) < c$

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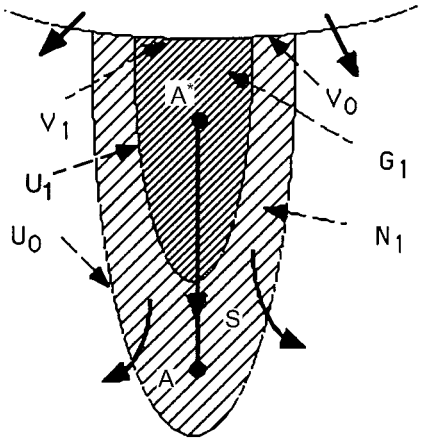


FIGURE 1.

(this is possible because $R_2 \subset R_1$). Let $U_1 = f_2^{-1}(c) \bigcap N_2$, $N_1 = f_2^{-1}((-\infty, c]) \bigcap N_2$, $L_1 = N_2 - (\mathrm{Int}(N_1) \bigcup U_0)$, $G_1 = \overline{N_2 - N_1}$ and $V_1 = \partial G_1 \bigcap V_0$, $U_2 = U_1 \bigcup (V_0 - V_1)$. With these notations the pairs (N_2, N_1), (L_1, U_2) and (G_1, U_1) are index pairs of A^* for the flow γ''; (L_1, V_0) and (G_1, V_1) are index pairs for A^* with respect to the flow $-\gamma''$; (N_1, U_0) is an index pair for A with respect to γ''; (N_2, L_1), (N_1, U_2) are index pairs for A with respect to $-\gamma''$. (See Figure 1.)

Duality. The basic argument that will be used is very classical in nature and is a variant of one that appears in [**2**]. For completeness we will formulate it as a lemma and we will indicate the idea of the proof.

LEMMA 3.6. *Assume that M is an $n + k$ manifold with boundary embedded in S^{n+k} such that ∂M admits a decomposition $\partial M = M_0 \bigcup M_1$ with M_0 and M_1 being $(n + k - 1)$-dimensional manifolds such that $M_1 \bigcap M_0 = \partial M_1 = \partial M_0$. Then M/M_0 and M/M_1 are $(n + k)$-Spanier–Whitehead duals.*

Proof of the lemma. Embed $S = S^{n+k}$ in S^{n+k+1} as the boundary of a disk $D = D^{n+k+1}$. Denote by D' the complementary disk. Consider a copy of M, M', in the exterior of D and paste it, in S^{n+k+1}, to D following M_0. (We use here and below the existence of collared neighborhoods of M, M_0, M_1 and $M_0 \bigcap M_1$.) The result of this pasting, M^*, is homotopy equivalent to M/M_0. As D' is the cone over the boundary of D it is easy to see that $M'' = S^{n+k+1} - M^*$ deforms to the union $M''' \bigcup_{M_1} D''$ with D'' a slightly smaller disk than D' and M''' a copy of M that does not touch M'. Hence, $M'' \simeq M/M_1$. (See Figure 2.) □

Notice that the argument of the lemma can also be used to prove the duality of some maps. Indeed assume that K is an $(n + k)$-dimensional submanifold of M such that ∂K decomposes, similarly to ∂M, as $K_0 \bigcup K_1$ and such that $K_0 \subset M_0$, K_1 separates and is an NDR in M, K_0 and $K_1 \bigcap \partial M$ are NDRs in ∂M. Then the maps $K/K_0 \longrightarrow M/M_0$ and $M/M_1 \longrightarrow K/K_1$ are Spanier–Whitehead duals. The reason is that, as $K_0 \subset M_0$

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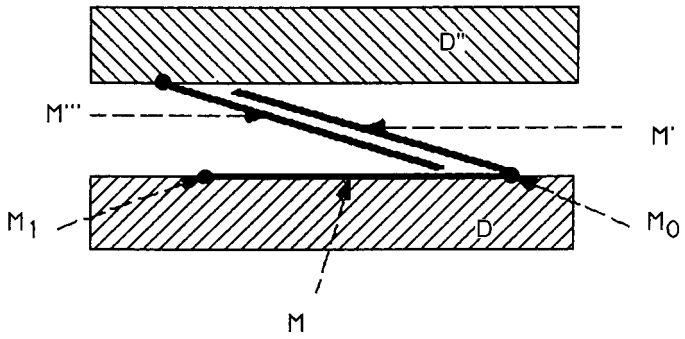


FIGURE 2.

and $K/K_1 = M/\big((M-K)\bigcup K_1\big)$, by the construction of the lemma the two maps are identified to a pair of dual inclusions.

We return now to the proof of the theorem. We intend to show that the cofibration sequences: $N_1/U_0 \xrightarrow{i} N_2/U_0 \xrightarrow{p} N_2/N_1$ and $L_1/V_0 \xrightarrow{i'} N_2/V_0 \xrightarrow{p'} N_2/L_1$ are Spanier–Whitehead duals. Consider first the pair of maps (i, p'). We have $N_2/L_1 = N_1/U_2$ and the duality is immediate by the above. Similarly, for the pair (p, i') we have $L_1/V_0 = G_1/V_1$ and $N_2/N_1 = G_1/U_1$.

Duality for connected simple systems. The proof is completed by showing that the duality induced by the two cofibration sequences described above does not depend on the choice of the Lyapounov functions or on the choice of the constants a, b and c. It is easily seen that this statement reduces to proving that if N_2', U_0' and V_0' are constructed in the same way as, respectively, N_2, U_0 and V_0 and we also have $N_2' \subset \mathrm{Int}(N_2)$, then the map $N_2'/U_0' \wedge N_2'/V_0' \xrightarrow{\mu'} S^{n+k}$ is homotopic to the composition $N_2'/U_0' \wedge N_2'/V_0' \xrightarrow{c\wedge c^*} N_2/U_0 \wedge N_2/V_0 \xrightarrow{\mu} S^{n+k}$.

Here μ and μ' are duality maps (here and above all duality maps given by the lemma are constructed with respect to the same embedding of S^{n+k} in S^{n+k+1}); the maps $c : N_2'/U_0' \longrightarrow N_2/U_0$ and $c^* : N_2'/V_0' \longrightarrow N_2/V_0$ are the standard comparison maps given, respectively, by the connected simple systems $c_{\gamma''}(S)$ and $c_{-\gamma''}(S)$.

This can be seen as follows. Let $N_2'' = \{x \in N_2 : \exists t \geq 0, \gamma''_{-t}(x) \in N_2'\}$, $U_0'' = N_2'' \bigcap U_0$ and $V_0'' = \overline{\partial N_2'' - U_0''}$. Notice that U_0' and U_0'' are homeomorhpic as well as the pairs V_0', V_0'' and N_2', N_2''. The comparison map c factors using these homeomorphisms $N_2'/U_0' \longrightarrow N_2''/U_0'' \xrightarrow{c''} N_2/U_0$ with c'' being induced by the inclusions. We apply the general duality argument in the lemma to obtain that c'' is dual to the map $N_2/V_0 \longrightarrow N_2''/V_0''$. This map is clearly homotopic to the comparison map $c''' : N_2/V_0 \longrightarrow N_2''/V_0''$.

Hence, if we denote by c_1^* the comparison map which is the homotopy inverse of c^* we obtain that c_1^* and c are duals and this implies the homotopy we look for. □

COROLLARY 3.7. *The two connection maps* $\overline{\delta} : \overline{c}_\gamma^{S(\eta)}(A^*) \longrightarrow \Sigma\overline{c}_\gamma^{S(\eta)}(A)$ *and* $\delta^* : c_{-\gamma}(A) \longrightarrow \Sigma c_{-\gamma}(A^*)$ *are Spanier–Whitehead duals. In particular, if* η *is trivial, then* δ^* *is dual to the connection map* $\delta : c_\gamma(A^*) \longrightarrow \Sigma c_\gamma(A)$.

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Remark 5. This clearly extends the relation, discovered by Franks [**13**], between the two connection maps in the case when A and A^* are two successive critical points of a Morse–Smale flow. The corollary also implies, via the Thom isomorphism, the (co)-homological results of McCord [**19**]. It also provides an extension of the duality result for succesive reasonable critical points of [**8**]. However, in the Morse–Smale case the understanding of the geometry of the problem is better because of the explicit description of the stable and unstable manifolds of the critical points. Also, more precise information can be deduced in the case of successive reasonable critical points with η non-trivial. In that case η is trivial in small enough neighborhoods of A and of A^*, hence $\Sigma^k c_\gamma(A) \simeq \overline{c}_{-\gamma}^{S(\eta)}(A)$ and similarly for A^*. One can then prove that δ and δ^* are Spanier–Whitehead duals *modulo some twisting* depending in a precise way on η.

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