NOTE ON A PROBLEM OF ERDŐS

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The purpose of this paper is to prove the following

THEOREM. *If a_i, c_i $(i = 1, 2, \ldots, n)$ are non-negative integers with $a_1 \neq 0$ and $c_1 = 0$, then there are an infinity of integers x such that*

$$\xi = \frac{(a_1x + a_2x + \ldots + a_nx)!}{(a_1x + c_1)!\,(a_2x + c_2)! \ldots (a_nx + c_n)!}$$

is an integer.

The particular case $n = 2$, $a_1 = a_2 = 1$ was Problem 4252 (set by Erdős) in the *Amer. Math. Monthly*; it was solved by Herzog (1) and others. Generalizations, contained in the above theorem, were produced by Mordell (2) and Wright (3), though both required that Σa_i should be prime.

We require two preliminary lemmas.

LEMMA 1. *For non-negative integers m_i,*

$$\frac{(m_1 + m_2 + \ldots + m_n)!}{m_1!\,m_2! \ldots m_n!}$$

is an integer.

This is the familiar property of the multinomial coefficient.

LEMMA 2. *For strictly positive integers p, q, r, there are an infinity of integers y for which $r \mid N$, where*

$$N = \frac{(py + qy)!}{(py)!\,(qy)!}.$$

N is an integer by Lemma 1.

Proof. Let π be any prime factor of r and suppose

$$\pi^l \parallel r. \tag{1}$$

Choose an integer m such that $\pi^m \geqslant p + q.$ (2)

Take y as any solution of the set of congruences, one to each prime factor of r,

$$y \equiv -1 \mod (\pi^{l+m}). \tag{3}$$

The power of π which divides N is π^t, where

$$t = \sum_{s=1}^{\infty} \left(\left[\frac{pq + qy}{\pi^s} \right] - \left[\frac{py}{\pi^s} \right] - \left[\frac{qy}{\pi^s} \right] \right), \tag{4}$$

from the standard form of the index of π in $n!$.

For $m < s \leqslant l + m$, $y \equiv -1 \pmod{\pi^s}$, from (3). Hence

$$py + qy \equiv -p - q \pmod{\pi^s},$$

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and, by (2), $\pi^s > p+q$. Therefore

$$\left[\frac{py+qy}{\pi^s}\right] = \frac{(p+q)(y+1)}{\pi^s} - 1.$$

Similarly,
$$\left[\frac{py}{\pi^s}\right] = \frac{p(y+1)}{\pi^s} - 1$$

and
$$\left[\frac{qy}{\pi^s}\right] = \frac{q(y+1)}{\pi^s} - 1.$$

Hence the sth term of (4) is $+1$ for $m < s \leqslant l+m$; also every term of (4) is non-negative. Therefore $t \geqslant l$, i.e. $\pi^l \mid N$. Since this is true for every prime factor π of s, $s \mid N$. Finally, the set of congruences (3) has an infinity of solutions in integers. This completes the proof of the lemma.

Proof of the theorem. Write

$$a = a_1, \quad b = \sum_2^n a_i, \quad c = \sum_2^n c_i.$$

The theorem is trivial if $c = 0$, so we assume $c > 0$. Write

$$x = c!\,(ac+bc)!\,y. \tag{5}$$

Then, by Lemma 2, with

$$p = c!(ac+bc)!\,a, \quad q = c!(ac+bc)!\,b, \quad r = c!,$$

there are an infinity of integers x, of the form (5), for which

$$A = \frac{(ax+bx)!}{(ax)!\,(bx)!\,c!}$$

is an integer. Now

$$B = \frac{(bx+c)!}{(bx)!\,c!}, \quad C = \frac{(ax+bx)!}{(ax-c)!\,(bx+c)!}, \quad D = \frac{(ax)}{(ax-c)!}$$

are integers, by Lemma 1. If π is a common prime factor of B and D, then

$$\pi \mid (bx+e) \quad \text{and} \quad \pi \mid (ax-f),$$

where
$$1 \leqslant e \leqslant c \quad \text{and} \quad 0 \leqslant f \leqslant c-1.$$

Therefore $\pi \mid (ae+bf)$. Since $ae+bf$ is non-zero,

$$\pi \leqslant ae+bf < ac+bc. \tag{6}$$

By (5), $x \equiv 0 \mod (c!(ac+bc)!)$. Therefore

$$(bx+1)!\,(bx+2)!\,\ldots\,(bx+c)! \equiv c! \mod (c!(ac+bc)!),$$

i.e., B has no prime factors less than $(ac+bc)$ and so, by (6), B and D are coprime. Therefore

$$\frac{(ax+bx)!}{(ax)!\,(bx+c)!} = \frac{A}{B} = \frac{C}{D}$$

is an integer.

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Finally,
$$\xi = \frac{(ax+bx)!}{(ax)!\,(bx+c)!}\,\frac{(bx+c)!}{\prod_2^n ((a_i x+c_i)!)};$$

and
$$\frac{(bx+c)!}{\prod_2^n ((a_i x+c_i)!)}$$

is an integer, by Lemma 1.

This completes the proof of the theorem.

The more general problem, without the restriction that c_1 should be zero, appears difficult, though one can reasonably conjecture that the theorem is still true. The case $n = 2$, $a_1 = a_2 = c_1 = c_2 = 1$ has been proved by Mordell (unpublished).

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