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Non-simple universal knots

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1. *Introduction*

A link or knot in S^3 is *universal* if it serves as common branching set for all closed, oriented 3-manifolds. A knot is *simple* if its exterior space is simple, i.e. any incompressible torus or annulus is parallel to the boundary. No iterated torus knot or link is universal, but we know of many knots and links that are universal. Thurston gave the first examples of universal links [**8**], and subsequently we proved that all 2-bridge knots and links that can be universal (no torus knots or links) are in fact universal [**3**]. Some other universal knots are described in [**1**] and [**2**], together with a general procedure for constructing such knots. For a general reference to knots see [**9**].

The natural problem is to describe the class of universal knots, and this was asked by one of the authors in his address to the 'Symposium of Kleinian groups, 3-manifolds and Hyperbolic Geometry' held in Durham, U.K., during July 1984. In the problem session of the same symposium W. Thurston asked if a non-simple knot can be universal and more concretely, if a cable knot can be universal. The question had the interest of testing whether the universality property has anything to do with the hyperbolic structure of some knots [**7**]. That this is not the case is shown in this paper, where we give infinitely many examples of double, composite and cable knots that are universal. While the examples of double and composite knots were relatively easy to obtain, the universal cable knots were very elusive. After all, a cable knot has a non-trivial Jaco-Shalen, Johannson decomposition, which in many cases must lift to a trivial one if the knot is universal.

2. *Universal doubles*

The idea is to take the trefoil knot K and its 3-fold cyclic covering Q^3 (quaternionic space). Then S^3 is an 8-fold unbranched covering of Q^3, and a 24-fold covering of S^3 branched over the trefoil knot. But then Q^3 is also an irregular 3-fold covering of S^3 branched over a 2-cable of K (in fact over infinitely many 2-cables of K). The modification shown in Fig. 1 changes these 2-cables (which are not universal) into doubles of K. The move M can be computed in the covering (which still is Q^3) and in most cases the preimage in S^3 of the so modified K is a universal link.

Here are the details. Let $p: S^3 \to Q^3$, $q: Q^3 \to S^3$ be the coverings of 8 and 3 sheets,

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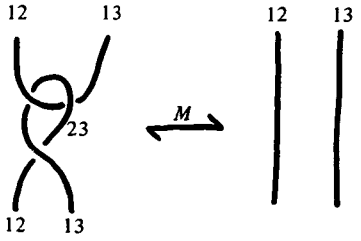


Fig. 1

where the second is branched over the left-handed trefoil knot K. We know describe these coverings. Let $S^3 = (O\circ 0|-1;(3,1),(2,1))$ in Seifert notation [**10**], so that the left-handed trefoil K is a general fibre. The quaternionic space is

$$Q^3 = (O\circ 0|-2;\,(2,1),\,(2,1),\,(2,1)).$$

Let $A: S^3 \to S^2$, $B: Q^3 \to S^2$ be the maps obtained by collapsing fibres of the Seifert fibrations. Then $q: Q^3 \to S^3$ induces a 3-fold cyclic covering $S^2 \to S^2$ (between the bases of A and B) branched over $A(K)$ and $A(E)$, where E is the exceptional fibre of order 3 in S^3. Thus we see that $q^{-1}(K)$ is a general fibre of Q^3. The covering $p: S^3 = (O\circ 0|-1) \to Q^3$ factors through three 2-fold unbranched coverings that we now describe. The first,

$$p_1: (O\circ 0|-2;\,(2,1),\,(2,1)) := Q_1^3 \to Q^3,$$

induces between the bases of $C: Q_1^3 \to S^2$ and $B: Q^3 \to S^2$ a 2-fold cyclic covering, whose branching set is the image under B of two of the three exceptional fibres of Q^3. Therefore $p_1^{-1}q^{-1}K$ consists of two general fibres of Q_1^3. The second, p_2: $(O\circ 0|-2) = \mathbb{R}P^3 \to Q_1^3$, induces between the bases of $D: RP^3 \to S^2$ and $C: Q_1^3 \to S^2$ a 2-fold covering branched over the images of the exceptional fibres of Q_1^3. Therefore $p_2^{-1}p_1^{-1}q^{-1}K$ are four general fibres of $\mathbb{R}P^3$. The third and last covering $p_3: S^3 \to \mathbb{R}P^3$, such that $p_3p_2p_1 = p$, is the well-known one induced by the antipodal map. Then $(qp)^{-1}K$ consists of 4 fibres of the left-Hopf fibration of S^3. These fibres project onto K, under the map qp, in a 2 to 1 fashion. Using [**6**], the covering q can be changed into a simple 3-fold covering $\hat{q}: Q^3 \to S^3$ branched over the 2-cable link L formed by K and a parallel longitude K' which defines the framing -6. For readers not well acquainted with [**6**], this change of q into $\hat{q}$ consists of a modification near the branching set K of q and its preimage $\tilde{K}$. Namely, the solid tori neighbouring K and $\tilde{K}$ are parametrized by $S^1 \times D^2$, so that the restriction of q to $S^1 \times D^2$ gives the map

$$\begin{aligned} Q: S^1 \times D^2 &\to S^1 \times D^2 \\ (x, z) &\mapsto (x, z^3), \end{aligned}$$

where we think of D^2 as the set of complex numbers of norm less than or equal to 1. The above parametrization of the neighbourhoods of K and $\tilde{K}$ is taken so that $(S^1 \times \{0\},\ S^1 \times \{\frac{1}{2}\})$ equals (K, K') where K' is another fibre of the fibration $(O\circ 0|-1;\,(3,1),\,(2,1))$. This is why K' defines the framing -6. In this way $q^{-1}(K, K')$ is a pair of general fibres of Q^3. Now the map Q is changed into

$$\begin{aligned} Q': S^1 \times D^2 &\to S^1 \times D^2 \\ (x, z) &\mapsto (x, rz), \end{aligned}$$

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where $r: D^2 \to D^2$ coincides with $z \mapsto z^3$ in ∂D^2 and is a simple 3-fold covering branched over two points, say 0 and $\frac{1}{2}$, of D^2. Recall that each one of $r^{-1}(0)$, $r^{-1}(\frac{1}{2})$ consists of two points, one with branching index one (*pseudo-branching cover*) and another with branching index two (*branching cover*). Therefore the branching set of $\hat{q}$ is $L = K \cup K'$, and $\hat{q}^{-1}(L)$ (Fig. 2) consists of four general fibres of Q^3. Consequently $(\hat{q}p)^{-1}(L)$ is

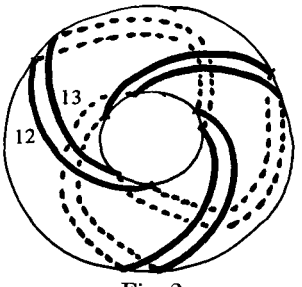


Fig. 2

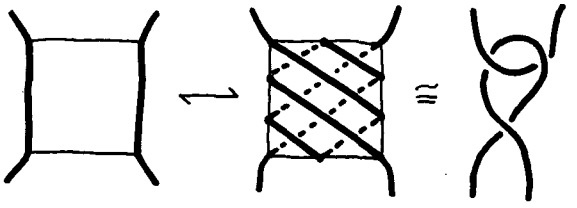


Fig. 3

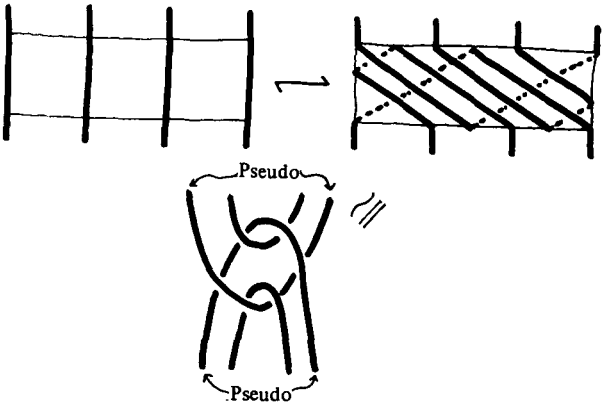


Fig. 4

formed by 16 fibres of the left-Hopf fibration of S^3. Now we apply to L the modification M (Fig. 3). The effect of M in the 4 curves $\hat{q}^{-1}(L)$ is Fig. 4 [**3**] (compare [**4**] and [**5**], page 82). Therefore M changes the link $(\hat{q}p)^{-1}(L)$; and this modification, in two of the eight curves of the pseudo-branching cover of $(\hat{q}p)^{-1}(L)$, is Fig. 5, which, being a non-toroidal 2-bridge link, is universal [**3**]. In a similar way the double $K(n)$ (Fig. 6) has a 24-fold covering $r: S^3 \to S^3$ such that $r^{-1}(K(n))$ contains the 2-bridge link (Fig. 7). This link is universal except for $n = 1$. Thus

THEOREM 1. *Infinitely many doubles of the trefoil knot are universal.*

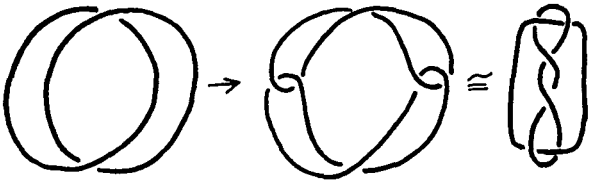


Fig. 5

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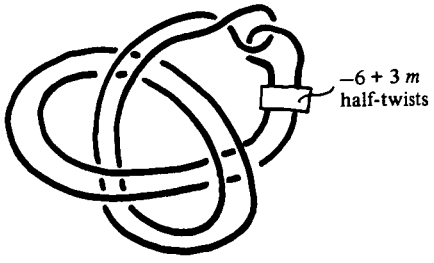


Fig. 6

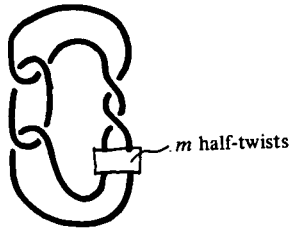


Fig. 7

In the last part of the proof we have made use of n modifications of type (Fig. 8). The reader might like to compute the effect of this move in the covering (see [**4**], [**5**]).

3. *Universal composite knots*

Consider the 4-fold simple covering $p: D_1^2 \to D_2^2$ obtained by folding D_1^2 by 180° rotations through the vertices of the equilateral triangle 1 (compare [**1**]). By thinking

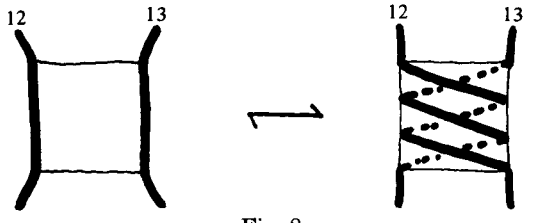


Fig. 8

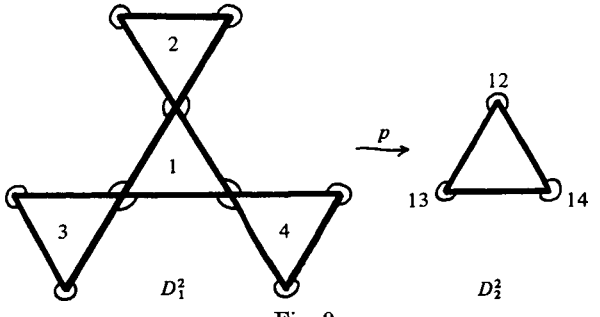


Fig. 9

of S_i^3 as an open book with leaf D_i^2, the covering p defines as 4-fold simple covering $P: S_1^3 \to S_2^3$ branched over the trivial link K of three components. The map P coincides with p in each leaf.

In Figure 10 we see the effect of two modifications, labelled 3/1 and 9/2, in the link K. The result of these two modifications is a 4-fold simple covering $\hat{P}: S^3 \to S^3$ branched over the composite knots (3/1) # (9/2) (= (trefoil knot) # (stevedore's knot)), where we are thinking of p/q as a rational knot, p odd. The preimage of (3/1) # (9/2) under $\hat{P}$ consists of the two components of the pseudo-branching cover (which are (3/1) # (3/1) and 9/2), and of the branching cover which is the trivial knot. Since 9/2 is universal [**3**], it follows that (3/1) # (9/2) is also universal. More

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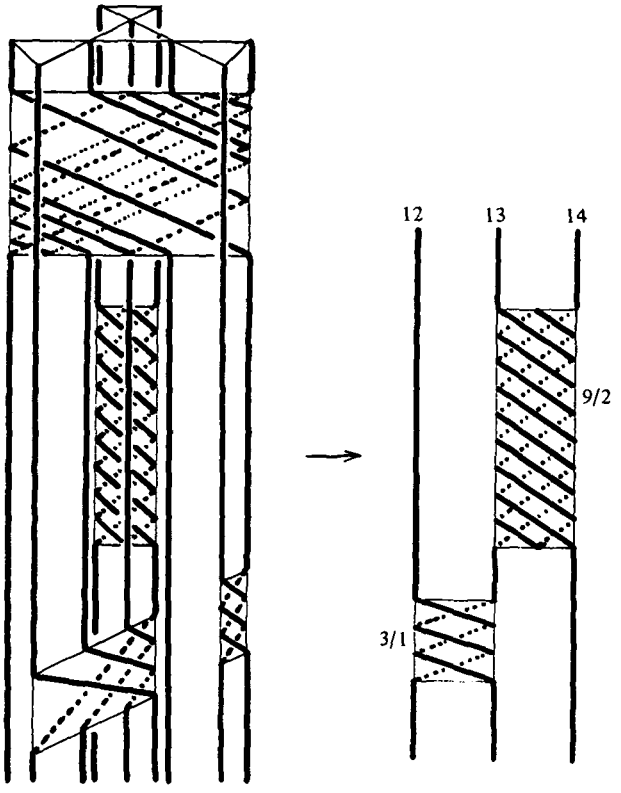


Fig. 10

generally, given $(3p/q) \# (3r/s)$, we have a 4-fold simple covering $S^3 \to S^3$ such that the pseudo-branching cover is the union of $(p/q) \# (3r/s)$ and $(3p/q) \# (r/s)$. For $p = 1$, we deduce as before that $(3/q) \# (3r/s)$ is universal whenever $3r/s$ is universal, i.e. when $s \not\equiv \pm 1 \bmod 3r$. Using induction, we see that:

THEOREM 2. *The knots* $(3p/q) \# (3r/s)$, *where* p *and* r *are odd,* $q = 1$ *or* 2 *and* $s \not\equiv \pm 1 \bmod 3r$, *are universal.*

4. *Universal cables*

In this case the idea is to find a covering $p: S^3 \to S^3$, branched over a knot K, such that

(i) The branching cover consists of one component of branching index $n \geqslant 3$.
(ii) The pseudo-branching cover is universal.
(iii) Each component of the pseudo-branching cover maps onto K in a $(n-1)m$ to 1 fashion.

Once such a p is found, we can replace it by $\hat{p}: S^3 \to S^3$ branched over a $(n-1)$-cable $\hat{K}$ of K [6]. (This replacement of $\hat{p}$ by p is analogous to the one explained in Section 2.) Then each component of the pseudo-branching cover is replaced by a $(n-1)$-cable *link* of $(n-1)$ components, so that the new pseudo-branching cover contains the old one. This would show that the $(n-1)$-cable $\hat{K}$ is universal.

Such an example is the Montesinos knot $K = (O|-2; (2, 1), (5, 1), (5, 1))$ with the representation shown in Fig. 11.

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$\omega: \pi_1(S^3 - K) \to S^7$
$x \longmapsto (123)$
$y \longmapsto (135)$
$z \longmapsto (167)$

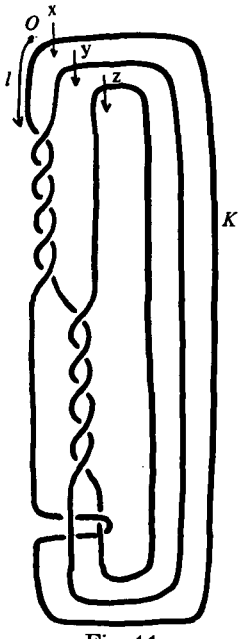


Fig. 11

The representation ω maps the longitude l based in O, and running on the paper parallel to the knot, into the permutation (132) (45) (67). This shows that the pseudo-branching cover consists of two components which map onto K in a 2 to 1 fashion. To see this, consider $p^{-1}(O) = O_1 \cup O_2 \cup \ldots \cup O_7$. Let $X \in K$ be the centre of the disc-meridian bounded by x. Thus $p^{-1}(X)$ consists of a point of branch index 3, around which O_1, O_2, O_3 are placed cyclically, and four other points belonging to the pseudo-branching cover. Since $\omega(l) = (132)\,(45)\,(67)$, $p^{-1}(l)$ consists of three components. One passes through O_1, O_2, O_3 and is mapped 3 to 1 onto K. The other two which connect O_4 with O_5 and O_6 with O_7 respectively, run parallel to the pseudo-branching cover and map 2 to 1 onto l.

Moreover the manifold covering S^3, branched over K corresponding to ω is S^3, because K is a 3-bridge knot. It only remains to show that the pseudo-branching cover is universal. This is a laborious calculation which we achieve by observing that K is a 3-braid which comes from the trivial 3-braid by performing three moves: two of type

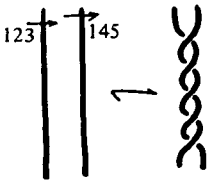


Fig. 12

and one of type

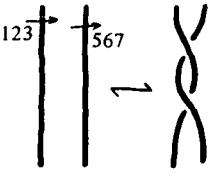


Fig. 13

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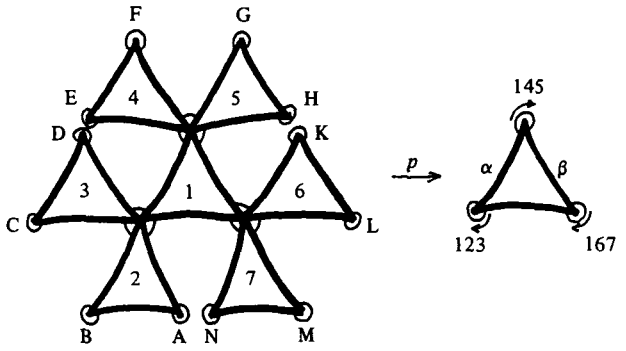


Fig. 14

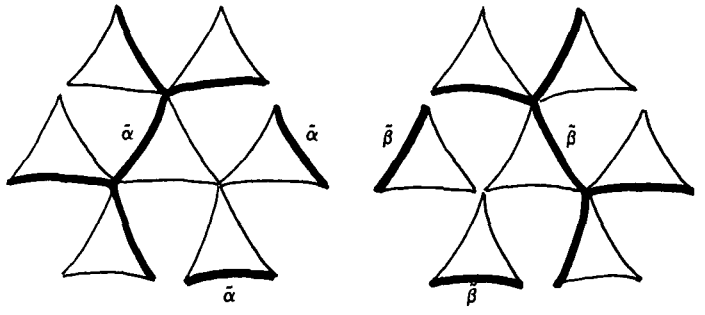


Fig. 15

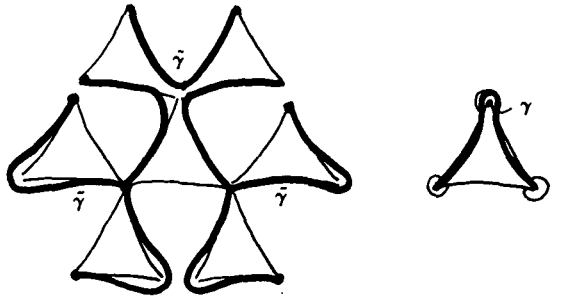


Fig. 16

As in the last section, we first consider the covering $p: D_1^2 \to D_2^2$ given by 120° rotations in the vertices of the disc 1 (Fig. 14).

The covering $P: S^3 \to S^3$ is obtained by the open-book construction. The moves of the first type affect the preimages $\tilde{\alpha}$, $\tilde{\beta}$ of the arcs α, β, and their effect is half a twist in a neighbourhood of the non-manifold components of $\tilde{\alpha}$ and $\tilde{\beta}$, and five half-twists in the neighbourhoods of the other components of $\tilde{\alpha}$ and $\tilde{\beta}$ (Fig. 15). The move of the second type affects the preimage $\tilde{\gamma}$ of γ, and its effect is $-\frac{1}{3}$ of a twist in the neighbourhood of the non-manifold components of $\tilde{\gamma}$ and a whole left twist in the other component of $\tilde{\gamma}$ (Fig. 16). The pseudo-branching cover over K is then the braid (Fig. 17). The component starting in N is the rational knot 31/7 which is universal [3]. Thus we have Fig. 18.

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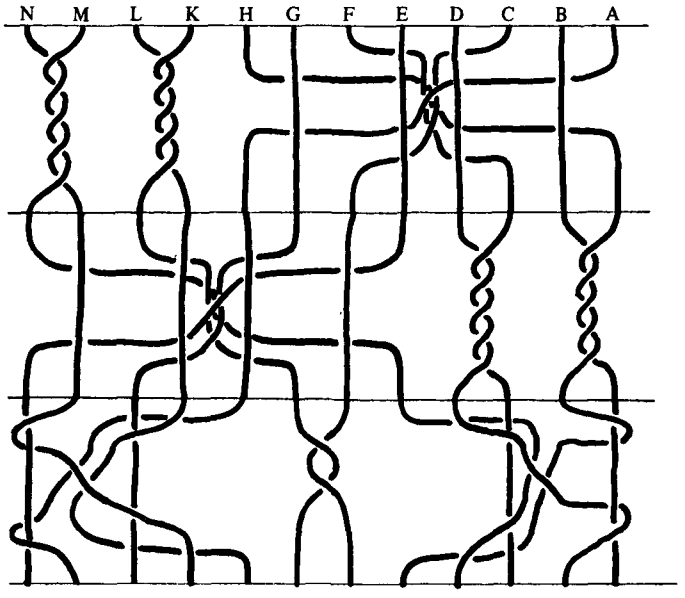


Fig. 17

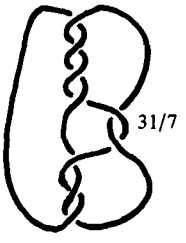


Fig. 18

THEOREM 3. *Infinitely many 2-cable knots of the knot* $(O|-2; (2, 1), (5, 1), (5, 1))$ *are universal.*

In fact, using [6], K can be converted to a 2-cable knot (in infinitely many different ways) so that the original covering branched over K is converted to a covering $S^3 \to S^3$ branched over this 2-cable. Moreover, the new pseudo-branching cover consists of two 2-cable *links* around the old pseudo-branching cover. Therefore, any of those 2-cable knots of K is universal.

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