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The boundary value problem for triholomorphic maps

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We introduce a natural boundary value problem for a triholomorphic map u from a compact almost hyper-Hermitian manifold M with smooth boundary ∂M into a closed hyperKähler manifold N with free boundary $u(\partial M) \subset \Gamma$ lying on some geometrically natural closed supporting submanifold $\Gamma \subset N$, called tri-isotropic submanifold. We establish partial regularity theory and energy quantization result in this boundary setting under some additional assumption on the $W^{2,1}$ norm of the weakly converging sequences.

Keywords: energy quantization; free boundary; partial regularity; tri-isotropic submanifold; triholomorphic maps

2020 Mathematics Subject Classification: 58E20; 53C26; 35F60

1. Introduction

The geometry and analysis of triholomorphic maps and their moduli space is a challenging problem which has been attracting more and more attentions in both mathematics and physics. Triholomorphic maps appear naturally in high-dimensional gauge theory [6, 10, 19]. They are closely related to the topological sigma models in 4 dimensions and hyperinstantons [1] and the classification of solutions to the 3 dimensional Beltrami equations for hydrodynamics [11]. They have been playing a very important role in understanding hyperKähler geometry [27]. To understand the sigma models and hyperintantons, it is natural to investigate boundary value problems for triholomorphic maps, see for instance, [11, Section 3, Section 6].

When the domain is a hyperKähler manifold, smooth triholomorphic maps are harmonic maps, e.g. Chen [6, Lemma 3.1] and weakly (respectively stationary) triholomorphic maps are weakly (respectively stationary) harmonic maps,

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e.g. Wang [28, Lemma 2.1]. The compactness problem and energy quantization for stationary triholomorphic maps from a closed domain was studied by Wang [28]. In the more general framework, when the domain manifold is merely almost hyper-Hermitian, the energy quantization for locally approximable triholomorphic maps from a closed domain was established by Bellettini-Tian [2].

In this paper, we introduce a natural boundary value problem for triholomorphic maps from a compact almost hyper-Hermitian manifold with smooth boundary into a closed hyperKähler manifold, which maps the boundary to some geometrically natural closed supporting submanifold in the target space, called *tri-isotropic submanifold*. Then we establish partial regularity theory and energy quantization result in this boundary value setting of triholomorphic maps.

Let (M, g, i, j, k) be a compact and almost hyper-Hermitian manifold of dimension $4m$ with smooth non-empty boundary ∂M and (N, h, I, J, K) be a closed hyperKähler manifold of dimension $4n$. Here i, j, k (respectively I, J, K) are three smooth almost complex structures compatible with the smooth Riemannian metric g (respectively h) on M (respectively N) and satisfy the quaternionic relation $ijk = -Id$ (respectively $IJK = -Id$). We denote by $\omega_i(\cdot, \cdot) := g(\cdot, i\cdot)$ the unique and non-degenerate 2-form associated with g and i on M. Similarly, we have the 2-forms ω_j, ω_k on M and the 2-forms $\Omega_I, \Omega_J, \Omega_K$ on N. Note that "hyperKähler" means that the 2-forms $\Omega_I, \Omega_J, \Omega_K$ are closed.

By Nash's embedding theorem, we embed (N, h) isometrically into some Euclidean space $\mathbb{R}^{\mathcal{K}}$ for some $\mathcal{K}$. Consider the Sobolev space

$$H^1(M, N) := \left\{ v \in H^1(M, \mathbb{R}^{\mathcal{K}}) \,:\, v(x) \in N \ \text{ for } \ \text{a.e. } x \in M \right\}.$$

To proceed, we need some definitions.

Definition 1.1. *A submanifold Γ of (N, h, I, J, K) is called a* tri-isotropic *submanifold if Γ is an isotropic submanifold of all three symplectic manifolds $(N, \Omega_I), (N, \Omega_J)$ and (N, Ω_K). In other words, Γ is a submanifold vanishing all three Kähler forms on N.*

Similarly to Lagrangian submanifolds of symplectic manifolds, a maximal tri-isotropic submanifold is called a *tri-Lagrangian* submanifold. The largest dimension of a tri-isotropic submanifold of N is n.

Now we give examples of tri-isotropic submanifolds and tri-Lagrangian submanifolds.

Example 1.2. Consider the following holomorphic symplectic form with respect to complex structure I on N

$$\Omega := \Omega_J + \sqrt{-1} \cdot \Omega_K.$$

Suppose there exists a complex Lagrangian submanifold L of the symplectic manifold (N, Ω) with real dimension $2n$. It is elementary to verify

$$\left(L, h' := h \llcorner L, I' := I \llcorner L, \Omega'_I := \Omega_I \llcorner L\right),$$

is a Kähler manifold, where "$\llcorner$" means restriction.

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To begin with, we claim that I' is a smooth complex structure on L, i.e., I preserve the tangent bundle TL. For any $q \in L$ and $X, Y \in T_qL \subset T_qN$, the definition of Lagrangian submanifolds leads to

$$h(q)(X, J(Y)) = \Omega_J(q)(X, Y) = 0 = \Omega_K(q)(X, Y) = h(q)(X, K(Y)),$$

which means $J(T_qL)$ and $K(T_qL)$ is the subspace of the orthogonal complement of T_qL with respect to the metric h, which is denoted by $T_q^{\perp}L$. Notice that J, K is linear isomorphism hence injective. Compare the dimensions between T_qL and $T_q^{\perp}L$, we obtain

$$J(T_qL) = T_q^{\perp}L = K(T_qL).$$

Then it is easy to see

$$I' = (J \circ K)\llcorner L : TL \to TL,$$

is a linear isomorphism.

Next, we show that (L, Ω'_I) is a symplectic manifold. It is sufficient to check that Ω'_I is closed and non-degenerate. Let $\mathfrak{I} : L \to N$ be the natural embedding map. Then

$$d\Omega'_I = d(\mathfrak{I}^*\Omega_I) = \mathfrak{I}^*(d\Omega_I) = 0,$$

where $\mathfrak{I}^*$ denotes the pull-back of $\mathfrak{I}$ and the closeness of Ω'_I is valid. Fix a point $q \in N$ and suppose $\Omega'_I(X, Y) = 0$ holds for all $Y \in T_qL$, we assert that $X \in T_qL$ must be zero. By the definition of Ω'_I,

$$\begin{aligned} 0 = \Omega'_I(X, Y) = \mathfrak{I}^*\Omega_I(X, Y) = \Omega_I(d\mathfrak{I}(X), d\mathfrak{I}(Y)) = h(d\mathfrak{I}(X), I \circ d\mathfrak{I}(Y)) \\ = h(d\mathfrak{I}(X), d\mathfrak{I} \circ I(Y)) = \mathfrak{I}^*h(X, I(Y)) = h'(X, I(Y)). \end{aligned}$$

Here, $I \circ d\mathfrak{I} = d\mathfrak{I} \circ I$ is deduced from the fact that I preserves TL in local coordinates directly. Besides, h' is the induced Riemannian metric hence non-degenerate. It is clear that $X = 0$ from $I(T_qL) = T_qL$. As a corollary, $\Omega'_I(\cdot, \cdot) = h'(\cdot, I'\cdot)$.

Let Γ be a Lagrangian submanifold of (L, Ω'_I). Then Γ is a tri-isotropic submanifold of N with $\dim L = n$, which means that Γ is actually a tri-Lagrangian submanifold of N.

DEFINITION 1.3. *Let (M, g, i, j, k) and (N, h, I, J, K) be as above. Suppose that $\Gamma \subset N$ is a closed tri-isotropic submanifold with $1 \leq \dim \Gamma \leq n$. A map $u \in H^1(M, N)$ is called a locally approximable triholomorphic map with respect to the tri-isotropic free boundary condition $u(\partial M) \subset \Gamma$ if u satisfies the followings:*

- ***(i)*** *Weakly triholomorphic maps equation:* $du = I \circ du \circ i + J \circ du \circ j + K \circ du \circ k$.
- ***(ii)*** *Free boundary condition:* $du(x)(\nu(x)) \perp T_{u(x)}\Gamma$ *for a.e.* $x \in \partial M$, *where $\nu(x)$ is the outward unit normal vector at $x \in \partial M$ and "$\perp$" means orthogonal.*
- ***(iii)*** *Locally approximable condition:* $d(u^*\Omega) = 0$ *for every closed 2-form Ω on N.*

REMARK 1.4. For the locally approximable condition, we actually only need it for $\Omega = \Omega_I, \Omega_J, \Omega_K$ in the present work.

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Before proceeding, we explain that Definition 1.3 is a natural definition. To be more precise, we shall show that, when M is a hyperKähler manifold, a locally approximable triholomorphic map u with respect to the tri-isotropic free boundary condition $u(\partial M) \subset \Gamma$ is a stationary harmonic map with respect to the free boundary condition $u(\partial M) \subset \Gamma$.

Let $\phi \in C^\infty(M, \mathbb{R}^{\mathcal{K}})$ satisfy $\phi(x) \in T_{u(x)}N$ for a.e. $x \in M$ and $\phi(x) \in T_{u(x)}\Gamma$ for a.e. $x \in \partial M$. Such a vectorfield is called an admissible target variation vectorfield with respect to the free boundary condition $u(\partial M) \subset \Gamma$. Let Π_N be the nearest point projection of some neighborhood of $N \subset \mathbb{R}^{\mathcal{K}}$. The variation of u is defined as

$$u_t := \Pi_N(u + t\phi).$$

According to the computations in [2] (see also [7, Proposition 2.2], [23, Proposition II.1], [28, Lemma 2.1], [4, Lemma 3.2] for more details), we have for every admissible target variation vectorfield ϕ with respect to the free boundary condition $u(\partial M) \subset \Gamma$

$$\begin{aligned} \frac{d}{dt} E(u_t)\Big|_{t=0} = \frac{1}{(2m-1)!} \Bigg(& - \int_M d\left(\omega_i^{2m-1} \wedge u^*(\iota_\phi \Pi_N^* \Omega_I)\right) \\ & + \int_M \left(d(\omega_i^{2m-1})\right) \wedge (u^*(\iota_\phi \Pi_N^* \Omega_I)) \Bigg) \\ & + (\omega_j, J) - \text{term} + (\omega_k, K) - \text{term}, \end{aligned} \tag{1.1}$$

and

$$u^*(\iota_\phi \Pi_N^* \Omega_I)(X) := u^* \Pi_N^* \Omega_I(\phi, X) = u^* \Omega_I(\phi, X),$$

for all $X \in C^\infty(M, TN)$ that are tangential along u and fulfill the same boundary condition as ϕ.

By the Stokes theorem and the definition of u and Γ in Definition 1.3,

$$\int_M d\left(\omega_i^{2m-1} \wedge u^*(\iota_\phi \Omega_I)\right) = \int_{\partial M} \omega_i^{2m-1} \wedge u^*(\iota_\phi \Omega_I) = 0.$$

The same result holds for (ω_j, J)-term and (ω_k, K)-term. Moreover, by canonical calculations for the equations of weakly harmonic maps with respect to the free boundary conditions, l.h.s. of (1.1) is equal to

$$\int_M \langle \nabla u, \nabla \phi \rangle \, \mathrm{d}Vol_g.$$

Consequently, we obtain (compare with [2, (32)])

$$\begin{aligned} \int_M \langle \nabla u, \nabla \phi \rangle \, \mathrm{d}Vol_g = & \frac{1}{(2m-1)!} \int_M \left(d(\omega_i^{2m-1})\right) \wedge (u^*(\iota_\phi \Omega_I)) \\ & + \left(d(\omega_j^{2m-1})\right) \wedge (u^*(\iota_\phi \Omega_J)) + \left(d(\omega_k^{2m-1})\right) \wedge (u^*(\iota_\phi \Omega_K)). \end{aligned} \tag{1.2}$$

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As in [2, § 3.3], (1.2) yields a perturbation of the classical harmonic map equation with respect to the free boundary condition $u(\partial M) \subset \Gamma$, namely

$$\begin{cases} \Delta_g u - A(u)(\nabla u, \nabla u) = f(x, u, \nabla u) \quad x \in M, \\ u(x) \in \Gamma, \ \dfrac{\partial u}{\partial \vec{n}}(x) \perp T_{u(x)}\Gamma \quad \text{for a.e. } x \in \partial M, \end{cases} \tag{1.3}$$

where $A(\cdot, \cdot)$ is the second fundamental form of $N \hookrightarrow \mathbb{R}^{\mathcal{K}}$, Δ_g is the Laplace-Beltrami operator on M which is defined by

$$\Delta_g := -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^l} \left(\sqrt{|g|} g^{ls} \frac{\partial}{\partial x^s} \right),$$

f is smooth with respect to the three variables and f has linear dependence on the first derivatives of u and $\vec{n}(x)$ is the outward unit normal vector at $x \in \partial M$.

Provided that M is a hyperKähler manifold, in other words, $\omega_i, \omega_j, \omega_k$ are closed (see [20, Lemma 6.8]), (1.2) reduces to

$$\int_M \langle \nabla u, \nabla \phi \rangle \, \mathrm{d}Vol_g = 0. \tag{1.4}$$

The integral identity (1.4) is exactly the one used for defining weakly harmonic maps with respect to the free boundary condition $u(\partial M) \subset \Gamma$. Hence we extend [28, Lemma 2.1] to the case of free boundary conditions thanks to the natural definition given in Definition 1.3.

Notice that the above discussions do not rely on the locally approximable condition **(iii)** in Definition 1.3. It can be seen from [2] that the locally approximable condition plays a role like "stationarity". In fact, from [23, Proposition II.2], we know that the locally approximable condition implies that weakly triholomorphic maps are stationary harmonic maps under the hyperKähler assumption on M. The same observation was made in [4, Lemma 3.3] for pseudoholomorphic maps on higher-dimensional domains. From this point of view, intuitively, we expect a map u which is as in Definition 1.3 to be a stationary harmonic map with respect to free boundary condition $u(\partial M) \subset \Gamma$ when M is a hyperKähler manifold.

Definition 1.5. ([25, Definition 1.1]) *A map $u \in H^1(M, N)$ is called a stationary harmonic map with respect to the free boundary condition $u(\partial M) \subset \Gamma$ if it is a weakly harmonic map satisfying the free boundary condition $u(\partial M) \subset \Gamma$ and*

$$\frac{d}{dt} E(u \circ \Psi_t)\Big|_{t=0} = 0, \tag{1.5}$$

holds for all 1-parameter families of C^∞-diffeomorphisms $\Psi_t : M \to M$ satisfying ***(i)*** *$\Psi_0 = Id_M$,* ***(ii)*** *$\Psi_t(\partial M) \subset \partial M$.*

The computation is similar to that of target variations (see also [2, 23]). Let X be a smooth vectorfield with compact support on M and satisfy that $X\big|_{\partial M}$ is a vectorfield on ∂M, that is $X(p) \in T_p \partial M (\subset T_p M)$ for all $p \in \partial M$. Denote $\Psi_t := Id_M + tX$ the corresponding 1-parameter families of domain variations. Then

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it is easy to see that Ψ_t are admissible domain variations vectorfields in the sense of Definition 1.5, and without loss of generality, we can assume that the admissible diffeomorphisms are of this form. We have

$$\begin{aligned}
&\frac{d}{dt}E\left(u\circ\Psi_t,M\right)\Big|_{t=0}=\frac{d}{dt}\int_M|\nabla(u\circ\Psi_t)|^2\,\mathrm{d}Vol_g\Big|_{t=0}\\
&=\frac{1}{(2m-1)!}\int_M\left(d(\omega_i^{2m-1})\right)\wedge\left(\iota_X(u^*\Omega_I)\right)+(\omega_j,J)-\text{term}+(\omega_k,K)-\text{term},\\
&\leq C(m,\omega_i,\omega_j,\omega_k)\int_M|X|\cdot|\nabla u|^2\,\mathrm{d}Vol_g,\\
&\leq C(m,g,\omega_i,\omega_j,\omega_k)\,\|X\|_{L^\infty}\int_{\mathrm{spt}(X)}|\nabla u|^2\,dx,
\end{aligned}\tag{1.6}$$

which is deduced from the following two facts:

$$\frac{d}{dt}(u\circ\Psi_t)^*\Omega_I\Big|_{t=0}=d\left(\iota_X(u^*\Omega_I)\right)+\iota_X\left(d(u^*\Omega_I)\right)=d\left(\iota_X(u^*\Omega_I)\right),\tag{1.7}$$

and

$$\int_M d\left((\omega_i^{2m-1})\wedge(\iota_X(u^*\Omega_I))\right)=\int_{\partial M}(\omega_i^{2m-1})\wedge(\iota_X(u^*\Omega_I))=0.\tag{1.8}$$

If $\omega_i,\omega_j,\omega_k$ are closed 2-forms, we immediately obtain that u is a stationary harmonic map with respect to the free boundary condition $u(\partial M)\subset\Gamma$ as what we have anticipated.

Our main result is as follows:

THEOREM 1.6 *Let*

$$u_l:(M,g,i,j,k)\to\left(N\hookrightarrow\mathbb{R}^{\mathcal{K}},h,I,J,K\right)\ \text{ for }\ l=1,2,\cdots$$

be a sequence of locally approximable triholomorphic maps with respect to the tri-isotropic free boundary condition $u_l(\partial M)\subset\Gamma$, and with a uniformly bounded $W^{2,1}$ norm, that is:

$$\|u_l\|_{W^{2,1}(M)}\leq\Upsilon$$

for some universal constant $\Upsilon>0$. If $u_l\rightharpoonup u$ weakly in $H^1(M,N)$ with a uniform energy bound $\Lambda>0$, then

$$|\nabla u_l|^2\,\mathrm{d}Vol_g\rightharpoonup|\nabla u|^2\,\mathrm{d}Vol_g+\nu\quad\text{as Radon measures on }M,$$

where u is a weakly triholomorphic map with respect to the tri-isotropic free boundary condition $u(\partial M)\subset\Gamma$ and is smooth away from a closed, $(4m-2)$-rectifiable set Σ with $\mathscr{H}^{4m-2}(\Sigma)<\infty$.

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Moreover, ν is a non-negative Radon measure on M of the form

$$\nu = \Theta(x)\mathscr{H}^{4m-2}\llcorner\Sigma. \tag{1.9}$$

Lastly, for $\mathscr{H}^{4m-2}$-a.e. $x \in \Sigma \cap \partial M$, we have the following energy identity:

$$\Theta(x) = \sum_{s=1}^{L_x} E(\phi_s) + \sum_{p=1}^{L'_x} E(\varphi_p),$$

where $\phi_s : S^2 \to N$ is a smooth, non-constant harmonic 2-sphere for $s = 1, \cdots, L_x$ and $\varphi_p : B^2 \to N$ is a smooth, non-constant harmonic 2-disc with respect to the free boundary condition $\varphi_p(\partial B^2) \subset \Gamma$ for $p = 1, \cdots, L'_x$.

The paper is organized as follows. In Section 2, we derive an almost-monotonicity formula in the boundary setting. In Section 3, we establish partial regularity theory in the boundary setting, including the ε-regularity (see Theorem 3.6) and a gradient estimate (see Proposition 3.8). In Section 4, we complete the proof of Theorem 1.6.

2. Almost-monotonicity formula

In this section, we will derive an almost-monotonicity formula in the boundary setting. In the spirit of the interior case [2, Proposition 1], we arrive at an almost-monotonicity formula for locally approximable triholomorphic maps u with tri-isotropic free boundary condition. Set

$$\begin{aligned} B_R(x_0) &= \left\{x \in \mathbb{R}^{4m} : |x - x_0| < R\right\}; \\ B_R^+(x_0) &= \{x \in \mathbb{R}^{4m} : |x - x_0| < R, x^{(4m)} \geq 0\}; \\ \partial^0 B_R^+(x_0) &= B_R^+(x_0) \cap \{x \in \mathbb{R}^{4m} : x^{(4m)} = 0\}; \end{aligned}$$

where $x^{(4m)}$ denotes the $(4m)$th coordinate of x. When $x_0 = 0$, we use B_R, B_R^+ and $\partial^0 B_R^+$ for simplicity. Firstly, we have

Proposition 2.1. *There exists $r_0 > 0$ such that for any $x \in \partial M$ and $0 < r < r_0$ the following quantity is monotonically decreasing in weak sense as $r \downarrow 0$:*

$$\frac{(1 + (4m-2)r)}{r^{4m-2}} \left[\int_{B_r^+(x)} (\alpha_1^{2m-1}) \wedge u^*\Omega_I + (\alpha_2^{2m-1}) \wedge u^*\Omega_J + (\alpha_3^{2m-1}) \wedge u^*\Omega_K \right], \tag{2.1}$$

where $\alpha_1, \alpha_2, \alpha_3$ stand for the standard Kähler forms on the m-dimensional quaternionic vector space $\mathbb{H}^m$. This immediately yields the following almost-monotonicity formula for the energy ratio:

$$\frac{1}{r^{4m-2}} \int_{B_r^+(x)} |\nabla u|^2 \, \mathrm{d}Vol_g = f(r) + O(rf(r)) \quad \textit{with } f \textit{ a non-decreasing function of } r.$$

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Proof. The proof is quite similar to that of [2, Proposition 1] except from applying some minor modifications. Note that for any $(4m-3)$-form η on ∂M, we have

$$\int_{\partial M} \eta \wedge u^*\Omega_I = \int_{\partial M} \eta \wedge u^*\Omega_J = \int_{\partial M} \eta \wedge u^*\Omega_K = 0. \tag{2.2}$$

To see (2.2), as u satisfies the free boundary condition $u(\partial M) \subset \Gamma$ and Γ is tri-isotropic, we have

$$\iota^*_{\partial M}(u^*\Omega_I) = \iota^*_{\partial M}(u^*\Omega_J) = \iota^*_{\partial M}(u^*\Omega_K) = 0 \quad \text{on } \partial M. \tag{2.3}$$

For every subset A of M, we denote $\partial^0 A := \partial A \cap \partial M$ and $\partial^+ A := \partial A \backslash \partial^0 A$. For each case in the proof of [2, Proposition 1], the calculation needs to be modified by (2.2) in only one place.

For **Case 1** in the proof of [2, Proposition 1],

$$\begin{aligned}
&\frac{1}{R^{4m-2}} \int_{B_R^+} (\alpha_1^{2m-1}) \wedge u^*\Omega_I - \frac{1}{s^{4m-2}} \int_{B_s^+} (\alpha_1^{2m-1}) \wedge u^*\Omega_I \\
&= \int_{\partial B_R^+} \frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)R^{4m-2}} \wedge u^*\Omega_I - \int_{\partial B_s^+} \frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)s^{4m-2}} \wedge u^*\Omega_I \\
&= \int_{\partial^+ B_R^+} \frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)|x|^{4m-2}} \wedge u^*\Omega_I - \int_{\partial^+ B_s^+} \frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)|x|^{4m-2}} \wedge u^*\Omega_I \\
&+ \int_{\partial^0 B_R^+} \frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)R^{4m-2}} \wedge u^*\Omega_I - \int_{\partial^0 B_s^+} \frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)s^{4m-2}} \wedge u^*\Omega_I \\
&\overset{(2.2)}{=\!=} \int_{\partial^+ B_R^+} \frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)|x|^{4m-2}} \wedge u^*\Omega_I - \int_{\partial^+ B_s^+} \frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)|x|^{4m-2}} \wedge u^*\Omega_I \\
&+ \int_{\partial^0 B_R^+} \frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)|x|^{4m-2}} \wedge u^*\Omega_I - \int_{\partial^0 B_s^+} \frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)|x|^{4m-2}} \wedge u^*\Omega_I \\
&= \int_{\partial(B_R^+ \backslash B_s^+)} \frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)|x|^{4m-2}} \wedge u^*\Omega_I \\
&= \int_{B_R^+ \backslash B_s^+} d\left(\frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)|x|^{4m-2}} \right) \wedge u^*\Omega_I = \int_{B_R^+ \backslash B_s^+} \frac{(\alpha_1^{2m-1})_{\tan}}{|x|^{4m-2}} \wedge u^*\Omega_I.
\end{aligned}$$

For **Case 2** in the proof of [2, Proposition 1],

$$\begin{aligned}
0 &\le \int_{B_R^+ \backslash B_s^+} d\left(\frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)|x|^{4m-2}} + \frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{|x|^{4m-3}} \right) \wedge u^*\Omega_I \\
&= \int_{\partial(B_R^+ \backslash B_s^+)} \left(\frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)|x|^{4m-2}} + \frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{|x|^{4m-3}} \right) \wedge u^*\Omega_I \\
&\overset{(2.2)}{=\!=} \int_{\partial B_R^+} \left(\frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)R^{4m-2}} + \frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{R^{4m-3}} \right) \wedge u^*\Omega_I
\end{aligned}$$

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$$
\begin{aligned}
&-\int_{\partial B_s^+}\left(\frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{(4m-2)s^{4m-2}}+\frac{\iota_{x\partial_x}(\alpha_1^{2m-1})}{s^{4m-3}}\right)\wedge u^*\Omega_I\\
&=\frac{(1+(4m-2)R)}{(4m-2)R^{4m-2}}\int_{\partial B_R^+}\iota_{x\partial_x}(\alpha_1^{2m-1})\wedge u^*\Omega_I\\
&-\frac{(1+(4m-2)s)}{(4m-2)s^{4m-2}}\int_{\partial B_s^+}\iota_{x\partial_x}(\alpha_1^{2m-1})\wedge u^*\Omega_I\\
&=\frac{(1+(4m-2)R)}{R^{4m-2}}\int_{B_R^+}(\alpha_1^{2m-1})\wedge u^*\Omega_I-\frac{(1+(4m-2)s)}{s^{4m-2}}\int_{B_s^+}(\alpha_1^{2m-1})\wedge u^*\Omega_I.
\end{aligned}
$$

The remaining proof follows from applying similar arguments as in the proof [2, Proposition 1], so we omit it. □

Moreover, it is straightforward to verify the following immediate consequences of almost monotonicity.

Corollary 2.2.

(i) For every $x\in\partial M$, the density of u at x defined by

$$\Theta(u,x):=\lim_{r\to 0^+}\frac{1}{r^{4m-2}}\int_{B_r^+(x)}|\nabla u|^2\,\mathrm{d}Vol_g,$$

exists.

(ii) $\Theta(u,x)$ is an upper semi-continuous function of x.

(iii) There exist $r_0>0$ and $C>0$, depending only on the geometric data, so that whenever $x\in\partial M$ and $0<r<r_0$, we have

$$\frac{1}{r^{4m-2}}\int_{B_r^+(x)}|\nabla u|^2\,\mathrm{d}Vol_g\geq\Theta(u,x)-Cr.$$

(iv) By the compactness of ∂M, there exist $R_0>0$ and $C>0$, depending only on the geometric data, so that whenever $x\in\partial M$ and $0<r_1<r_2<R_0$, we have

$$\frac{1}{r_1^{4m-2}}\int_{B_{r_1}^+(x)}|\nabla u|^2\,\mathrm{d}Vol_g\leq\frac{1+Cr_2}{r_2^{4m-2}}\int_{B_{r_2}^+(x)}|\nabla u|^2\,\mathrm{d}Vol_g.$$

The same holds for $y\in B_{R_0}^+(x)\backslash\partial^0 B_{R_0}^+(x)$ as long as $B_{r_2}^+(y)=B_{r_2}(y)\subset B_{R_0}^+(x)$.

3. Partial regularity

In this section, we will establish partial regularity theory of u in the boundary setting, inspired by Scheven's work [24, 25]. Without loss of generality (up to a dilation), we set $R_0=1$ in **(iv)** of Corollary 2.2.

Lemma 3.1. (cf. [24, Korollar 1.13], [25, Corollary 1.3]) *There exists a constant $C>0$, depending only on the geometric data, such that for all locally approximable*

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triholomorphic maps u with respect to the tri-isotropic free boundary condition $u(\partial M) \subset \Gamma$, it holds that

$$\sup_{\substack{x\in B^+_{1/2}\\ 0<r<1-|x|}} \frac{1}{r^{4m-2}} \int_{B^+_r(x)} |\nabla u|^2 \, \mathrm{d}Vol_g \le C \int_{B^+_1} |\nabla u|^2 \, \mathrm{d}Vol_g.$$

Proof. Consider $x \in \partial^0 B^+_{1/2}$ first. If $0 < r < 1/2$, it holds from **(iv)** of Corollary 2.2

$$\begin{aligned}\frac{1}{r^{4m-2}} \int_{B^+_r(x)} |\nabla u|^2 \, \mathrm{d}Vol_g &\le (1 + C/2)2^{4m-2} \int_{B^+_{1/2}(x)} |\nabla u|^2 \, \mathrm{d}Vol_g \\ &\le (1 + C/2)2^{4m-2} \int_{B^+_1} |\nabla u|^2 \, \mathrm{d}Vol_g.\end{aligned}$$

If $1/2 \le r < 1 - |x|$, then

$$\frac{1}{r^{4m-2}} \int_{B^+_r(x)} |\nabla u|^2 \, \mathrm{d}Vol_g \le 2^{4m-2} \int_{B^+_1} |\nabla u|^2 \, \mathrm{d}Vol_g.$$

Next, we assume $x = \left(x', x^{(4m)}\right)$ with $d := x^{(4m)} > 0$. If $d \le r < 1/4$, we have

$$\begin{aligned}&\frac{1}{r^{4m-2}} \int_{B^+_r(x)} |\nabla u|^2 \, \mathrm{d}Vol_g \le \frac{1}{r^{4m-2}} \int_{B^+_{r+d}((x',0))} |\nabla u|^2 \, \mathrm{d}Vol_g, \\ &\le \frac{2^{4m-2}}{(2r)^{4m-2}} \int_{B^+_{2r}((x',0))} |\nabla u|^2 \, \mathrm{d}Vol_g \\ &\le 2^{4m-2}(1 + C) \left(1 - |(x', 0)|\right)^{2-4m} \int_{B^+_{(1-|(x',0)|)}((x',0))} |\nabla u|^2 \, \mathrm{d}Vol_g \\ &\le 4^{4m-2}(1 + C) \int_{B^+_1} |\nabla u|^2 \, \mathrm{d}Vol_g.\end{aligned}$$

If $1/4 \le r < 1 - |x|$, we arrive at

$$\frac{1}{r^{4m-2}} \int_{B^+_r(x)} |\nabla u|^2 \, \mathrm{d}Vol_g \le 4^{4m-2} \int_{B^+_1} |\nabla u|^2 \, \mathrm{d}Vol_g.$$

If $r < d$, we obtain

$$\begin{aligned}&\frac{1}{r^{4m-2}} \int_{B^+_r(x)} |\nabla u|^2 \, \mathrm{d}Vol_g \\ &= \frac{1}{r^{4m-2}} \int_{B_r(x)} |\nabla u|^2 \, \mathrm{d}Vol_g \le \frac{1 + Cd}{d^{4m-2}} \int_{B_d(x)} |\nabla u|^2 \, \mathrm{d}Vol_g.\end{aligned}$$

If $1/4 \le d < 1/2$,

$$\frac{1 + Cd}{d^{4m-2}} \int_{B_d(x)} |\nabla u|^2 \, \mathrm{d}Vol_g \le (1 + C/2)4^{4m-2} \int_{B^+_1} |\nabla u|^2 \, \mathrm{d}Vol_g.$$

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If $0 < d < 1/4$,

$$\begin{aligned}\frac{1+Cd}{d^{4m-2}}\int_{B_d(x)}|\nabla u|^2\,\mathrm{d}Vol_g &\le \frac{1+C/4}{d^{4m-2}}\int_{B^+_{2d}((x',0))}|\nabla u|^2\,\mathrm{d}Vol_g\\ &\le (1+C/4)2^{4m-2}(1+C)\,(1-|(x',0)|)^{2-4m}\\ &\quad\times\int_{B^+_{(1-|(x',0)|)}((x',0))}|\nabla u|^2\,\mathrm{d}Vol_g\\ &\le (1+C/4)(1+C)4^{4m-2}\int_{B^+_1}|\nabla u|^2\,\mathrm{d}Vol_g.\end{aligned}$$

□

Lemma 3.2. (cf. [24, Satz 2.1], [25, Lemma 3.1]) *There exists a constant $\varepsilon_0 > 0$, depending only on the geometric data, such that for all locally approximable triholomorphic maps u with respect to the tri-isotropic free boundary condition $u(\partial M) \subset \Gamma$ and*

$$\sup_{\substack{x\in B^+_{1/2}\\ 0<r<1/2-|x|}}\frac{1}{r^{4m-2}}\int_{B^+_r(x)}|\nabla u|^2\,\mathrm{d}Vol_g \le \varepsilon_0, \tag{3.1}$$

it holds $\mathrm{dist}^N(u(x),\Gamma) \le C\varepsilon_0^{1/2}$ for all $x \in B^+_{1/4}$ with a constant $C > 0$, depending only on the geometric data.

Proof. From the interior regularity for locally approximable triholomorphic maps [2, § 3.3], we know that u is smooth in the interior of $B^+_{1/2}$ and the following interior gradient estimate holds (see [2, (33)]):

$$|\nabla u| \le \frac{C\sqrt{\varepsilon_0}}{\mathrm{dist}\left(x,\partial^0 B^+_{1/2}\right)}.$$

Then the proof is almost identical to that of [25, Lemma 3.1]. □

On account of Lemma 3.2, we may choose $\delta > 0$ sufficiently small such that

(i) The geodesic reflection σ across Γ defined on the δ-neighborhood of $\Gamma \subset N$

$$U_\delta := \left\{x \in N : \mathrm{dist}^N(x,\Gamma) < \delta\right\}$$

is a diffeomorphism.

(ii) If the scaled energies of u are sufficiently small, that is (3.1) holds, then $u(B^+_{1/4}) \subset U_\delta$.

The geodesic reflection σ across Γ is defined as follows: for $y \in U_\delta$, the nearest point $\pi(y) \in \Gamma$ is uniquely defined and connected to y by a unique shortest geodesic.

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Hence

$$\sigma : U_\delta \to U_\delta,\ y = \exp_{\pi(y)}(v) \longmapsto \exp_{\pi(y)}(-v),$$

where $v \in T_{\pi(y)}N$ is uniquely determined by the constraint condition $|v| = \mathrm{dist}^N(y, \Gamma)$. Hence we can extend u onto $B_{1/4}$ by σ as follows:

$$\bar{u}(x) := \begin{cases} u(x) & , x \in B^+_{1/4}, \\ \sigma(u(\rho(x))) & , x \in B^-_{1/4}, \end{cases}$$

where $\rho(x)$ is the reflection

$$\rho\left(x = \left(x^1, \cdots, x^{4m-1}, x^{4m}\right)\right) := \left(x^1, \cdots, x^{4m-1}, -x^{4m}\right),\ x \in B_{1/4}.$$

Then we extend the Riemannian metric g onto $B_{1/4}$ by ρ as follows:

$$\bar{g}(x)(X,Y) := \begin{cases} g(x)(X,Y) & , x \in B^+_{1/4}, \\ \rho^* g(x)(X,Y) & \text{for } \forall\, X, Y \in T_x B_{1/4}. \\ \quad = g(\rho(x))(d\rho(x)(X), d\rho(x)(Y)) & , x \in B^-_{1/4}, \end{cases}$$

Such extension of the Riemannian metric is also used to prove the partial regularity for stationary harmonic maps with free boundary, see [24, (2.4)].

In the local coordinates $\bar{g} = (\bar{g}_{ls})$, for $x \in B^-_{1/4}$,

$$\bar{g}_{ls}(x) = (-1)^{\delta_{lm'} + \delta_{sm'}} g_{ls}(\rho(x)), \tag{3.2}$$

where $m' := 4m$ for simplicity and

$$\delta_{ls} = \begin{cases} 1 & , \text{if } l = s, \\ 0 & , \text{if } l \neq s, \end{cases}$$

denotes the Kronecker symbol. By the nature of the coordinates system at the boundary, $g_{m'm'} = 1$ in the neighborhood of ∂M which yields $\bar{g}_{m'm'}$ is smooth. By definition, for every $x \in \partial M$ the normal vector $\nu(x)$ is orthogonal to $T_x(\partial M)$. Therefore, we have $g_{lm'} = g_{m'l} = 0$ for $l = 1, \cdots, m' - 1$ and

$$\frac{\partial g_{lm'}}{\partial x_{m'}} = \frac{\partial}{\partial x_{m'}} \langle \partial_l, \partial_{m'} \rangle = \left\langle \nabla_{\partial_{m'}} \partial_l, \partial_{m'} \right\rangle + \left\langle \partial_l, \nabla_{\partial_{m'}} \partial_{m'} \right\rangle,$$

where ∇ is the Levi-Civita connection on (M, g). By the geodesic construction of the neighborhood, $\nabla_{\partial_{m'}} \partial_{m'} = 0$ and we obtain

$$\frac{\partial g_{lm'}}{\partial x_{m'}} = \left\langle \nabla_{\partial_{m'}} \partial_l, \partial_{m'} \right\rangle.$$

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From the properties of the Levi-Civita connection, we know for any three smooth vectorfields X, Y, Z defined on M

$$\begin{aligned}\langle Z, \nabla_Y X\rangle = \frac{1}{2}\left(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle [X,Z],Y\rangle\right.\\ \left. - \langle [Y,Z],X\rangle - \langle [X,Y],Z\rangle\right).\end{aligned}$$

Hence,

$$\left\langle \nabla_{\partial_{m'}}\partial_l, \partial_{m'}\right\rangle = \frac{1}{2}\frac{\partial g_{m'm'}}{\partial x_l} = 0,$$

which means that $g_{lm'} = g_{m'l} = 0$ near the boundary such that $\bar{g}_{lm'}$ and $\bar{g}_{m'l}$ are smooth for $l = 1, \cdots, m'-1$. Moreover, $\bar{g}$ is continuous and the extension is actually the even extension.

Owing to isometrically embedding $N \hookrightarrow \mathbb{R}^{\mathcal{K}}$, we still use h to denote the metric on TN which is induced by the Euclidean metric on $\mathbb{R}^{\mathcal{K}}$. Then we define

$$\bar{h}(x) := \begin{cases} h(\bar{u}(x)) & , x \in B^+_{1/4}, \\ \sigma^* h(\bar{u}(x)) & , x \in B^-_{1/4}, \end{cases}$$

and

$$\Sigma(x) := \nabla\sigma\left(\bar{u}(x)\right).$$

According to the arguments in [25, § 3], we know $\sigma^{-1} = \sigma$, $\Sigma^{-1}(x) = \Sigma(\rho(x))$ and for $x_0 \in \partial^0 B^+_{1/4}$, we have

$$\Sigma(x_0) = \Pi_\Gamma(u(x_0)) - \Pi^\perp_\Gamma(u(x_0)),$$

where $\Pi_\Gamma(y)$ and $\Pi^\perp_\Gamma(y)$ for $y \in \Gamma$ are the orthogonal projections from T_yN on $T_y\Gamma$ and $T^\perp_y\Gamma$, respectively.

The covariant derivative of an H^1-vectorfield V tangential to N along $\bar{u}$ is defined by

$$\nabla^{\bar{h}} V(x) := \begin{cases} \nabla^N V(x) & , x \in B^+_{1/4}, \\ \Sigma(\rho(x))\nabla^N(\Sigma(x)V(x)) & , x \in B^-_{1/4}, \end{cases}$$

with the Levi-Civita connection ∇^N on (N, h).

Then we obtain the following equation for the extended map $\bar{u}$.

Lemma 3.3. (cf. [24, Lemma 2.4], [25, Lemma 3.2]) *Let $V \in H^1 \cap L^\infty(B_{1/4}, TN)$ be a vectorfield tangential to N along $\bar{u}$ with compact support in $B_{1/4}$ and $V(x) \in T_{\bar{u}(x)}N$ for a.e. $x \in B_{1/4}$. Then it holds*

$$\int_{B_{1/4}} \nabla\bar{u} \cdot^{\bar{g}}_{\bar{h}} \nabla^{\bar{h}} V \,\mathrm{d}Vol_{\bar{g}} = \int_{B_{1/4}} \bar{f}(x, \bar{u}, \nabla\bar{u}) \cdot_{\bar{h}} V \,\mathrm{d}Vol_{\bar{g}}, \tag{3.3}$$

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where '$\cdot\,\overset{\bar{g}}{\bar{h}}\,$' stands for the Hilbert-Schmidt scalar product with respect to the metric $\bar{h}$ on TN and

$$\bar{f}(x,y,\xi) := \begin{cases} f(x,y,\xi) & ,x \in B^+_{1/4}, \\ \Sigma(\rho(x)) \cdot f\left(\rho(x), \sigma(y), \Sigma(x)\xi(x)\nabla\rho(\rho(x))\right) & ,x \in B^-_{1/4}. \end{cases}$$

Here, f is the function in (1.3).

Proof. As in the proof of [25, Lemma 3.2]. We decompose the vectorfield V into two parts $V = V_e + V_a$ for $x \in B_{1/4}$

$$V_e(x) := \frac{1}{2}\left[V(x) + \Sigma(\rho(x))V(\rho(x))\right]$$
$$V_a(x) := \frac{1}{2}\left[V(x) - \Sigma(\rho(x))V(\rho(x))\right]$$

Then it holds

$$V_e(\rho(x)) = \Sigma(x)V_e(x) \text{ and } \ V_a(\rho(x)) = -\Sigma(x)V_a(x).$$

It is clear that for $x_0 \in \partial^0 B^+_{1/4}$, $V_e(x_0) \in T_{u(x_0)}\Gamma$ and $V_e|_{B^+_{1/4}}$ is an admissible target variation vectorfield for $\bar{u}|_{B^+_{1/4}}$ with respect to the free boundary condition $\bar{u}(\partial^0 B^+_{1/4}) \subset \Gamma$. Hence from (1.2) and (1.3) we have

$$\int_{B^+_{1/4}} \nabla\bar{u} \cdot\overset{\bar{g}}{\bar{h}}\, \nabla^{\bar{h}} V_e \,\mathrm{d}Vol_{\bar{g}} = \int_{B^+_{1/4}} \nabla u \cdot^g \nabla^N V_e \,\mathrm{d}Vol_g = \int_{B^+_{1/4}} f(x,u,\nabla u) \cdot_h V_e \mathrm{d}Vol_g.$$

By the definition of $\bar{h}$ and $\nabla^{\bar{h}}$, we have

$$\begin{aligned} \int_{B^-_{1/4}} \nabla\bar{u} \cdot\overset{\bar{g}}{\bar{h}}\, \nabla^{\bar{h}} V_e \,\mathrm{d}Vol_{\bar{g}} &= \int_{B^-_{1/4}} \Sigma(x)\nabla\bar{u}(x) \cdot^{\bar{g}(x)} \nabla^N(\Sigma(x)V_e(x)) \,\mathrm{d}Vol_{\bar{g}} \\ &= \int_{B^-_{1/4}} \nabla[u(\rho(x))] \cdot^{\bar{g}(x)} \nabla^N[V_e(\rho(x))] \,\mathrm{d}Vol_{\bar{g}} \\ &\overset{(3.2)}{=\!=} \int_{B^-_{1/4}} \nabla u(\rho(x)) \cdot^{g(\rho(x))} \nabla^N V_e(\rho(x)) \,\mathrm{d}Vol_g \\ &= \int_{B^+_{1/4}} \nabla u \cdot^g \nabla^N V_e \,\mathrm{d}Vol_g. \end{aligned}$$

Analogously, one can easily show that

$$\int_{B^-_{1/4}} \nabla\bar{u} \cdot\overset{\bar{g}}{\bar{h}}\, \nabla^{\bar{h}} V_a \,\mathrm{d}Vol_{\bar{g}} = -\int_{B^+_{1/4}} \nabla\bar{u} \cdot\overset{\bar{g}}{\bar{h}}\, \nabla^{\bar{h}} V_a \,\mathrm{d}Vol_{\bar{g}}.$$

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Next, we may check

$$\begin{aligned}\int_{B^-_{1/4}} \bar{f}(x,\bar{u},\nabla\bar{u})\cdot_{\bar{h}} V_e \,\mathrm{d}Vol_{\bar{g}} &= \int_{B^+_{1/4}} \bar{f}(x,\bar{u},\nabla\bar{u})\cdot_{\bar{h}} V_e \,\mathrm{d}Vol_{\bar{g}}\\ &= \int_{B^+_{1/4}} f(x,u,\nabla u)\cdot_h V_e \mathrm{d}Vol_g.\end{aligned}$$

Because of the definition of $\bar{f}$,

$$\begin{aligned}\int_{B^-_{1/4}} \bar{f}(x,\bar{u},\nabla\bar{u})\cdot_{\bar{h}} V_e \,\mathrm{d}Vol_{\bar{g}} &= \int_{B^-_{1/4}} \Sigma(x)\bar{f}(x,\bar{u},\nabla\bar{u})\cdot\Sigma(x)V_e(x)\,\mathrm{d}Vol_{\bar{g}}\\ &= \int_{B^-_{1/4}} f\left(\rho(x),u(\rho(x)),\nabla u(\rho(x))\right)\cdot V_e(\rho(x))\,\mathrm{d}Vol_{\bar{g}}\\ &= \int_{B^+_{1/4}} f(x,u,\nabla u)\cdot_h V_e \,\mathrm{d}Vol_g.\end{aligned}$$

Similarly, we can acquire

$$\int_{B^-_{1/4}} \bar{f}(x,\bar{u},\nabla\bar{u})\cdot_{\bar{h}} V_a \,\mathrm{d}Vol_{\bar{g}} = -\int_{B^+_{1/4}} f(x,u,\nabla u)\cdot_h V_a \,\mathrm{d}Vol_g.$$

Now, the statement follows by a direct check. □

REMARK 3.4. It is obvious that $\bar{f}(x,y,\xi)$ still depends linearly on ξ and

$$|\bar{f}(x,y,\xi)| \le C(1+|\xi|) \text{ for all } \ x\in B_{1/4}, y\in N, \xi\in \mathbb{R}^{m'*}\otimes TN,$$

with a constant

$$\begin{aligned}C = C\Big(&\|\omega_i\|_{C^1(M)}, ||\omega_j||_{C^1(M)}, \|\omega_k\|_{C^1(M)}, \|\Omega_I\|_{C^0(N)}, \|\Omega_J\|_{C^0(N)},\\ &\|\Omega_K\|_{C^0(N)}, \|\sigma\|_{C^1\left(\overline{U_\delta}\right)}\Big) > 0,\end{aligned}$$

depending only on the geometric data.

Furthermore, we have

LEMMA 3.5. *There exist constants $C,\varepsilon_0>0$, depending only on the geometric data, such that for all locally approximable triholomorphic maps u with respect to the tri-isotropic free boundary condition $u(\partial M)\subset\Gamma$ satisfying*

$$\int_{B^+_1} |\nabla u|^2 \,\mathrm{d}Vol_g < \varepsilon_0,$$

it holds

$$\sup_{\substack{x\in B_{1/4}\\ 0<r<1/4-|x|}} \frac{1}{r^{4m-2}}\int_{B_r(x)} |\nabla\bar{u}|^2 \,\mathrm{d}Vol_{\bar{g}} \le C\varepsilon_0.$$

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Proof. For $x \in B^-_{1/4}$, we have

$$\begin{aligned}
|\nabla \bar{u}|^2(x) &= \nabla \bar{u}(x) \cdot^{\bar{g}(x)}_{\bar{h}(x)} \nabla \bar{u}(x) \\
&= \Sigma(x)\nabla \bar{u}(x) \cdot^{\bar{g}(x)} \Sigma(x)\nabla \bar{u}(x) \\
&= \nabla\left[u(\rho(x))\right] \cdot^{\bar{g}(x)} \nabla\left[u(\rho(x))\right] \\
&= \nabla u(\rho(x)) \cdot^{g(\rho(x))} \nabla u(\rho(x)) \\
&= |\nabla u|^2(\rho(x)),
\end{aligned}$$

and $\sqrt{|\bar{g}(x)|} = \sqrt{|g(\rho(x))|}$.

Hence, for $x \in B^+_{1/4}$

$$\frac{1}{r^{4m-2}} \int_{B_r(x)} |\nabla \bar{u}|^2 \, \mathrm{d}Vol_{\bar{g}} \leq \frac{2}{r^{4m-2}} \int_{B^+_r(x)} |\nabla u|^2 \, \mathrm{d}Vol_g,$$

and, for $x \in B^-_{1/4}$

$$\frac{1}{r^{4m-2}} \int_{B_r(x)} |\nabla \bar{u}|^2 \, \mathrm{d}Vol_{\bar{g}} \leq \frac{2}{r^{4m-2}} \int_{B^+_r(\rho(x))} |\nabla u|^2 \, \mathrm{d}Vol_g.$$

By Lemma 3.1, the conclusion is valid. □

Combining all above results and [24, Satz 3.4], we have

THEOREM 3.6 *There exists a constant $\varepsilon_0 > 0$, depending only on the geometric data, such that for all locally approximable triholomorphic maps u with respect to the tri-isotropic free boundary condition $u(\partial M) \subset \Gamma$ satisfying*

$$\int_{B^+_1} |\nabla u|^2 \, \mathrm{d}Vol_g < \varepsilon_0,$$

it holds

$$\bar{u} \in C^{1,\alpha}(B_{1/16})$$

with a Hölder exponent $\alpha \in (0,1)$ and bounded norm

$$||\bar{u}||_{C^{0,\alpha}(B_{1/8})} + ||\bar{u}||_{C^{1,\alpha}(B_{1/16})} \leq C,$$

where $\alpha \in (0,1)$ and $C > 0$ only depend on the same data like ε_0.

REMARK 3.7. The Hölder continuity is obtained from [24, Satz 3.9] and Morrey's well-known lemma (see e.g. [5, 14]). Following the calculation in [24, Lemma 2.5], in local coordinates near $\bar{u}(B_\rho(x))$ with $B_\rho(x) \subset B_{1/4}$ and $0 < \rho \ll 1/4$, (3.3) is of

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the following type:

$$\int_{B_\rho(x)} \bar{g}^{sl}\partial_s\bar{u}^\beta\partial_l V^\beta\sqrt{|\bar{g}|}\,\mathrm{d}y = \int_{B_\rho(x)} F_\beta(y,\bar{u}(y),\nabla\bar{u}(y))V^\beta\,\mathrm{d}y,$$

where, for all $y \in B_\rho(x)$ and $\beta = 1,\cdots,4n$,

$$|F_\beta(y,\bar{u}(y),\nabla\bar{u}(y))| \le C\left(1+|\nabla\bar{u}|^2\right)$$

with a universal constant $C > 0$, depending only on the geometric data. Then [24, Satz 1.18] yields $C^{1,\alpha}$-regularity (see also [12, § VI]).

Assume that $u \in C^{k,\alpha}\left(B^+_{1/16}\right)$ for some integer $k \ge 1$. Note that $f(x,u,\nabla u)$ in (1.3) has smooth dependence on the three variables. Therefore, $u \in C^{k,\alpha}\left(B^+_{1/16}\right)$ implies $f \in C^{k-1,\alpha}\left(B^+_{1/16}\right)$ and $A(u)\left(\nabla u,\nabla u\right) \in C^{k-1,\alpha}\left(B^+_{1/16}\right)$. In view of (1.3), $\Delta_g u \in C^{k-1,\alpha}\left(B^+_{1/16}\right)$ in the weak sense. As stated in [13], in appropriate local coordinates for N (see [24, Satz 1.19] for detailed construction), the free boundary condition **ii** of Definition 1.3 prescribes Neumann conditions for n of the coordinate functions $u^1,\cdots,u^n$ and Dirichlet conditions for the remaining coordinate functions $u^{n+1},\cdots,u^{4m}$. Thus, the higher regularity immediately follows from the Schauder theory at the boundary (see [24, Satz 1.19]) with the following estimate, for each compact subset $B \subset B^+_{1/16}$

$$\begin{aligned}\|u\|_{C^{k+1,\alpha}(B)} \le C\Bigg(&\|u\|_{L^\infty(M)} + \|A(u)\left(\nabla u,\nabla u\right)\|_{C^{k-1,\alpha}\left(B^+_{1/16}\right)}\\ &+\|f(x,u,\nabla u)\|_{C^{k-1,\alpha}\left(B^+_{1/16}\right)}\Bigg),\end{aligned}$$

where the constant $C > 0$ only depends on the geometric data.

From the blow-up arguments along the lines of [21, Theorem 2.2] (cf. [15, §3 Proposition], [8, Proposition 3]), we have

PROPOSITION 3.8. *There exists $\varepsilon_0 > 0$, depending only on the geometric data, such that for every $0 < r < 1$ and all locally approximable triholomorphic maps u with respect to the tri-isotropic free boundary condition $u(\partial M) \subset \Gamma$ satisfying*

$$\frac{1}{r^{4m-2}}\int_{B^+_r} |\nabla u|^2\,\mathrm{d}Vol_g < \varepsilon_0,$$

we have $\bar{u}$ is smooth on $B_{r/2}$ and the following gradient estimate is valid:

$$\sup_{x\in B_{r/2}} |\nabla\bar{u}(x)| \le \frac{C\sqrt{\varepsilon_0}}{r},$$

where $C > 0$ is a constant depending only on the geometric data.

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Proof. Note that the proof of [21, Theorem 2.2] does not rely on the full information of energy monotonicity ([21, (2.4)]), but only the control of the scaled energies ([21, (2.5)]). Hence, by Lemma 3.5 and Theorem 3.6, this proposition can be proved in the same way as the interior case, that is as in [2, (33)]. □

Let the singular set $\mathrm{sing}(u)$ of u be defined as follows:

$$\mathrm{sing}(u) := M \backslash \{x \in M : u \text{is smooth in a neighborhood of } x\}.$$

It is a standard covering argument (see e.g. [3, VII], [26, Corollary 2.7]) to show that the ε-regularity and the almost energy monotonicity reveal $\mathrm{sing}(u)$ is a $\mathscr{H}^{4m-2}$-null set (compare with [25, Theorem 2.2]).

PROPOSITION 3.9. *With the same assumptions as in Proposition 3.8, we have that* $\mathrm{sing}(u)$ *is a closed set and* $\mathscr{H}^{4m-2}(\mathrm{sing}(u)) = 0$.

4. Energy quantization

In this section, we will prove Theorem 1.6. Let

$$u_l : (M, g, i, j, k) \to \left(N \hookrightarrow \mathbb{R}^{\mathcal{K}}, h, I, J, K\right) \text{ for } \ l = 1, 2, \cdots,$$

be a sequence of locally approximable triholomorphic maps with respect to the tri-isotropic free boundary condition $u_l(\partial M) \subset \Gamma$, and with a uniform $W^{2,1}$ norm bound $\Upsilon > 0$ such that $u_l \rightharpoonup u$ weakly in $H^1(M, N)$ with a uniform energy bound $\Lambda > 0$ for some $u \in H^1(M, N)$ with the boundary condition $u(\partial M) \subset \Gamma$. Hence, as Radon measures on M,

$$|\nabla u_l|^2 \mathrm{d}Vol_g \rightharpoonup |\nabla u|^2 \mathrm{d}Vol_g + \nu' := \mu,$$

where the defect measure $\nu' \geq 0$ is a non-negative Radon measure on M.

As in [2], we define the blow-up set at the boundary ∂M:

$$\begin{aligned}
\Sigma := \bigcap_{0<r<r_0} \Bigg\{ & x \in \partial M \text{such that} \\
& \liminf_{l\to\infty} \frac{(1+(4m-2)r)}{r^{4m-2}} \left[\int_{B_r^+(x)} (\alpha_1^{2m-1}) \wedge u^*\Omega_I + (\alpha_2^{2m-1}) \wedge u^*\Omega_J \right. \\
& \left. + (\alpha_3^{2m-1}) \wedge u^*\Omega_K \right] \geq \varepsilon_0 \Bigg\} \\
= \bigcap_{0<r<r_0} & \left\{ x \in \partial M : \liminf_{l\to\infty} \frac{1}{r^{4m-2}} \int_{B_r^+(x)} |\nabla u_l|^2 \, \mathrm{d}Vol_g \geq \varepsilon_0 - Cr \right\} \\
= & \left\{ x \in \partial M : \lim_{r\downarrow 0} r^{2-4m} \mu\left(B_r^+(x)\right) =: \Theta_\mu^+(x) \geq \varepsilon_0 \right\},
\end{aligned}$$

where $C > 0, \varepsilon_0 > 0$ and $r_0 > 0$ are constants depending only on the geometric data as in **(iii)** of Corollary 2.2, Theorem 3.6 and Proposition 2.1 respectively.

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Referring to the careful and detailed blow-up analysis in [2, § 4] (cf. also the proof of [25, Lemma 5.2] in the boundary case) and exploiting the almost energy monotonicity result, namely Proposition 2.1, we immediately conclude that

PROPOSITION 4.1. *Let* $\{u_l\}_{l\in\mathbb{N}^*}$, *and* u *and* Σ *be as above and* $\nu := \nu' \llcorner \partial M$. *Then*

(a) $u_l \to u$ *locally smoothly away from* Σ.
(b) Σ *is closed in* ∂M *and* Σ *has a locally finite* $\mathscr{H}^{4m-2}$*-measure.*
(c) $spt(\nu) \subset \Sigma$ *and* $\Sigma \backslash spt(\nu) \subset sing(u)$.
(d) $\nu(x) = \Theta(x)\mathscr{H}^{4m-2}\llcorner\Sigma$, *where* $0 < \varepsilon_1 \le \Theta(x) \le C < +\infty$ *for* $\mathscr{H}^{4m-2}$*-a.e.* $x \in \Sigma$ *with some constants* ε_1 *and* C *which depend only on the geometric data.*
(e) Define

$$\Theta^+_\nu(x) := \lim_{r\downarrow 0} r^{2-4m}\nu\left(B^+_r(x)\right),$$

then $0 < \varepsilon_0 \le \Theta^+_\nu(x) \le C < +\infty$ *for* ν*-a.e.* $x \in M$ *with some constants* ε_0 *and* C *which depend only on the geometric data. Thus,* ν *is* $\mathscr{H}^{4m-2}$*-rectifiable and so is* Σ.

REMARK 4.2. The constant ε_0 in **(e)** can be actually chosen as in Theorem 3.6 and the constant ε_1 in **(d)** is $\varepsilon_0/\omega_{4m-2}$, where ω_{4m-2} denotes the Lebesgue measure in $\mathbb{R}^{4m-2}$ of B^{4m-2}. The proof of Proposition 4.1 is quite similar to [2, § 4], so we omit it.

By the property of rectifiable sets, for $\mathscr{H}^{4m-2}$-a.e. $x \in \Sigma$, Σ has a unique classical tangent space P_x. For ease of notations, let P_x be the $(4m-2)$-dimensional plane

$$\mathbb{R}^{4m-2} \times \{(0,0)\} \subset \mathbb{R}^{4m}$$

and x be the origin in suitable coordinates. We choose such $x \in \Sigma$. As in [25, Lemma 3.1] and up to a diagonal argument, we can find a sequence of locally approximable triholomorphic maps $\{\widetilde{u}_l\}_{l\in\mathbb{N}^*}$ defined on $B^+_{S_l}$ for a sequence $0 < S_l \to \infty$ with respect to the tri-isotropic free boundary condition $\widetilde{u}_l(\partial M) \subset \Gamma$ and with a uniform energy bound such that

$$|\nabla \widetilde{u}_l|^2_{g_l} \mathrm{d}Vol_{g_l} \rightharpoonup \Theta(x)\mathscr{H}^{4m-2}\llcorner\left(\mathbb{R}^{4m-2} \times \{(0,0)\}\right) \text{ as Radon measures on } \mathbb{R}^{4m},$$

where $g_l(y) := g(x + s_l y)$ for some $s_l > 0$ with $s_l \to 0$ as $l \to \infty$. For simplicity, we will drop the tilde $\widetilde{\cdot}$ for the rest of arguments.

To avoid confusion in notations, we use the superscript to denote the dimension of balls. For example, $B^p_R(x)$ is the p-dimensional ball in $\mathbb{R}^p$ with radius $R > 0$ and center $x \in \mathbb{R}^p$. If $x = 0$, we also use B^p_R for simplicity. To begin, we restrict each map u_l to

$$Q^+_1 := B^{4m-2}_1 \times B^{2+}_1 = \left\{(x_1, \cdots, x_{4m}) \in \mathbb{R}^{4m} : |(x_1, \cdots, x_{4m-2})| < 1, |(x_{4m-1}, x_{4m})| < 1, x_{4m} \ge 0\right\}.$$

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For $R>0$, let $Q_R := B_R^{4m-2}\times B_R^2$ and $Q_R^+ := B_R^{4m-2}\times B_R^{2+}$. Then we have

$$u_l \rightharpoonup const, \text{ weakly in } W^{1,2}\left(Q_1^+, N\right),$$

the convergence is C^1 locally in the complement of $B_1^{4m-2}\times\{(0,0)\}$ and

$$|\nabla u_l|^2_{g_l} dVol_{g_l} \rightharpoonup \Theta(x)\mathscr{H}^{4m-2}\llcorner\left(B_1^{4m-2}\times\{(0,0)\}\right) \text{ as Radon measures on } Q_1^+.$$

From the arguments in [17, §2] (see [24, Behauptung 1] for details), we have

$$\int_{Q_1^+}\sum_{s=1}^{4m-2}\left|\frac{\partial u_l}{\partial x_s}\right|^2(X)\,dX\to 0, \tag{4.1}$$

as $l\to\infty$. This conclusion remains valid for our case thanks to the almost energy monotonicity, namely Lemma 2.1.

By convention, let $X_1=(x_1,\cdots,x_{4m-2}), X_2=(x_{4m-1},x_{4m})$, and

$$f_l(X_1) := \int_{B_{1/2}^{2+}}\sum_{s=1}^{4m-2}\left|\frac{\partial u_l}{\partial x_s}\right|^2(X_1,X_2)\,dX_2,\quad X_1\in B_{1/2}^{4m-2}. \tag{4.2}$$

Due to (4.1) and Fubini's theorem, it holds that $f_l(X_1)\to 0$ in $L^1\left(B_{1/2}^{4m-2}\right)$. By Proposition 3.9, we obtain that for $\mathscr{H}^{4m-2}$-a.e. $X_1\in B_{1/2}^{4m-2}$,

$$u_l \text{ is smooth near all the points of } (X_1,X_2)\in Q_{1/2}^+.$$

From the weakly L^1-estimation for the Hardy-Littlewood maximal function of f_l, there is a sequence of points $\{X_1^l\}_{l\in\mathbb{N}^*}\in B_{1/2}^{4m-2}$ such that

$$u_l \text{is smooth near all the points } (X_1^l,X_2)\in Q_{1/2}^+, \tag{4.3}$$

and

$$\sup_{0<r\le 1/2} r^{2-4m}\int_{B_r^{4m-2}(X_1^l)} f_l(X_1)\,dX_1\to 0 \text{ as } l\to\infty. \tag{4.4}$$

Moreover, for all l sufficiently large, we may find (see [24, p.55-p.56]) $\delta_l\in(0,1/2)$ satisfying $\delta_l\to 0$ as $l\to\infty$ and $X_2^l\to(0,0)$ as $l\to\infty$ such that X_2^l achieve the maximum value

$$\max_{X_2\in\overline{B_{1/2}^{2+}}}\delta_l^{2-4m}\int_{B_{\delta_l}^{4m-2}(X_1^l)\times B_{\delta_l}^{2+}(X_2)}|\nabla u_l|^2_{g_l}(X_1,X_2)\,dVol_{g_l}=\frac{\varepsilon_0}{8\cdot 2^{4m}}. \tag{4.5}$$

Let $X_l=\left(X_1^l,X_2^l\right)$,

$$\mathbb{R}_l^{4m}=\left\{(x_1,\cdots,x_{4m})\in\mathbb{R}^{4m}:\, x_{4m}\ge -\frac{X_l^{(4m)}}{\delta_l}\right\}, \tag{4.6}$$

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where $X_l^{(4m)}$ denotes the $(4m)$th coordinate of X_l and $d_l := X_l^{(4m)} > 0$. Then, we define scaled versions of the u_l

$$v_l(y) := u_l\left(X_l + \delta_l y\right). \tag{4.7}$$

From the discussions in [17, p.818] (see also [24, Behauptung 4] for the free boundary case and compared with [24, Behauptung 4.a]), for every compact set $\mathscr{K} \subset \mathbb{R}^{4m}$

$$\sup_{l \geq l_0(\mathscr{K})} \|v_l\|_{C^{1,\alpha}(\mathscr{K} \cap \mathbb{R}_l^{4m})} \leq C = C(\mathscr{K}, m, \Gamma, N). \tag{4.8}$$

It is worth mention that (4.8) uses more information of the almost stationarity (1.6) than the almost energy monotonicity (Proposition 2.1). Then by the Arzelà-Ascoli theorem, we have (For detailed proof, see e.g. [24, p.58-p.59], [25, p.151-p.153])

If $\overline{\lim\limits_{l\to\infty}} \frac{d_l}{\delta_l} = a < \infty$, then there exists a $v_a \in C^{1,\alpha}\left(\mathbb{R}_a^{4m+}, N\right)$

$$v_l \to v_a \text{ in } C^1_{loc}\left(\mathbb{R}_a^{4m+}, N\right) \text{ as } l \to \infty, \tag{4.9}$$

where

$$\mathbb{R}_a^{4m+} := \left\{(x_1, \cdots, x_{4m}) \in \mathbb{R}^{4m} : x_{4m} \geq -a\right\}. \tag{4.10}$$

Then v_a can be extended to a smooth, non-constant harmonic 2-disc with respect to the free boundary condition $\nu_a\left(\partial B_1^2\right) \subset \Gamma$.

If $\overline{\lim\limits_{l\to\infty}} \frac{d_l}{\delta_l} \to \infty$, then there exists a $v \in C^{1,\alpha}\left(\mathbb{R}^{4m}, N\right)$ such that

$$v_l \to v \text{ in } C^1_{loc}\left(\mathbb{R}^{4m}, N\right) \text{ as } l \to \infty, \tag{4.11}$$

and we obtain a smooth, non-constant harmonic 2-sphere $\nu : S^2 \to N$.

Similarly, let

$$\widetilde{v}_l(X_2) := u_l\left(X_1^l, X_2^l + \delta_l X_2\right). \tag{4.12}$$

As in (4.9) and (4.11), one has

$$\widetilde{v}_l \to \varphi_1 \text{ strongly in } C^1_{loc}\left(\mathbb{R}_a^{2+}, N\right) \text{ as } l \to \infty \text{ if } \overline{\lim_{l\to\infty}} \frac{d_l}{\delta_l} = a < \infty, \tag{4.13}$$

where

$$\varphi_1 = v_a|_{\{\mathbf{0}\}\times\mathbb{R}_a^{2+}},$$

can be extended to a non-trivial harmonic 2-disc with respect to the free boundary condition $\varphi_1\left(\partial B_1^2\right) \subset \Gamma$ (still denoted by φ_1) and

$$\widetilde{v}_l \to \phi_1 \text{ strongly in } C^1_{loc}\left(\mathbb{R}^2, N\right) \text{ as } l \to \infty \text{ if } \overline{\lim_{l\to\infty}} \frac{d_l}{\delta_l} \to \infty, \tag{4.14}$$

where

$$\phi_1 = v|_{\{\mathbf{0}\}\times\mathbb{R}^2}$$

can be extended to a non-trivial harmonic 2-sphere (still denoted by ϕ_1). As a result, we get the first bubble.

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By the standard induction argument in as in [9] (see also [18, § 4]), we may assume that there is only one bubble without loss of generality.

To estimate the Lorentz norm $L^{(2,1)}$, we need the following embedding result.

LEMMA 4.3. ([14, Theorem 3.3.10]) *For each integer $q \geq 2$, the space $W^{1,1}(\mathbb{R}^q)$ is continuously embedded in $L^{\left(\frac{q}{q-1},1\right)}(\mathbb{R}^q)$.*

From the uniformly bounded $W^{2,1}$ norm assumption, we know

$$\|\nabla u_l\|_{W^{1,1}\left(Q^+_{2/3}\right)} \leq \|u_l\|_{W^{2,1}\left(Q^+_{2/3}\right)} \leq \Upsilon.$$

By Fubini's theorem, we may assume for all $X_1^l \in E_l \subset B^{4m-2}_{2/3}, l = 1, 2, \cdots$, and for any $l \in \mathbb{N}^*$ ($\mathscr{L}^{4m-2}$ stands for the $(4m-2)$-dimensional Lebesgue measure) that

$$\mathscr{L}^{4m-2}(E_l) \geq 0.99\mathscr{L}^{4m-2}\left(B^{4m-2}_{2/3}\right),$$

such that

$$\|\nabla u_l\|_{W^{1,1}\left(B^{2+}_{2/3}\right)}\left(X_1^l, \cdot\right) \leq C\|\nabla u_l\|_{W^{1,1}\left(Q^+_{2/3}\right)},$$

where $C > 0$ is a universal constant.

Finally, according to the Lemma 4.3, we arrive at

$$\|\nabla u_l\|_{L^{(2,1)}\left(B^{2+}_{2/3}\right)}\left(X_1^l, \cdot\right) \leq C^*\|\nabla u_l\|_{W^{1,1}\left(B^{2+}_{2/3}\right)}\left(X_1^l, \cdot\right) \leq CC^*\Upsilon, \tag{4.15}$$

where $C^* > 0$ is a uniform constant. Moreover, we can find a subset $F_l \subset B^{4m-2}_{1/2}$ for each $l \in \mathbb{N}^*$ such that

$$\mathscr{L}^{4m-2}(F_l) \geq 0.99\mathscr{L}^{4m-2}\left(B^{4m-2}_{1/2}\right)$$

and for all $X_1^l \in F_l$, both (4.3) and (4.4) hold. Next, we would only work on slices

$$\{X_1 = X_1^l : X_1^l \in E_l \cap F_l\},$$

so that (4.3), (4.4) and (4.15) are valid and define

$$\widetilde{u}_l(X_2) := u_l(X_1^l, X_2).$$

As in [17, § 3], [18, (2.9)] and [22, Lemma II.1.], we can apply Allard's strong constancy lemma [29] to directly conclude that for $\mathscr{L}^{4m-2}$-a.e. $X_1^l \in E_l \cap F_l$,

$$\lim_{l\to\infty} E\left(\widetilde{u}_l, B^{2+}_{1/2}\right) := \lim_{l\to\infty}\int_{B^{2+}_{1/2}} |\nabla u_l|^2\left(X_1^l, X_2\right)\,\mathrm{d}X_2 = \Theta(x), \tag{4}$$

which means that the energy identity is equal to

$$\lim_{l\to\infty} E\left(\widetilde{u}_l, B^{2+}_{1/2}\right) = \sum_{s=1}^{L_x} E(\phi_s) + \sum_{p=1}^{L'_x} E(\varphi_p), \tag{4.16}$$

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where $\phi_s : S^2 \to N$ is a smooth, non-constant harmonic 2-sphere for $s = 1, \cdots, L_x$ and $\varphi_p : B_1^2 \to N$ is a smooth, non-constant harmonic 2-disc with respect to the free boundary condition $\varphi_p\left(\partial B_1^2\right) \subset \Gamma$ for $p = 1, \cdots, L_x'$.

According to the "one bubble" assumption and the construction of the first bubble, the energy identity (4.16) is reduced to

$$\lim_{R\to\infty}\lim_{l\to\infty} E\left(\widetilde{u}_l, A^+(R,l)\right) = 0, \tag{4.17}$$

here $A^+(R,l) := B^{2+}_{1/2}\left(X_2^l\right) \Big\backslash B^{2+}_{\delta_l R}\left(X_2^l\right)$.

Claim: For any $\varepsilon > 0$, there are sufficiently large R and L such that, for all $l \geq L$, we have

$$r^{2-4m}\int_{B_r^{4m-2}(X_1^l)}\int_{B^{2+}_{2r}(X_2^l)\backslash B^{2+}_r(X_2^l)} |\nabla u_l|^2 \,\mathrm{d}X_1\mathrm{d}X_2 < \varepsilon, \tag{4.18}$$

for every $r \in [\delta_l R, 1/4]$.

We prove the **Claim** by contradiction. If the **Claim** is false, then for any $R > 0$, we would find a sequence of $\{r_l\}_{l\in\mathbb{N}^*}$ such that

$$r_l^{2-4m}\int_{B_{r_l}^{4m-2}(X_1^l)}\int_{B^{2+}_{2r_l}(X_2^l)\backslash B^{2+}_{r_l}(X_2^l)} |\nabla u_l|^2 \,\mathrm{d}X_1\mathrm{d}X_2 \geq \varepsilon_1 > 0. \tag{4.19}$$

The following lemma can be easily derived from the fact that the energy can not accumulate close to the boundaries of the domain.

LEMMA 4.4. *For R sufficiently large,*

$$r_l \to 0 \quad \textit{and} \quad \frac{r_l}{\delta_l} \to \infty \ \textit{as} \ \ l \to \infty.$$

Now, we face a dichotomy, that is either (4.13) or (4.14) holds.

Case I. $\lim\limits_{l\to\infty}\frac{d_l}{\delta_l} = a < \infty$ and (4.13) holds.

Thus, by Lemma 4.4, we have

$$\lim_{l\to\infty}\frac{d_l}{r_l} = 0.$$

We set

$$\omega_l(y) := u_l\left(X_l + r_l y\right)$$

and (4.19) becomes

$$\int_{B_1^{4m-2}}\int_{\left(B_2^2\backslash B_1^2\right)\cap\left\{y^{(4m)}\geq -\frac{d_l}{r_l}\right\}} |\nabla\omega_l|^2 \mathrm{d}Y_1\mathrm{d}Y_2 \geq \varepsilon_2 > 0. \tag{4.20}$$

Besides, from (4.4), for all $0 < \rho \leq \rho_l = \frac{1}{2r_l}$ we have

$$\rho^{2-4m}\int_{B_\rho^{4m-2}}\int_{B^{2+}_{\rho_l}} |\nabla_{Y_1}\omega_l|^2 \,\mathrm{d}Y_1\mathrm{d}Y_2 \to 0 \ \text{as} \ \ l \to \infty. \tag{4.21}$$

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By the energy bound on $\{u_l\}$, we assume $\omega_l \rightharpoonup \omega$ weakly in $H^1_{loc}\left(\mathbb{R}^{4m+}, N\right)$. Now, from (4.21) and Proposition 4.1, it is easy to see that ω is a harmonic map with finite energy on $\mathbb{R}^{2+}$ and with respect to the free boundary condition $\omega\left(\partial\mathbb{R}^{2+}\right) \subset \Gamma$. In fact, ω is a triholomorphic map from $\mathbb{H}^{m+}$, the m-dimensional upper half quaternionic space (a hyperKähler manifold with boundary equipped with the Euclidean metric and the standard hyperKähler structure), to N with respect to the free boundary condition $\omega(\mathbb{H}^{m+}) \subset \Gamma$ and independent of the first $(4m-2)$ variables. Then by the conformal invariance of harmonic maps in dimension 2 and the removability of a local singularity at the free boundary (see e.g. [16, Theorem 3.6]), we obtain a smooth harmonic 2-disc with respect to the free boundary condition.

Suppose $\omega_l \to \omega$ strongly on $B_1^{4m-2} \times \left(B_2^{2+} \backslash B_1^{2+}\right)$, then (4.20) indicates that ω is another non-trivial bubble. This is a contradiction to the "one bubble" assumption. If the convergence of ω_l to ω is not strong on $B_1^{4m-2} \times \left(B_2^{2+} \backslash B_1^{2+}\right)$. Hence we can again employ the blow-up arguments on the construction of the first bubble in [2] for the interior case or (4.9) and (4.11) for the boundary case to obtain another non-trivial bubble, which leads to a contradiction. Thus, the **Claim** is true for the **Case I**.

Recall that

$$X_l = \left(X_1^l, X_2^l\right) = \left(X_l^{(1)}, X_l^{(2)}, \cdots, X_l^{(4m-1)}, X_l^{(4m)}\right).$$

We denote that $\widetilde{X}_2^l := \left(X_l^{(4m-1)}, 0\right)$ and decompose the neck domain $B^{2+}_{1/2}\left(X_2^l\right) \Big\backslash B^{2+}_{\delta_l R}\left(X_2^l\right)$ as follows when l and R are sufficiently large

$$\begin{aligned} B^{2+}_{1/2}\left(X_2^l\right) \Big\backslash B^{2+}_{\delta_l R}\left(X_2^l\right) &= B^{2+}_{1/2}\left(X_2^l\right) \Big\backslash B^{2+}_{1/4}\left(\widetilde{X}_2^l\right) \cup B^{2+}_{1/4}\left(\widetilde{X}_2^l\right) \Big\backslash B^{2+}_{2\delta_l R}\left(\widetilde{X}_2^l\right) \\ &\quad \cup B^{2+}_{2\delta_l R}\left(\widetilde{X}_2^l\right) \Big\backslash B^{2+}_{\delta_l R}\left(X_2^l\right), \\ &:= D_1 \cup D_2 \cup D_3. \end{aligned}$$

It is easy to see that

$$D_1 \subset B^{2+}_{1/2}\left(X_2^l\right) \Big\backslash B^{2+}_{1/8}\left(X_2^l\right) \qquad \text{and} \qquad D_3 \subset B^{2+}_{4\delta_l R}\left(X_2^l\right) \Big\backslash B^{2+}_{\delta_l R}\left(X_2^l\right),$$

and for any $2\delta_l R \le r \le 1/8$, it holds that

$$B^{2+}_{2r}\left(\widetilde{X}_2^l\right) \Big\backslash B^{2+}_{r}\left(\widetilde{X}_2^l\right) \subset B^{2+}_{4r}\left(X_2^l\right) \Big\backslash B^{2+}_{r/2}\left(X_2^l\right).$$

Then in view of Lemma 4.4, **Claim** (4.18) implies that (compared with [18, (2.13)], [2, (47)])

$$E\left(\widetilde{u}_l, D_1\right) + E\left(\widetilde{u}_l, D_3\right) < \varepsilon \text{for all large } R \text{and } l, \tag{4.22}$$

and

$$r^{2-4m} \int_{B_r^{4m-2}\left(X_1^l\right)} \int_{B^{2+}_{2r}\left(\widetilde{X}_2^l\right) \backslash B^{2+}_{r}\left(\widetilde{X}_2^l\right)} |\nabla u_l|^2 \, \mathrm{d}X_1 \mathrm{d}X_2 < \varepsilon, \tag{4.23}$$

for all $r \in [2\delta_l R, 1/8]$.

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Hence by (4.22), the energy identity (4.17) is equal to

$$\lim_{R\to\infty}\lim_{l\to\infty} E\left(\widetilde{u}_l, D_2\right) = 0. \tag{4.24}$$

According to the small energy regularity theory and from (4.23), we obtain

$$\left|X_2 - \widetilde{X}_2^l\right| \left|\nabla_{X_2}\widetilde{u}_l(X_2)\right| \leq C\sqrt{\varepsilon}, \tag{4.25}$$

for all $X_2 \in D_2, l \geq L, R \geq R_0$.

To be more precise, as in the interior case [18], if we define

$$W_l(X_1, t, \theta) := u_l\left(X_1 + X_1^l, \mathrm{e}^{-t}, \theta\right)$$

with $X_2 = (r = \mathrm{e}^{-t}, \theta)$ in polar coordinates on D_2,

$$\nabla W_l := \left(\mathrm{e}^{-t}\nabla_{X_1} W_l, \frac{\partial W_l}{\partial t}, \frac{\partial W_l}{\partial \theta}\right)$$

and $B_l := [3\log 2, |\log(2\delta_l R)|] \times [0, \pi]$, then (4.23) becomes

$$\mathrm{e}^{(4m-2)t}\int_{|X_1|\leq \mathrm{e}^{-t}}\int_r^{r+1}\int_0^{\pi} \left|\nabla W_l\right|^2 (X_1, t, \theta)\, \mathrm{d}\theta\, \mathrm{d}t\, \mathrm{d}X_1 < \varepsilon$$

for all $r \in [3\log 2, |\log(2\delta_l R)| - 1]$, which implies that

$$\left|\nabla W_l\right|^2 = \mathrm{e}^{-2t}\left|\nabla_{X_1} W_l\right|^2 + \left|\frac{\partial W_l}{\partial t}\right|^2 + \left|\frac{\partial W_l}{\partial \theta}\right|^2 \leq C\varepsilon$$

on $\{\mathbf{0}\} \times B_l$ for all large l, R. Hence (4.25) is true.

Then, by the extension argument in [18, (3.9)] and a direct computation, we get

$$\left\|\nabla_{X_2}\widetilde{u}_l\right\|_{L^{(2,\infty)}(D_2)} \leq C_1\sqrt{\varepsilon}, \tag{4.26}$$

where the constant $C_1 > 0$ is independent of the left-hand side of (4.26).

Meanwhile, (4.15) yields that

$$\left\|\nabla_{X_2}\widetilde{u}_l\right\|_{L^{(2,1)}(D_2)} \leq C_2\Upsilon. \tag{4.27}$$

Combine (4.26), (4.27) and the $L^{(2,1)} - L^{(2,\infty)}$ duality, we see that

$$\left\|\nabla_{X_2}\widetilde{u}_l\right\|_{L^2(D_2)} \leq \left\|\nabla_{X_2}\widetilde{u}_l\right\|^{1/2}_{L^{(2,1)}(D_2)} \left\|\nabla_{X_2}\widetilde{u}_l\right\|^{1/2}_{L^{(2,\infty)}(D_2)} \leq C(\Upsilon, C_1, C_2)\varepsilon^{1/4},$$

which completes the proof of the energy identity for **Case I**.

Case II. $\lim\limits_{l\to\infty} \frac{d_l}{\delta_l} \to \infty$ and (4.14) holds.

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The proof is quite similar to that for **Case I**. Assuming the **Claim** is false. Then, by Lemma 4.4 we may find $r_l \to 0$ with $\lim_{l\to\infty} \frac{r_l}{\delta_l} = \infty$ and (4.19) holds. Moreover,

$$\lim_{l\to\infty} \frac{d_l}{r_l} = b \in [0, \infty].$$

We need to consider the following two cases:

(i) $b < \infty$.

The proof is almost identical to **Case I** where $b = 0$, the major change being the substitution of

$$B^2_{\rho_l} \cap \left\{ y^{(4m)} \geq -\frac{d_l}{r_l} \right\},$$

$\mathbb{R}^{4m+}_b$ as in (4.10) and

$$\left(B^2_2 \backslash B^2_1\right) \cap \mathbb{R}^{4m+}_b$$

for $B^{2+}_{\rho_l}$ in (4.21), $\mathbb{R}^{4m+}$ and $B^{2+}_2 \backslash B^{2+}_1$ in the proof of **Case I** respectively.

(ii) $b = \infty$.

Then the situation amounts to the interior blow-up. For l sufficiently large, (4.20) and (4.21) become

$$\int_{B^{4m-2}_1} \int_{\left(B^2_2 \backslash B^2_1\right)} |\nabla \omega_l|^2 \mathrm{d}Y_1 \mathrm{d}Y_2 \geq \varepsilon_2 > 0.$$

and for all $0 < \rho \leq \rho_l = \frac{1}{2r_l}$,

$$\rho^{2-4m} \int_{B^{4m-2}_\rho} \int_{B^2_{\rho'}} |\nabla_{Y_1} \omega_l|^2 \, \mathrm{d}Y_1 \mathrm{d}Y_2 \to 0 \text{ as } \ l \to \infty,$$

for any prescribed $\rho' > 0$. The rest of the proof is analogous to that in [2, p.31-p.32] and we omit it. Hence, the **Claim** is proved for **Case II**.

We still use the same notations in **Case I** and decompose the neck domain $B^{2+}_{1/2}\left(X^l_2\right) \Big\backslash B^{2+}_{\delta_l R}\left(X^l_2\right)$ as follows when l and R are sufficiently large

$$\begin{aligned} B^{2+}_{1/2}\left(X^l_2\right) \Big\backslash B^{2+}_{\delta_l R}\left(X^l_2\right) &= B^{2+}_{1/2}\left(X^l_2\right) \Big\backslash B^{2+}_{1/4}\left(\widetilde{X}^l_2\right) \cup B^{2+}_{1/4}\left(\widetilde{X}^l_2\right) \Big\backslash B^{2+}_{2d_l}\left(\widetilde{X}^l_2\right) \\ &\quad \cup B^{2+}_{2d_l}\left(\widetilde{X}^l_2\right) \Big\backslash B^{2+}_{d_l}\left(X^l_2\right) \cup B^{2+}_{d_l}\left(X^l_2\right) \Big\backslash B^{2+}_{\delta_l R}\left(X^l_2\right), \\ &:= D_1 \cup D_2 \cup D_3 \cup D_4. \end{aligned}$$

It is easy to see that

$$D_1 \subset B^{2+}_{1/2}\left(X^l_2\right) \Big\backslash B^{2+}_{1/8}\left(X^l_2\right) \qquad \text{and} \qquad D_3 \subset B^{2+}_{4d_l}\left(X^l_2\right) \Big\backslash B^{2+}_{d_l}\left(X^l_2\right),$$

and for any $2\delta_l R \leq r \leq 1/8$, it holds

$$B^{2+}_{2r}\left(\widetilde{X}^l_2\right) \Big\backslash B^{2+}_{r}\left(\widetilde{X}^l_2\right) \subset B^{2+}_{4r}\left(X^l_2\right) \Big\backslash B^{2+}_{r/2}\left(X^l_2\right).$$

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As in **Case I**,

$$E\left(\widetilde{u}_l, D_1\right) + E\left(\widetilde{u}_l, D_3\right) \leq \varepsilon \text{for all large } R \text{and } l,$$

and

$$r^{2-4m} \int_{B_r^{4m-2}\left(X_1^l\right)} \int_{B_{2r}^{2+}\left(\widetilde{X}_2^l\right) \setminus B_r^{2+}\left(\widetilde{X}_2^l\right)} |\nabla u_l|^2 \, \mathrm{d}X_1 \mathrm{d}X_2 < \varepsilon, \tag{4.28}$$

for every $r \in [2d_l, 1/8]$.

Similarly to **Case I**, (4.28) leads to

$$\lim_{l\to\infty} E\left(\widetilde{u}_l, D_2\right) = 0.$$

Hence it remains to show

$$\lim_{R\to\infty} \lim_{l\to\infty} E\left(\widetilde{u}_l, D_4\right) = 0. \tag{4.29}$$

Note that $D_4 = B_{d_l}^{2+}\left(X_2^l\right) \setminus B_{\delta_l R}^{2+}\left(X_2^l\right) = B_{d_l}^{2}\left(X_2^l\right) \setminus B_{\delta_l R}^{2}\left(X_2^l\right)$ and

$$E\left(\widetilde{u}_l, D_4\right) = \int_{B_{d_l}^2\left(X_2^l\right) \setminus B_{\delta_l R}^2\left(X_2^l\right)} |\nabla \widetilde{u}_l|^2 \left(X_1^l, X_2\right) \, \mathrm{d}X_2.$$

Recall the definition of $\widetilde{v}_l$, that is (4.12), then we have

$$E\left(\widetilde{u}_l, D_4\right) = \int_{B_{d_l/\delta_l}^2 \setminus B_R^2} |\nabla \widetilde{v}_l|^2 \left(X_1^l, X_2\right) \, \mathrm{d}X_2.$$

By (4.14), it is easy to show that

$$\lim_{l\to\infty} \int_{B_{d_l/\delta_l}^2} |\nabla \widetilde{v}_l|^2 \left(X_1^l, X_2\right) \, \mathrm{d}X_2 = E\left(\phi_1\right),$$

and

$$\lim_{R\to\infty} \lim_{l\to\infty} \int_{B_R^2} |\nabla \widetilde{v}_l|^2 \left(X_1^l, X_2\right) \, \mathrm{d}X_2 = E\left(\phi_1\right).$$

Thus (4.29) is valid and the energy identity is proved for **Case II**.

Hence we complete the proof of Theorem 1.6.

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