102.11 Four congruent right triangles

The following (somewhat surprising) result about a square is possibly not as well-known as it should be.

Result (Square): The diagram below shows four directly congruent right-angled triangles *PAB*, *QCB*, *RCD* and *SAD*, with right angles at *P*, *Q*, *R* and *S*, such that *ABCD* is a square.

In this configuration *P*, *Q*, *R* and *S* are collinear.

(Two triangles are *directly congruent* if neither of them needs to be turned over to establish the congruence. Points are *collinear* if they lie in a straight line.)

The reader may like to try to prove this result before reading on.

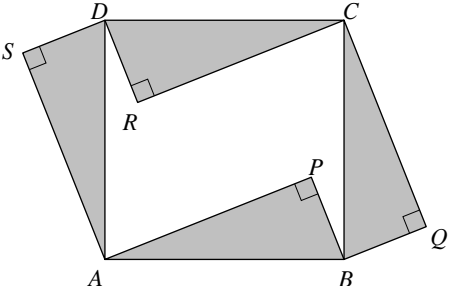


FIGURE 1

Is the result still true when the square is replaced by a rhombus? In that case, the triangles may overlap, as shown in Figure 2.

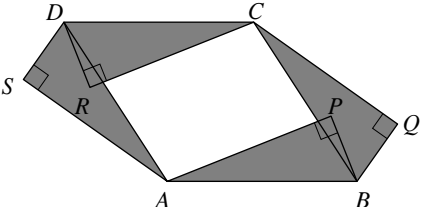


FIGURE 2

We give an elementary proof of this generalisation, in which *ABCD* is a rhombus and the triangles may overlap. Thereby we automatically prove the result for a square.

Result (Rhombus):

Figure 3 shows four directly congruent right-angled triangles *PAB*, *QCB*, *RCD* and *SAD*, with right angles at *P*, *Q*, *R* and *S*.

In this configuration *P*, *Q*, *R* and *S* are collinear.

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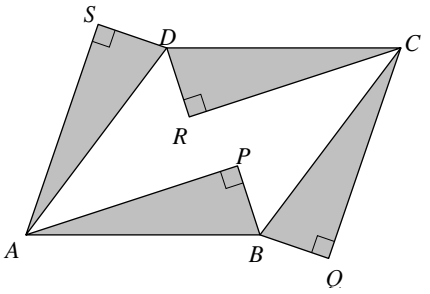


FIGURE 3

Proof: From the congruency, $PB = BQ$ and $\angle CBQ = \angle ABP$, so that $\angle PBQ = \angle ABC$. Join P to Q, as shown in Figure 4, and consider the marked angle of the isosceles triangle PBQ.

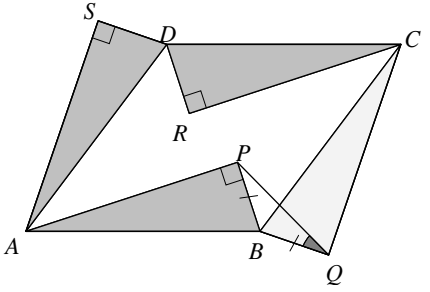


FIGURE 4

We have

$$\begin{aligned}\angle BQP &= \tfrac{1}{2}\left(180^\circ - \angle PBQ\right)\\ &= \tfrac{1}{2}\left(180^\circ - \angle ABC\right).\end{aligned}$$

In a similar way,

$$\angle RQC = \tfrac{1}{2}\left(180^\circ - \angle BCD\right).$$

Therefore

$$\angle BQP + \angle RQC = 180^\circ - \tfrac{1}{2}\left(\angle ABC + \angle BCD\right).$$

But AB and DC are parallel, so that

$$\angle ABC + \angle BCD = 180^\circ.$$

Hence

$$\begin{aligned}\angle BQP + \angle RQC &= 90^\circ\\ &= \angle BQC.\end{aligned}$$

It follows that P, Q and R are collinear.

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By repeating this argument 'on the other side', it follows that P, R and S are collinear. Therefore P, Q, R and S are collinear, as claimed.

Note that exactly the same argument applies whether or not PQ is shorter than QR, as in all our diagrams, or if the triangles overlap.

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102.12 On circles inscribed in an ellipse

In this Note we shall consider the radii of circles which have their centres on the x-axis, and are tangential to each other and to the circumscribing ellipse whose equation can be written as

$$(\sin^2\theta)x^2 + y^2 = c^2, \qquad (c > 0,\ 0 < \theta < \tfrac{\pi}{2}). \tag{1}$$

For some integer n, the circle C_n, centre $(c_n,\ 0)$, radius r_n, has equation

$$(x - c_n)^2 + y^2 = r_n^2, \qquad (c_n < c_{n+1},\ r_n > 0). \tag{2}$$

In particular let

$$r_0 = c\cos\omega, \qquad (0 \leqslant \omega < \tfrac{\pi}{2}). \tag{3}$$

The circles C_n and C_{n+1} are tangential to each other.

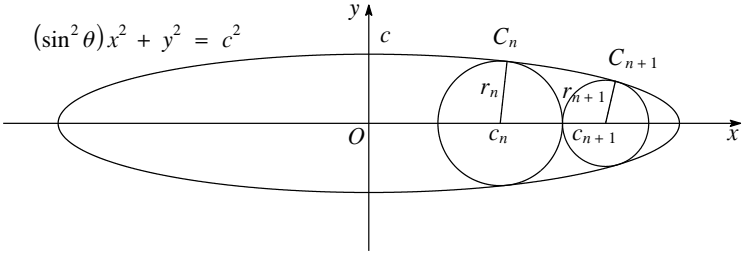


FIGURE 1: The circles in the ellipse

Then we have the following theorem.

Theorem: $r_n = c\cos(2n\theta + \omega)$, where n is an integer and $|2n\theta + \omega| < \frac{\pi}{2}$.

Proof: The intersections of C_n and the ellipse, ((1), (2)), are given by

$$(\cos^2\theta)x^2 - 2c_n x + c_n^2 - r_n^2 + c^2 = 0.$$

Tangency at these intersections yields

$$c_n^2 = \frac{-1}{\tan^2\theta}(r_n^2 - c^2). \tag{4}$$

Then

$$c_{n+1}^2 - c_n^2 = \frac{-1}{\tan^2\theta}(r_{n+1}^2 - r_n^2). \tag{5}$$

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