The Lebesgue-Stieltjes Integral. By Mr E. C. FRANCIS, Peterhouse.

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It has long been realized that a Stieltjes integral can be defined and developed on the same lines as a Lebesgue integral, using in place of the measure of a set E the variation of an increasing function $\phi(x)$ upon E; but only the more obvious—and less useful—properties of such an integral seem to have been stated. The original integral defined by Stieltjes* possesses the remarkable Integration by Parts property that if either of $\int_a^b f d\phi$ and $\int_a^b \phi df$ exists, so does the other, and their sum is equal to

$$f(b)\,\phi(b) - f(a)\,\phi(a).$$

Stieltjes' definition does not lend itself to systematic analysis and has been largely replaced by definitions similar to Darboux's definition of the "Riemann integral"; but for the integrals so produced this Integration by Parts result—in which lies the chief claim to importance of the Stieltjes integral—has usually either been ignored or been proved in a form shorn of its generality†. For the Lebesgue-Stieltjes integral it is, in fact, true under most general conditions‡. An attempt has here been made to give a systematic sketch of the development of the integral, leading up to three main theorems—Change of Variable, Integration by Parts, and the Second Mean Value Theorem—theorems which display in striking manner the fundamental character of this conception of integration with respect to a function.

Making use of the theory of variation and of differentiation with respect to a function contained in an earlier paper§, I have first proved the fundamental differential properties of the *LS*-integral; and have used these as a basis for the theorem on change

* T. J. Stieltjes, "Recherches sur les Fractions Continues," *Ann. de la Fac. des Sc. de Toulouse*, vol. VIII (1894).

† E.g. W. H. Young, "Integration with respect to a Function of Bounded Variation," *Proc. L. M. S.* vol. XIII (1913), proves that if u and v are positive increasing functions such that both u has an *RS* (Riemann-Stieltjes) integral with respect to v and v with respect to u, then

$$\int u\,dv + \int v\,du = uv + \text{constant},$$

where the constant is zero if either function is continuous.

‡ The Integration by Parts theorem proved in this paper has also been given—proved by quite different methods—by P. J. Daniell, "On differentiation with regard to a function of limited variation," *Trans. Amer. Math. Soc.* vol. XIX, p. 353 (1918).

§ E. C. Francis, "On Differentiation with respect to a Function," *Proc. Camb. Phil. Soc.* vol. 22, p. 924.

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of variable. The Integration by Parts property has been proved without making use of the idea of differentiation, by establishing the connection between the *LS*-integral and the original Stieltjes integral.

§ 1. The Original Stieltjes-Integral.

Some reasons have already been given for the fact that the original Stieltjes-integral (or, as we shall call it, the *S*-integral) has something more than an historical interest. This integral has been dealt with somewhat exhaustively in a recent paper by S. Pollard*, and we shall confine ourselves here to a statement, for reference, of the definition and of the Integration by Parts property.

Definition. Take any division of (ab) *by a finite number of points*

$$a = x_0 < x_1 < x_2 \ldots < x_n = b.$$

Let ξ_ι *be any point such that* $x_{\iota-1} \leqslant \xi_\iota \leqslant x_\iota$.

Consider the sum $\mathscr{S} = \Sigma f(\xi_\iota)\{\phi(x_\iota) - \phi(x_{\iota-1})\}$.

If $\mathscr{S}$ *tends to a limit, for all possible* ξ_ι, *as* $\max \mid x_\iota - x_{\iota-1} \mid \to 0$, *then* $f(x)$ *is integrable* (S) *on* (ab) *and*

$$S\int_a^b f(x)\, d\phi(x) = \lim \mathscr{S}.$$

As an immediate consequence of this definition we have the remarkable result:

If $S\int_a^b f\,d\phi$ *exists so does* $S\int_a^b \phi\, df$ *and*

$$S\int_a^b f\,d\phi + S\int_a^b \phi\, df = f(b)\phi(b) - f(a)\phi(a).$$

It is upon this formula that we base our proof of an even more general theorem for the *LS*-integral.

§ 2. The Lebesgue-Stieltjes Integral.

We define and develop a Lebesgue-Stieltjes integral $LS\int_E f\,d\phi$ by straightforward methods, following closely Young's development† of the Lebesgue-integral. Since *variation* is involved we must restrict $\phi(x)$, and shall, in the first place, take it to be an *increasing function*, later extending our theory to embrace a function of bounded variation. $f(x)$ we assume *bounded*.

Definition. Consider any division of E *into a finite number of measurable* (ϕ) *sets* $E_1, E_2 \ldots E_n$. *Let* M_ι, m_ι *be the upper and lower bounds of* $f(x)$ *in* E_ι. *Define the Partial Sums* $S = \Sigma M_\iota . v(\phi \mid E_\iota)$,

* S. Pollard, "The Stieltjes Integral and its Generalizations," *Q. J.* vol. xlix, 1920–21.

† W. H. Young, "On the general theory of integration," *Phil. Trans. R. S.* (A), 204, pp. 221–252 (1904).

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$s = \Sigma m_i . v(\phi \mid E_i)$. *It is easily seen that every* $S \geqslant$ *every* s, *so that the sums* S *have a lower bound* I *and the sums* s *have an upper bound* I', *where* $I \geqslant I'$. *If* $I = I'$ *we say that* $f(x)$ *is integrable* (ϕ) *on* E *and define*

$$LS \int_E f d\phi = I = I'.$$

A measurable function is easily shown to be integrable and the converse can be proved.

For some purposes it is convenient to use only integrals over an interval. This is always possible, for it is easy to establish the equivalence of the integral above to $\int_I f_1 d\phi$, where the interval I contains E and.

$$f_1(x) \begin{cases} = f(x) \text{ on } E, \\ = 0 \text{ elsewhere.} \end{cases}$$

The LS-integral thus defined for bounded functions $f(x)$ can be extended, as in the case of the L-integral, to certain *unbounded functions*. We say that $f(x)$ is *summable* on E in each of the following three cases:

(i) $f(x) \geqslant 0$ on E and $\lim_{n \to \infty} \left[LS \int_E f_n d\phi \right]$ exists, where

$$f_n(x) \begin{cases} = f(x) \text{ when } f(x) \leqslant n, \\ = n \text{ elsewhere.} \end{cases}$$

(ii) $f(x) \leqslant 0$ on E and $-f(x)$ is summable on E.

(iii) $f(x)$ is the difference of two functions each positive and summable on E.

We give the obvious interpretation to $LS \int_E f d\phi$ in each case.

§ 3. Elementary Properties.

The following properties of the LS-integral extended as above are deduced by straightforward methods, and proofs will be omitted:

A. *If* E *is measurable* $\int_E d\phi$ *exists and is equal to* $v(\phi \mid E)$.

B. *If* $\int_E f d\phi$ *exists so does* $\int_E cf d\phi$ *and*

$$\int_E cf d\phi = c \int_E f d\phi,$$

c *being a constant.*

C. *The First Additive Theorem. If* $\int_E f d\phi$ *exists and* E *can be*

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split up into two measurable sets E_1, E_2, *then* $\int_{E_1} f d\phi$ *and* $\int_{E_2} f d\phi$ *both exist and are together equal to* $\int_E f d\phi$.

D. *The Second Additive Theorem. If* $\int_{E_1} f d\phi$, $\int_{E_2} f d\phi$ *both exist, and if* E_1, E_2 *are distinct sets with sum* E, *then* $\int_E f d\phi$ *exists and is equal to the sum of the other two integrals.*

E. *The f-Sum Theorem. If* $\int_E f_1 d\phi$, $\int_E f_2 d\phi$ *both exist, then* $\int_E (f_1 + f_2)\, d\phi$ *exists and is equal to their sum.*

F. *The φ-Sum Theorem. If* $\int_E f d\phi_1$, $\int_E f d\phi_2$ *both exist, then* $\int_E f d(\phi_1 + \phi_2)$ *exists and is equal to their sum.*

G. *The integral is "absolutely convergent." If* $\int_E f d\phi$ *exists so does* $\int_E |f|\, d\phi$ *and* $\int_E |f|\, d\phi \geqslant \left| \int_E f d\phi \right|$.

H. *If* $\int_E f_1 d\phi$, $\int_E f_2 d\phi$ *both exist, and* $f_1(x) \geqslant f_2(x)$ *on* E, *then* $\int_E f_1 d\phi \geqslant \int_E f_2 d\phi$.

I. *The First Mean Value Theorem* is a special case of **H**.

If $\int_E f d\phi$ *exists and* $f(x) \geqslant m$ *on* E, *then* $\int_E f d\phi \geqslant m v(\phi \mid E)$: a similar result holding for the reverse inequality.

J. *Lebesgue's Theorem.* (True in a more general case than that stated):

If $f_1(x), f_2(x), \ldots$ *is a bounded sequence of measurable functions which tend to* $f(x)$ *on* E, *then* $\int_E f d\phi$ *exists and* $= \lim_{n \to \infty} \int_E f_n d\phi$.

K. *The First Absolutely Additive Theorem. If* $\int_E f d\phi$ *exists and* E *is split up into a denumerable infinity of measurable subsets* $E_1, E_2, \ldots$, *then* $\int f d\phi$ *exists over each of these sets and*

$$\int_E f d\phi = \Sigma \int_{E_n} f d\phi.$$

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L. *The Absolute Continuity Property. If* $LS\int_E f d\phi$ *exists, then given any positive number* ϵ *there is an* $\eta\ (= \eta(\epsilon))$ *such that for all measurable subsets* S *of* E *for which* $v(\phi \mid S) < \eta$ *we shall have*

$$\left|\int_S f d\phi\right| < \epsilon.$$

An indefinite integral $F(x)$ is defined as follows at points x of an interval $\langle ab\rangle$ on which $f(x)$ is summable:

$$F(x-0) = \int_{\langle ax\rangle} f d\phi,$$

$$F(x+0) = \int_{\langle ax\rangle} f d\phi,$$

$$F(x) = \tfrac{1}{2}\{F(x+0) + F(x-0)\}.$$

The notation is free from ambiguity since we can show without difficulty that

$$F(x \pm 0) = \lim_{\xi \to x \pm 0} F(\xi).$$

We have the following properties.

M. $F(x)$ *is of bounded variation.*

N. $F(x)$ *has discontinuities of the first species only and these only where* $\phi(x)$ *is discontinuous.* It is thus a "*strictly smooth*" *function* continuous with* ϕ†.

O. *If* E *is any measurable*‡ *subset of* $\langle ab\rangle$, *then*

$$v(F \mid E) = LS\int_E f d\phi.$$

This theorem is of considerable importance in what follows. It is clearly true when E is an interval and extends at once to the case of an open or a closed set. For the general set it can be deduced without difficulty from the definition of measurability and the absolute continuity property **L**.

The case when $\phi(x)$ *is a function of bounded variation.*

Since a function of bounded variation is the difference of two increasing functions it is possible to extend our definition of $LS\int_E f d\phi$ to the case when $\phi(x)$ is such a function.

The new integral retains the majority of the properties of the old. Theorems **A** to **F**, **J**, **K**, **M**, **N** and **O** all remain true.

G, **I** and **L** may be modified in a fairly obvious manner by introducing a function $\bar{\phi}(x)$ to represent the "total variation" of $\phi(x)$.

* E. C. Francis, *loc. cit.* † E. C. Francis, *ibid.*

‡ I.e. measurable with respect to ϕ.

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§ 4. The Integration by Parts Formula.

We shall deduce the "Integration by Parts" result for the *LS*-integral by the straightforward process—free from all appeal to differential properties—of connecting the *LS*-integral with the *S*-integral (of which we have already noticed the Integration by Parts property). We establish the following theorem:

P. *If $\phi(x)$ is continuous and of bounded variation, and $S\int_a^b f\,d\phi$ exists, then $LS\int_a^b f\,d\phi$* exists and has the same value.*

Let $$S\int_a^b f\,d\phi = J.$$

With the notation of § 1 and § 2, we know that, given $\epsilon > 0$, we can find a division $x_1, \ldots x_n$ such that

$$|\mathscr{S} - J| < \epsilon$$

for this division and for every choice of ξ_i. Hence

$$|\Sigma M_i \{\phi(x_i) - \phi(x_{i-1})\} - J| \leqslant \epsilon,$$

$$|\Sigma m_i \{\phi(x_i) - \phi(x_{i-1})\} - J| \leqslant \epsilon,$$

and so $$S - s \leqslant 2\epsilon.$$

This is a sufficient condition that $LS\int_a^b f\,d\phi$ exists and is equal to J.

We have therefore at once the following result:

Q. *If $f(x)$, $\phi(x)$ are both continuous and of bounded variation in I, then*

$$LS\int_I f\,d\phi + LS\int_I \phi\,df = v(f\phi \mid I).$$

The assumption of continuity, however, is unnecessary and we proceed to the general theorem:

R. *Integration by Parts Formula:*

If $f(x)$, $\phi(x)$ are strictly smooth and of bounded variation on I,

then $$LS\int_I f\,d\phi + LS\int_I \phi\,df = v(f\phi \mid I).$$

From the Sum and Additive Theorems it is seen to be sufficient

* When $\phi(x)$ is continuous we have seen that the *LS*-integral is continuous. It is therefore possible to revert to the simpler notation $LS\int_a^b f\,d\phi$ in place of $LS\int_{\langle ab\rangle} f\,d\phi$ or $LS\int_{(ab)} f\,d\phi$.

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to prove the theorem when $f(x)$, $\phi(x)$ are monotonic. In this case we can resolve them in a manner suggested by Lebesgue*, giving

$$f(x) = f'(x) + f''(x),$$
$$\phi(x) = \phi'(x) + \phi''(x),$$

where $f'(x)$, $\phi'(x)$ are monotonic continuous functions; and $f''(x)$, $\phi''(x)$ are typified by $\phi''(x)$, where

$$\phi''(x-0) = \sum_{\langle ax \rangle} o_n, \qquad \phi''(x+0) = \sum_{\langle ax \rangle} o_n;$$

$o_1, o_2, \ldots$ being the oscillations of $\phi(x)$ over the denumerable (or finite) set of its discontinuities $q_1, q_2, \ldots$ denumerated in order of magnitude of the corresponding o's.

Again applying the Sum and Addition Theorems it is sufficient to prove the theorem in the three following cases:

(*a*) $f(x)$, $\phi(x)$ continuous and monotonic.

(*b*) $f(x)$ continuous and monotonic: $\phi(x)$ of type $\phi''(x)$.

(*c*) $f(x)$, $\phi(x)$ both of type $\phi''(x)$.

Case (*a*). The theorem has just been proved (**Q**).

Case (*b*). For simplicity suppose $\phi''(x)$ increasing. Let $\overline{M}$ be the upper bound of $f(x)$ in I. Given $\epsilon > o$, choose N such that

$$\sum_{N+1}^{\infty} o_n < \frac{\epsilon}{5\overline{M}} \qquad \ldots\ldots(1).$$

Let q be any point of I, then

$$\int_q f\,d\phi = f(q).o(q),$$

$$\int_q \phi\,df = o,$$

and
$$v(f\phi \mid q) = f(q).o(q),$$

where $o(q) = \operatorname{osc}(\phi \mid q)$. Hence if E denotes the set $q_1, q_2 \ldots q_N$ together with those end-points of I (if any) at which I is closed, we have on adding

$$\int_E f\,d\phi + \int_E \phi\,df = v(f\phi \mid E).$$

Remove the set E from I. There remain a finite number of open intervals J {$= (hk)$, say}.

By the first mean value theorem

$$\left| \int_J f\,d\phi \right| \leqslant \overline{M}.v(\phi \mid J) \qquad \ldots\ldots(3),$$

* Lebesgue, *Leçons sur l'Intégration*, p. 58.

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and $$\int_J \phi\, df = \phi(h+o)\, v(f \mid J) + \eta \qquad \text{.....(4),}$$

where $$|\eta| \leqslant 2\bar{M}.v(\phi \mid J) \qquad \text{.....(5).}$$

Also $$v(f\phi \mid J) = f(k)\phi(k-o) - f(h)\phi(h+o)$$
$$= \phi(h+o).v(f \mid J) + \theta \qquad \text{.....(6),}$$

where $$|\theta| \leqslant 2\bar{M}.v(\phi \mid J) \qquad \text{.....(7).}$$

Combining equations (3), (4), (5), (6) and (7) we have

$$\left| \int_J f\, d\phi + \int_J \phi\, df - v(f\phi \mid J) \right| \leqslant 5\bar{M}.v(\phi \mid J),$$

and, summing this for all intervals J, and remembering equation (2), we have

$$\left| \int_I f\, d\phi + \int_I \phi\, df - v(f\phi \mid I) \right| \leqslant 5\bar{M}.\Sigma v(\phi \mid J)$$
$$< \epsilon$$

by equation (1). Since ϵ is arbitrary the left side must be zero and the theorem is proved for case (*b*).

Case (*c*). Let $\omega_1, \omega_2, \ldots, p_1, p_2, \ldots$ bear the same relation to $f(x)$ as do $o_1, o_2, \ldots, q_1, q_2, \ldots$ to $\phi(x)$. Given $\epsilon > o$ we can choose N so large that

$$\sum_{N+1}^{\infty} o_n < \frac{\epsilon}{4\bar{M}} \qquad \text{.....(8),}$$

$$\sum_{N+1}^{\infty} |\omega_n| < \frac{\epsilon}{4\bar{\mu}} \qquad \text{.....(9),}$$

where $\bar{M}$, $\bar{\mu}$ are the upper bounds of $|f(x)|$, $|\phi(x)|$. [$\phi(x)$ is supposed increasing as before: $f(x)$ may be either increasing or decreasing.]

Let x be any point of I. Using the strictly smooth properties we have:

$$\int_x f\, d\phi = \tfrac{1}{2}[f(x+o) + f(x-o)][\phi(x+o) - \phi(x-o)],$$

$$\int_x \phi\, df = \tfrac{1}{2}[f(x+o) - f(x-o)][\phi(x+o) + \phi(x-o)],$$

$$v(f\phi \mid x) = f(x+o)\phi(x+o) - f(x-o)\phi(x-o).$$

Hence if E denotes the set $q_1, q_2, \ldots, q_N, p_1, p_2 \ldots p_N$, together with those end-points of I (if any) at which I is closed, we have on adding:

$$\int_E f\, d\phi + \int_E \phi\, df = v(f\phi \mid E) \qquad \text{.....(10).}$$

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Remove the set E from I. There remain a finite number of open intervals J [such as (hk)] and we have

$$\int_J f\,d\phi \leqslant \bar{M}.v(\phi \mid J),$$

$$\int_J \phi\,df \leqslant \bar{\mu} \mid v(f \mid J) \mid,$$

$$\mid v(f\phi \mid J) \mid \leqslant \bar{M}.v(\phi \mid J) + \bar{\mu}.\mid v(f \mid J) \mid.$$

Hence

$$\left| \int_J f\,d\phi + \int_J \phi\,df - v(f\phi \mid J) \right| \leqslant 2\bar{M}.v(\phi \mid J) + 2\bar{\mu}.\mid v(f \mid J) \mid.$$

Adding we have, on making use of equations (8), (9), (10),

$$\begin{aligned} \left| \int_I f\,d\phi + \int_I \phi\,df - v(f\phi \mid I) \right| &\leqslant 2\bar{M}.\Sigma v(\phi \mid J) + 2\bar{\mu}.\Sigma \mid v(f \mid J) \mid \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Since ϵ is arbitrary the left side must be zero and the theorem is proved for case (c), and therefore for all the cases.

Only in case (c) had we to make use of the "strictly smooth" condition; and case (c) appears only when *both* $f(x)$ *and* $\phi(x)$ *are discontinuous*. We have therefore:

S. *If* $f(x)$, $\phi(x)$ *are any functions of bounded variation, never discontinuous together,*

$$\int_I f\,d\phi + \int_I \phi\,df = v(f\phi \mid I).$$

The main Theorem **R** suggests at once the following:

T. *If* $f(x)$, $\phi(x)$ *are strictly smooth and of bounded variation, and if* E *is any set measurable* (f) *and* (ϕ), *then*

$$\int_E f\,d\phi + \int_E \phi\,df = v(f\phi \mid E).$$

This surprisingly general theorem is true, but not immediate, for we cannot make use at once of the facts that, if $f(x)$ is monotonic,

$$\int_E f\,d\phi = \int_I \bar{f}\,d\phi,$$

where $$\bar{f}(x) \begin{cases} = f(x) \text{ on } E, \\ = 0 \text{ elsewhere,} \end{cases} \text{ etc.,}$$

for $\bar{f}(x)$, $\bar{\phi}(x)$, defined in this way, are no longer strictly smooth. The theorem may be proved by considering the changes in the

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various terms on making $\bar{f}(x)$, $\bar{\phi}(x)$ strictly smooth, but it is much easier to proceed as follows:

If E is an open set the theorem follows at once from Theorem **R** and the absolutely additive properties of variation and of the *LS*-integral.

For any set E measurable (f) and (ϕ) there is an open set O containing E and such that

$$v(f \mid O - E), \quad v(\phi \mid O - E), \quad v(f\phi \mid O - E)$$

are as small as we please*. But the theorem is true for O. By the absolute continuity property it is therefore true for E.

As in Theorem **S** we may dispose of the "strictly smooth" condition if one of $f(x)$, $\phi(x)$ is continuous.

§ 5. Differential Properties.

We first prove the following Lemma:

U. *If $\phi(x)$ is an increasing function, $\int_E f d\phi$ exists, and $f(x) > k$ on E; then either $v(\phi \mid E) = 0$ or $\int_E f d\phi > k.v(\phi \mid E)$.*

Take any strictly decreasing sequence of numbers $k_1, k_2 \ldots$ which tend to k. Put $E_n = E\{f \geqslant k_n\}$.

Then either $v(\phi \mid E_n) = 0$ for all n, in which case $v(\phi \mid E) = 0$; or some $v(\phi \mid E_n) > 0$ in which case

$$\int_{E_n} f d\phi \geqslant k_n.v(\phi \mid E_n) > k.v(\phi \mid E_n).$$

But
$$\int_{E-E_n} f d\phi \geqslant k.v(\phi \mid E - E_n).$$

Hence
$$\int_E f d\phi > k.v(\phi \mid E).$$

Using the definitions of derivatives with regard to a strictly increasing function given in the paper quoted we can now extend the fundamental theorems of the integral calculus from the Lebesgue to the *LS*-integral. Throughout we assume—unless otherwise stated—$\phi(x)$ to be increasing, $LS\int_I f d\phi$ to exist, and $F(x)$ to be the corresponding indefinite integral. The proofs apply whether $f(x)$ is bounded or not.

V. *The First Fundamental Theorem:*

$F'(x)$ exists and is equal to $f(x)$ almost everywhere (ϕ).

Take $D(x)$ to be any derivative of $F(x)$. $D(x)$ is measurable since it is the limit of measurable functions.

* We have supposed $f(x)$, $\phi(x)$ increasing. For other cases the modification required is obvious.

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The set of points in the neighbourhood of which $\phi(x)$ is stationary clearly form an open set O on which $v(\phi \mid O) = v(F \mid O) = 0$. If Δ is the set of those end-points of intervals of O at which $\phi(x)$ is continuous, then Δ is finite or denumerable, so that

$$v(\phi \mid \Delta) = v(F \mid \Delta) = 0.$$

Neglecting the sets O, Δ we are therefore left with a set G at each point of which $D(x)$ has a definite meaning.

Now let E_k be the set of points of G at which

$$D(x) > k > f(x),$$

k being some rational number. E_k is measurable since $D(x), f(x)$ and G are all measurable, and by **O** $v(F \mid E_k)$ exists and is equal to $\int_{E_k} f d\phi$.

Hence, by our Lemma **U**, either $v(\phi \mid E_k) = 0$ or

$$v(F \mid E_k) < k \cdot v(\phi \mid E_k),$$

and the latter alternative is impossible since by the Generalized Variation Theorem*

$$v(F \mid E_k) \geqslant k \cdot v(\phi \mid E_k).$$

But the sum of the sets E_k for all rational values of k is the set E on which $D(x) > f(x)$. Hence

$$v(\phi \mid E) = 0.$$

Similarly if $\bar{E}$ is the set on which $f(x) > D(x)$,

$$v(\phi \mid \bar{E}) = 0,$$

and the theorem is proved, for the points at which $F'(x)$ either does not exist or is not equal to $f(x)$ are contained in E, $\bar{E}$, O, and Δ, all sets of measure zero.

There follows immediately:

W. *The Second Fundamental Theorem:*

If $D(x)$ is any derivative of $F(x)$, and if $F'(x)$ is considered only where it exists, then $D(x)$ and $F'(x)$ are summable and

$$\int_I D(x)\, d\phi = \int_I F'(x)\, d\phi = \int_I f d\phi = v(F \mid I).$$

These theorems have been stated with reference to an interval I. In virtue of a remark made in § 2, whereby $\int_E f d\phi$ can be regarded as an integral over an interval, they are true when E replaces I.

Our methods serve to prove the following extension of a well-known theorem on Increasing Functions. We take $\phi(x)$ to be strictly increasing.

* E. C. Francis, *loc. cit.*

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X. *If $\Psi(x)$ is an increasing function on I and $D(x)$ any of its derivatives, then $D(x)$ is summable (ϕ) and*

$$\int_I D(x)\, d\phi \leqslant v(\Psi \mid I).$$

As before $D(x)$ is measurable.

Put $$\{D(x)\}_n \begin{cases} = D(x) & \text{when } D(x) \leqslant n, \\ = n & \text{when } D(x) > n. \end{cases}$$

Then $\int_I \{D\}_n\, d\phi$ exists and defines an increasing indefinite integral $\Phi_n(x)$ which, by **V**, has a differential coefficient $\{D\}_n$ except on a set E where $v(\phi \mid E) = 0$.

Hence $\{\Phi_n(x) - \Psi(x)\}$ has a derivative $\leqslant 0$ at points of CE (the complement of E). Hence we deduce from Theorem **Q** of the earlier paper

$$v_*(\Phi_n - \Psi \mid CE) \leqslant 0,$$
$$v_*(\Phi_n \mid CE) \leqslant v^*(\Psi \mid CE).$$

But $v(\Phi_n \mid E)$ exists and is equal to $LS\int_E \{D\}_n\, d\phi = 0$. Hence

$$v(\Phi_n \mid I) = v(\Phi_n \mid CE) = v_*(\Phi_n \mid CE) \leqslant v^*(\Psi \mid CE) \leqslant v(\Psi \mid I),$$

i.e. $$LS\int_I \{D\}_n\, d\phi \leqslant v(\Psi \mid I).$$

The theorem is therefore proved.

As a corollary we have:

Y. *An increasing function has almost everywhere a summable differential coefficient.*

For let $D(x)$ be any derivative of an increasing function $\Psi(x)$. Let $F(x)$ be the indefinite integral corresponding to the integral $\int_I D\, d\phi$, which we know to exist. It follows from **X** that for any sub-interval J of I

$$v(F \mid J) \leqslant v(\Psi \mid J),$$

and therefore that, if D' be *any* derivative of $\Psi(x)$,

$$D' \geqslant \text{the corresponding derivative of } F(x).$$

But by **V** $F(x)$ has almost everywhere a differential coefficient D. Hence, almost everywhere

$$D' \geqslant D.$$

Similarly, almost everywhere

$$D \geqslant D',$$

i.e. the derivatives of $\Psi(x)$ are almost everywhere equal to one another.

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The last part of the theorem follows at once since $D(x)$ is summable.

§ 6. CHANGE OF VARIABLE.

We make use of the differential properties of the *LS*-integral to prove the following important theorem:

Z. *If $\phi(x)$ is of bounded variation and $\Psi(x)$ is the indefinite integral corresponding to $\int_I \psi(x)\, d\phi(x)$, then*

$$\int_E f(x)\, d\Psi(x) = \int_E f(x)\, \psi(x)\, d\phi(x),$$

provided that the right-hand side exists.

By a remark in § 2 it is sufficient to consider the case when E is an interval I.

We shall first assume:

(i) $f(x)$ bounded $\geqslant 0$.

(ii) $\psi(x) > 0$.

(iii) $\phi(x)$ is strictly increasing.

$\Psi(x)$ is therefore strictly increasing.

Since $f(x)\,\psi(x)$ is measurable (ϕ) and $\psi(x)$ is positive and measurable (ϕ) it follows that $f(x)$ is measurable (ϕ).

But, by definition of Ψ, if E is any set measurable (ϕ), E is also measurable (Ψ). Hence $f(x)$ is measurable (Ψ), and, since $f(x)$ is bounded, $\int_E f d\Psi$ exists.

Let $F(x)$, $\mathcal{F}(x)$ be the indefinite integrals corresponding to $\int f\psi d\phi$ and $\int f d\Psi$ respectively. Both are strictly increasing functions.

By Theorem **V**, except on a set E_1 for which $v(\phi \mid E_1) = 0$, $F_\phi'(x)$ and $\Psi_\phi'(x)$ exist and

$$F_\phi'(x) = f(x)\,\psi(x),$$
$$\Psi_\phi'(x) = \psi(x).$$

Hence by a result in our theory of differentiation* we deduce that, except on E_1, $F_\Psi'(x)$ exists and

$$F_\Psi'(x) = f(x).$$

This is therefore true almost everywhere (Ψ) for

$$v(\Psi \mid E_1) = LS \int_{E_1} \psi d\phi = 0.$$

* E. C. Francis, *loc. cit.*

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Hence by **X**

$$v(F \mid I) \geqslant \int_I f(x)\, d\Psi = v(\mathscr{F} \mid I) \qquad \ldots\ldots(1).$$

By an exactly similar argument $\mathscr{F}_\phi'(x)$ exists and

$$\mathscr{F}_\phi'(x) = f(x)\,\psi(x)$$

except on a set E_2 on which $v(\Psi \mid E_2) = 0$, and therefore

$$\int_{E_2} \psi\, d\phi = 0.$$

Since $\psi(x) > 0$ it follows from the lemma **U** that

$$v(\phi \mid E_2) = 0.$$

Hence, almost everywhere (ϕ)

$$\mathscr{F}_\phi'(x) = f(x)\,\psi(x),$$

and again applying **X** we have

$$v(\mathscr{F} \mid I) \geqslant \int_I f(x)\,\psi(x)\, d\phi(x) = v(F \mid I) \qquad \ldots\ldots(2).$$

Equations (1) and (2) prove the theorem in the case considered.

The relaxation of our assumptions one by one, thus extending our result by means of the sum and addition theorems to the general case, is almost trivial.

(*a*) *If* $f(x) \geqslant 0$ *but is unbounded,*

$$\int_E f_n\, d\Psi = \int_E f_n \psi\, d\phi,$$

where

$$f_n(x) \begin{cases} = f(x) & \text{when } f(x) \leqslant n, \\ = n & \text{when } f(x) > n, \end{cases}$$

$$\leqslant \int_E f\psi\, d\phi$$

which exists by hypothesis. Hence $\int_E f\, d\Psi$ exists and we can complete the theorem for this case either by repeating the remainder of the argument used above, or by quoting the theorem on limits under the integral sign.

(*b*) *If* $f(x)$ *takes negative values* we can express it as the difference of two positive functions, and the theorem follows at once from the f-Sum Theorem **E**.

(*c*) *If* $\psi(x) \geqslant 0$ *but takes the value zero on a set* G.

G is measurable (ϕ) and therefore measurable (Ψ), and

$$v(\Psi \mid G) = \int_G \psi\, d\phi = 0.$$

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Both $\int_G f d\Psi$ and $\int_G f\psi d\phi$ are therefore zero and the theorem still holds.

(*d*) *If* $\psi(x)$ *takes negative values.*

We can express $\psi(x)$ as the difference of two positive functions:

$$\psi(x) = \psi_1(x) - \psi_2(x),$$

where $\psi_1(x)$ is zero when $\psi(x)$ is negative, and $\psi_2(x)$ is zero when $\psi(x)$ is positive.

Now by Theorem **G** $\int_E |f|\,|\psi|\,d\phi$ and $\int_E |\psi|\,d\phi$ exist so that $\int_E f\psi_1 d\phi$, $\int_E f\psi_2 d\phi$, $\int_E \psi_1 d\phi$, $\int_E \psi_2 d\phi$ all exist.

Taking $\Psi_1(x)$, $\Psi_2(x)$ as the indefinite integrals corresponding to the last two integrals we have

$$\begin{aligned}\int_E f\psi d\phi &= \int_E f\psi_1 d\phi - \int_E f\psi_2 d\phi \\ &= \int_E f d\Psi_1 - \int_E f d\Psi_2 \text{ by case } (c) \\ &= \int_E f d\Psi \text{ by the } \phi\text{-Sum Theorem } \mathbf{F}.\end{aligned}$$

(*e*) *If* $\phi(x)$ *is increasing in the broad sense only* its constant stretches form a denumerable (or finite) set of intervals over each of which both $\int f\psi d\phi$ and $\int f d\Psi$ are zero; so the theorem remains true.

(*f*) *If* $\phi(x)$ *is of bounded variation only* the theorem follows at once from the ϕ-Sum Theorem **F**.

The theorem is therefore completely proved.

There is one interesting case—that when $f(x)$ is bounded and measurable (ϕ)—when the existence of the integral

$$\int_E f(x)\,\psi(x)\,d\phi(x)$$

is known*. We then have in all cases

$$\int_E f d\Psi = \int_E f\psi d\phi.$$

In particular this is true if $f(x)$ is a function of bounded variation.

* If $\psi(x)$ is not positive we can express it as the difference of two positive functions $\psi_1(x)$, $\psi_2(x)$. $f(x)\,\psi_1(x)$, $f(x)\,\psi_2(x)$ are measurable (ϕ) and it is easily seen that the existence of $\int_E |f\psi_1|\,d\phi$, $\int_E |f\psi_2|\,d\phi$, and therefore of $\int_E f\psi_1 d\phi$, $\int_E f\psi_2 d\phi$, follows at once from that of $\int_E \psi_1 d\phi$, $\int_E \psi_2 d\phi$.

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§ 7. THE SECOND MEAN VALUE THEOREM.

Combining the formula **T** for Integration by Parts with that for change of variable **Z**, we have:

α. *If $f(x)$ is a strictly smooth function of bounded variation, while $\psi(x)$ is summable (ϕ) on E and defines an indefinite integral $\Psi(x)$, then*

$$\int_E f\psi\, d\phi = v(f\Psi \mid E) - \int_E \Psi\, df.$$

β. In the case when E is an interval I ($\langle ab\rangle$, say, for convenience) and $f(x)$ *is monotonic* this can be re-written as the *Second Mean Value Theorem*

$$\int_I f\psi\, d\phi = f(a-o)(P - \Psi(a-o)) + f(b+o)(\Psi(b+o) - P),$$

where P is some number lying between the upper and lower bounds of $\Psi(x)$ on I.

For by the first mean value theorem

$$\int_I \Psi\, df = P\{f(b+o) - f(a-o)\}.$$

γ. In the case when $f(x)$ *is strictly smooth, positive, and decreasing*, we have *Bonnet's Form*

$$\int_I f\psi\, d\phi = f(a-o)\{P - \Psi(a-o)\}.$$

Both these theorems assume a more simple form when $\phi(x)$ *is continuous*. We then have

δ. *If $f(x)$ is a monotonic function, while $\psi(x)$ is summable (ϕ) on (ab), then*

$$\int_a^b f\psi\, d\phi = f(a \pm o)\int_a^\xi \psi\, d\phi + f(b \pm o)\int_\xi^b \psi\, d\phi,$$

where any of the ambiguous signs may be taken, and $a \leqslant \xi \leqslant b$.

ε. *If also $f(x)$ is decreasing and positive*

$$\int_a^b f\psi\, d\phi = f(a \pm o)\int_a^X \psi\, d\phi,$$

where $a \leqslant X \leqslant b.$

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