LINEAR DIFFERENCE-DIFFERENTIAL EQUATIONS

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1. The general linear difference-differential equation takes the form

$$\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} A_{\mu\nu}(x)\, y^{(\nu)}(x+b_\mu) = v(x), \tag{1}$$

where x is a real variable, $v(x)$ and $A_{\mu\nu}(x)$ are known functions and

$$0 = b_0 < b_1 < \ldots < b_m.$$

As boundary conditions we suppose given the values of $y^{(\nu)}(x_0)$ for $\nu \leqslant n-1$ and the value of $y^{(n)}(x)$ for all x in some 'initial interval' $x_0 \leqslant x < x_0 + b_m$, with the proviso that $y^{(n)}(x)$ is integrable* in this interval. We define $y^{(\nu)}(x)$ $(\nu \leqslant n-1)$ by

$$y^{(\nu)}(x) = y^{(\nu)}(x_0) + \int_{x_0}^{x} y^{(\nu+1)}(t)\, dt,$$

so that (1) is in fact an integral equation in $y^{(n)}(x)$.

I shall show that, subject to wide conditions on $v(x)$ and $A_{\mu\nu}(x)$, (1) has a unique integrable solution $y^{(n)}(x)$. By a slight narrowing of the conditions on $v(x)$ and $A_{\mu\nu}(x)$ for $x \geqslant x_0$ and on $y^{(n)}(x)$ in the initial interval, I am able to deduce a set of properties of $y^{(n)}(x)$ outside this interval and, in particular, to obtain order results as $x \to \infty$.

Apart from any interest they may have in themselves, the results here are a useful preliminary to a more detailed development of the theory of equation (1). For example, my order results justify the application of transforms in their simplest form and obviate any assumptions as to the order of $y^{(n)}(x)$ at infinity. They also enable me to develop (16) a theory for this equation when $A_{\mu\nu}(x) \to a_{\mu\nu}$ as $x \to \infty$, analogous to that for ordinary differential and difference equations due to Poincaré (11), Perron (6, 7, 8) and Bochner (3). This again plays a part in the theory of small solutions of non-linear difference-differential equations (15). These applications oblige me to prove the results of the present paper under wide conditions. This adds a little to the length of the work, which could be somewhat shortened if I were to assume all functions continuous, since the lemma of the next section would then be classical.

The special case of (1) in which the $A_{\mu\nu}(x)$ are constants has been discussed by Schmidt (12), Hilb (4), Bochner (2), Titchmarsh (13, 14), Pitt (9, 10) and others. Hoheisel (5) studied the case in which $A_{\mu\nu}(x)$ is a linear or quadratic polynomial in x, only one $A_{\mu n}(x)$ differs from zero and $y^{(n)}(x) = O(|x|^C)$ as $x \to \pm\infty$. These authors use methods considerably deeper than mine here, including the use of Cauchy's theorem and the theory of transforms, neither of which I require. Their results are naturally more specialized and detailed than mine. I may remark that the importance of the non-vanishing of $A_{mn}(x)$ (or $A_{0n}(x)$), which appears in the sequel, is recognized for the constant coefficient case by Hilb (4) and by Pitt (10).

* Lebesgue (here and throughout).

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2. In what follows μ and ν are whole numbers satisfying $0 \leqslant \mu \leqslant m$, $0 \leqslant \nu \leqslant n$; any statement involving either μ or ν holds for all such values, unless the contrary is expressly stated. $\sum_{\mu}$, $\sum_{\nu}$ denote the corresponding summations.

THEOREM 1. *If, for* $x_0 \leqslant x \leqslant X_0$, (i) $A_{mn}(x) = 1$, (ii) $A_{\mu\nu}(x)$ *and* $v(x)$ *are integrable and* (iii) $A_{\mu n}(x)$ *and* $A_{m\nu}(x)$ *are bounded, then* (1) *has a unique integrable solution* $y^{(n)}(x)$ *for* $x_0 \leqslant x \leqslant X_0 + b_m$.

We first need the following

LEMMA. *If, in the interval* $0 \leqslant \xi \leqslant a$, $u(\xi)$ *is integrable and* $B_l(\xi)$ $(0 \leqslant l \leqslant L)$ *is bounded and integrable, then the equation*

$$\phi(\xi) = u(\xi) + \sum_{l=0}^{L} B_l(\xi) \int_0^{\xi} t^l \phi(t)\, dt \tag{2}$$

has a unique integrable solution $\phi(\xi)$ *throughout the same interval.*

If our functions were continuous, this would be a special case of a classical result*. The standard proof by successive approximation extends trivially to the case of bounded $u(\xi)$, but requires a minor modification to cover our case.

Let
$$D_1 = \int_0^a |\, u(t)\,|\, dt, \quad D_2 = \sum_{l=0}^{L} a^l \max_{\xi} |\, B_l(\xi)\,|,$$

$$\psi_1(\xi) = \sum_{l=0}^{L} B_l(\xi) \int_0^{\xi} t^l u(t)\, dt,$$

$$\psi_N(\xi) = \sum_{l=0}^{L} B_l(\xi) \int_0^{\xi} t^l \psi_{N-1}(t)\, dt \quad (N \geqslant 2).$$

Obviously $|\, \psi_1(\xi)\,| \leqslant D_1 D_2$ and, by induction,

$$|\, \psi_N(\xi)\,| \leqslant \frac{D_1 D_2^N \xi^{N-1}}{(N-1)!}.$$

Hence the series
$$u(\xi) + \sum_{N=1}^{\infty} \psi_N(\xi) \tag{3}$$

is uniformly and absolutely convergent and so provides an integrable solution of (2). This solution is unique; for if $\phi(\xi)$ is an integrable solution of (2) and $S_N(\xi)$ is the sum of the first $N+1$ terms of (3), we have

$$S_N(\xi) = u(\xi) + \sum_{l=0}^{L} B_l(\xi) \int_0^{\xi} t^l S_{N-1}(t)\, dt,$$

$$\phi(\xi) - S_N(\xi) = \sum_{l=0}^{L} B_l(\xi) \int_0^{\xi} t^l \{\phi(t) - S_{N-1}(t)\}\, dt.$$

Hence, by induction,

$$|\, \phi(\xi) - S_N(\xi)\,| \leqslant \frac{D_2^N \xi^{N-1}}{(N-1)!} \int_0^a |\, \phi(t) - S_0(t)\,|\, dt \to 0$$

as $N \to \infty$, and so $\phi(\xi)$ is given by (3). This completes the proof of the lemma.

* See, for example, (1).

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Theorem 1 is trivial for $x_0 \leqslant x \leqslant x_0 + b_m$. We suppose the result true for $x_0 \leqslant x \leqslant X$, where $x_0 + b_m \leqslant X < X_0 + b_m$, and prove it true for $X \leqslant x \leqslant X + a$, where

$$a = X_0 - X \quad (m = 0), \quad a = \min(b_m - b_{m-1},\ X_0 + b_m - X) \quad (m \geqslant 1).$$

A finite number of repetitions completes the proof of the theorem.

Let $X \leqslant x \leqslant X + a$, so that $x + b_\mu - b_m \leqslant X$ for $\mu \leqslant m-1$. Hence, for $\mu \leqslant m-1$, $y^{(\nu)}(x + b_\mu - b_m)$ is unique and integrable for $\nu \leqslant n$ and bounded for $\nu \leqslant n-1$. From (1) we have

$$y^{(n)}(x) + \sum_{\nu=0}^{n-1} A_{m\nu}(x - b_m)\, y^{(\nu)}(x) = w(x), \tag{4}$$

where

$$w(x) = v(x - b_m) - \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{n} A_{\mu\nu}(x - b_m)\, y^{(\nu)}(x + b_\mu - b_m)$$

is unique and integrable, since the product of two integrable functions, one of which is bounded, is itself integrable. In (4) we write $x = X + \xi$ and $y^{(n)}(x) = \phi(\xi)$ and observe that

$$y^{(\nu)}(X + \xi) = \sum_{\lambda=0}^{n-\nu-1} \frac{y^{(\lambda+\nu)}(X)}{\Gamma(\lambda+1)} \xi^\lambda + \int_0^\xi \frac{(\xi - t)^{n-\nu-1} \phi(t)\, dt}{\Gamma(n-\nu)}$$

when $\nu \leqslant n-1$. Thus (4) reduces to (2) with $L = n-1$ and

$$B_l(\xi) = \frac{(-1)^{l+1}}{\Gamma(l+1)} \sum_{\nu=0}^{n-1-l} \frac{A_{m\nu}(X - b_m + \xi)}{\Gamma(n-l-\nu)} \xi^{n-1-l-\nu}.$$

Our lemma then leads at once to the result.

3. We say that an integrable function is P_1 if it is of integrable square, P_2 if it is bounded, P_3 if it is of bounded variation and P_4 if it is continuous, each in a stated closed finite interval. The symbol j denotes 1, 2, 3 or 4. We observe that a P_4 or P_3 function is P_2 and a P_2 function is P_1. Also the sum of two P_j functions is P_j; and so is their product, except when $j = 1$. The product of a P_1 and a P_2 function is P_1, however.

THEOREM 2. *If* (i) *the conditions of Theorem* 1 *are satisfied,* (ii) $A_{\mu\nu}(x)$ *and* $v(x)$ *are* P_j *for* $x_0 \leqslant x \leqslant X_0$, *and* (iii) $y^{(n)}(x)$ *is* P_j *for* $x_0 \leqslant x \leqslant x_0 + b_m$, *then* $y^{(n)}(x)$ *is* P_j *for* $x_0 \leqslant x \leqslant X_0 + b_m$.

From (1) we have

$$y^{(n)}(x) = T_1 + T_2,$$

where

$$T_1 = -\sum_{\mu=0}^{m-1} A_{\mu n}(x - b_m)\, y^{(n)}(x + b_\mu - b_m),$$

$$T_2 = v(x - b_m) - \sum_{\mu} \sum_{\nu=0}^{n-1} A_{\mu\nu}(x - b_m)\, y^{(\nu)}(x + b_\mu - b_m).$$

Every $y^{(\nu)}$ in T_2 is an integral and is therefore P_j for every j; hence, by (ii), T_2 is P_j, for the particular j of the theorem, when $x_0 + b_m \leqslant x \leqslant X_0 + b_m$.

Now suppose $x_0 + b_m \leqslant X \leqslant X_0 + b_m$ and the theorem true for $x_0 \leqslant x \leqslant X$ and take the same a as before. Let $X \leqslant x \leqslant X + a$, so that $x + b_\mu - b_m \leqslant X$ for $\mu \leqslant m-1$. Every $y^{(n)}$ in T_1 is thus P_j, while every $A_{\mu n}$ is P_j and P_2 (the last by the conditions of Theorem 1). It follows that T_1, and so $y^{(n)}(x)$, is P_j. By a repetition of this argument we cover the whole interval $(x_0, X_0 + b_m)$.

4. We now discuss the behaviour of the y-functions at infinity. We consider only the behaviour as $x \to +\infty$; the results as $x \to -\infty$ are an immediate corollary. As before we prove a variety of results valid under different conditions on $y^{(n)}(x)$ in the initial

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interval. We use C to denote a positive number, not always the same at each occurrence, independent of x, X, ξ and η, but possibly dependent on any other parameters. C_1, C_2, C_3, C_4 are numbers of the same type as C, but C_1, for example, has the same value at each occurrence.

THEOREM 3. *Let the conditions of Theorem* 1 *be satisfied for every* $X_0 > x_0$ *and let* $|A_{\mu\nu}(x)| < C$ *for every* $x \geqslant x_0$. (i) *If*

$$\int_{x_0}^{X} |v(x)|\,dx < Ce^{CX}$$

for all $X > x_0$, *then*

$$\int_{x_0}^{X} |y^{(n)}(x)|\,dx < Ce^{CX}$$

for all $X > x_0$. (ii) *If*

$$\int_{x_0}^{x_0+b_m} |y^{(n)}(x)|^2\,dx < C, \quad \int_{x_0}^{X} |v(x)|^2\,dx < Ce^{CX},$$

for all $X > x_0$, *then*

$$\int_{x_0}^{X} |y^{(n)}(x)|^2\,dx < Ce^{CX}$$

for all $X > x_0$. (iii) *If*

$$|v(x)| < Ce^{Cx}$$

for all $x > x_0$ *and*

$$|y^{(n)}(x)| < C \quad (x_0 \leqslant x \leqslant x_0 + b_m),$$

then

$$|y^{(n)}(x)| < Ce^{Cx}$$

for all $x > x_0$.

Finally, if any one of the three sets of conditions is satisfied, $|y^{(\nu)}(x)| < Ce^{Cx}$ *for all* $x > x_0$ *and all* $\nu \leqslant n-1$.

We write $a = b_m - b_{m-1}$ if $m \geqslant 1$ and write a for some fixed positive number if $m = 0$. We take $0 < \eta < a$. Suppose that the conditions of (i) are satisfied. By Theorem 1 we may write

$$\lambda_\nu(X) = \int_{x_0}^{X} |y^{(\nu)}(x)|\,dx, \quad \lambda(x) = \max_\nu \lambda_\nu(x).$$

For $\nu \leqslant n-1$, $|y^{(\nu)}(x)| = \left|y^{(\nu)}(x_0) + \int_{x_0}^{x} y^{(\nu+1)}(t)\,dt\right| \leqslant C + \lambda_{\nu+1}(x)$,

$$\lambda_\nu(X+\eta) - \lambda_\nu(X) = \int_{X}^{X+\eta} |y^{(\nu)}(x)|\,dx \leqslant \eta\{C + \lambda_{\nu+1}(X+\eta)\} \leqslant C + \eta\lambda_{\nu+1}(X+\eta). \tag{5}$$

From (1) we have

$$|y^{(n)}(x+b_m)| \leqslant C\sum_{\nu=0}^{n-1} |y^{(\nu)}(x+b_m)| + C\sum_{\mu=0}^{m-1}\sum_{\nu} |y^{(\nu)}(x+b_\mu)| + |v(x)|. \tag{6}$$

Integrating from $x = x_0$ to $x = X + \eta - b_m$, we have

$$\lambda_n(X+\eta) \leqslant C\sum_{\nu=0}^{n-1} \lambda_\nu(X+\eta) + C\lambda(X) + Ce^{CX}. \tag{7}$$

Using (5) repeatedly in this, we obtain

$$\lambda_n(X+\eta)\{1 - C_1(\eta + \eta^2 + \eta^3 + \ldots + \eta^n)\} \leqslant C\lambda(X) + Ce^{CX}.$$

C_1 is independent of η and so we can take $\eta < C_2 = \dfrac{1}{1+2C_1}$, so that

$$1 - C_1(\eta + \eta^2 + \ldots + \eta^n) > 1 - \frac{C_1\eta}{1-\eta} > 1 - \frac{C_1C_2}{1-C_2} = \frac{1}{2},$$

and so

$$\lambda_n(X+\eta) \leqslant C\lambda(X) + Ce^{CX}.$$

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Using (5) repeatedly we obtain

$$\lambda(X+\eta) \leqslant C + C\lambda(X) + C\lambda_n(X+\eta) \leqslant C\lambda(X) + Ce^{CX}.$$

From this it follows that $\lambda(X) < Ce^{CX}$. Hence $\lambda_n(X) < Ce^{CX}$, which is the result of (i).

Next we suppose the conditions of (ii) satisfied, write

$$\lambda_\nu(X) = \int_{x_0}^{X} |y^{(\nu)}(x)|^2 dx, \quad \lambda(X) = \max_\nu \lambda_\nu(X),$$

and take a as before and $0 < \eta < a$. From (1) we have

$$|y^{(n)}(x+b_m)|^2 \leqslant C\sum_{\nu=0}^{n-1} |y^{(\nu)}(x+b_m)|^2 + C\sum_{\mu=0}^{m-1}\sum_{\nu} |y^{(\nu)}(x+b_\mu)|^2 + |v(x)|^2.$$

Integrating from $x = x_0$ to $x = X + \eta - b_m$, we have

$$\lambda_n(X+\eta) \leqslant C\sum_{\nu=0}^{n-1} \lambda_\nu(X+\eta) + C\lambda(X) + Ce^{CX}. \tag{8}$$

If $\nu \leqslant n-1$,

$$y^{(\nu)}(x+\eta) = y^{(\nu)}(x) + \int_x^{x+\eta} y^{(\nu+1)}(t)\,dt,$$

$$|y^{(\nu)}(x+\eta)|^2 \leqslant 2|y^{(\nu)}(x)|^2 + 2\left(\int_x^{x+\eta} |y^{(\nu+1)}(t)|\,dt\right)^2$$

$$\leqslant 2|y^{(\nu)}(x)|^2 + 2\eta\int_x^{x+\eta} |y^{(\nu+1)}(t)|^2 dt.$$

Hence, integrating from x_0 to X, we obtain

$$\lambda_\nu(X+\eta) \leqslant 2\lambda_\nu(X) + C + 2\eta\int_{x_0}^{X}\int_0^{\eta} |y^{(\nu+1)}(x+t)|^2 dx\,dt.$$

Since $$\int_{x_0}^{X}\int_0^{\eta} |y^{(\nu+1)}(x+t)|^2 dx\,dt = \int_0^{\eta}\int_{x_0}^{X} |y^{(\nu+1)}(x+t)|^2 dt\,dx$$

$$\leqslant \int_0^{\eta} \lambda_{\nu+1}(X+t)\,dt \leqslant \eta\lambda_{\nu+1}(X+\eta),$$

we have $$\lambda_\nu(X+\eta) \leqslant 2\eta^2\lambda_{\nu+1}(X+\eta) + C + C\lambda_\nu(X).$$

We now use this repeatedly in (8) and argue as before to obtain

$$\lambda_n(X) \leqslant \lambda(X) < Ce^{CX}.$$

If the conditions of (iii) are satisfied, we write

$$\lambda_\nu(X) = \max_{x_0 \leqslant x \leqslant X} |y^{(\nu)}(x)|, \quad \lambda(X) = \max_\nu \lambda_\nu(X).$$

From (1) we obtain

$$\lambda_n(X+\eta) \leqslant C\sum_{\nu=0}^{n-1} \lambda_\nu(X+\eta) + C\lambda(X) + Ce^{CX}. \tag{9}$$

Again, if $\nu \leqslant n-1$, we have

$$\lambda_\nu(X+\eta) \leqslant \lambda_\nu(X) + \int_X^{X+\eta} |y^{(\nu+1)}(x)|\,dx \leqslant \lambda_\nu(X) + \eta\lambda_{\nu+1}(X+\eta).$$

We use this repeatedly in (9); the argument follows the same lines as before and

$$\lambda_n(X) \leqslant \lambda(X) < Ce^{CX}.$$

The final statement that $|y^{(\nu)}(x)| < Ce^{Cx}$ for $\nu \leqslant n-1$ is an immediate corollary of the result of (i). The conditions of (ii) and (iii) imply those of (i).

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5. From Theorem 3 we can deduce corresponding results valid as $x \to -\infty$. In this case we require that $A_{0n}(x) = 1$ (or, what is the same thing after division, that $|A_{0n}(x)| > C$) for $x \leqslant x_0$. We now prove a result in the opposite direction and show that solutions of the homogeneous equation

$$\sum_{\mu} \sum_{\nu} A_{\mu\nu}(x)\, y^{(\nu)}(x + b_\mu) = 0 \tag{10}$$

cannot be too small as $x \to +\infty$ without vanishing identically. Here $A_{0n}(x)$ is the coefficient whose modulus must be bounded below, while $A_{mn}(x)$ plays the same part in a corresponding result as $x \to -\infty$.

THEOREM 4. *If, for all* $x \geqslant x_0$, (i) $A_{0n}(x) = 1$, (ii) $|A_{\mu\nu}(x)| < C$, (iii) $y^{(n)}(x)$ *is a solution of* (10), (iv) $\int_x^\infty |y^{(n)}(x)|\, e^{kx}\, dx$ *converges for every* k *and* (v) $|y^{(0)}(x)|$ *does not tend to infinity nor to any non-zero limit as* $x \to +\infty$, *then* $y^{(n)}(x) = 0$ *for almost all* $x \geqslant x_0$ *and* $y^{(\nu)}(x) = 0$ *for* $\nu \leqslant n-1$ *and all* $x \geqslant x_0$.

It follows from (iv) that $y^{(n-1)}(x)$ tends to a limit l as $x \to \infty$ and that, in fact,

$$|y^{(n-1)}(x) - l| = \left| \int_x^\infty y^{(n)}(t)\, dt \right| \leqslant e^{-kx} \int_x^\infty |y^{(n)}(t)|\, e^{kt}\, dt < Ce^{-kx}$$

for every k. By repeated use of a similar argument we find that

$$y^{(0)}(x) = P(x) + O(e^{-kx})$$

for every k, where $P(x)$ is a polynomial. By (v), $P(x)$ is identically zero and so

$$|y^{(\nu)}(x)| < Ce^{-kx} \tag{11}$$

for $\nu \leqslant n-1$ and every k.

For $X \geqslant x_0$, we write

$$\rho_\nu(X) = \int_X^\infty |y^{(\nu)}(x)|\, dx, \quad \rho(X) = \max_\nu \rho_\nu(X).$$

(11) and (iv) show that the integral $\rho_\nu(X)$ converges. Clearly $\rho(X)$ decreases steadily to zero, in the non-strict sense, as $X \to \infty$. By (10),

$$|y^{(n)}(x)| \leqslant C\left(\sum_{\nu=0}^{n-1} |y^{(\nu)}(x)| + \sum_{\mu=1}^{m} \sum_{\nu=0}^{n} |y^{(\nu)}(x + b_\mu)| \right),$$

and so

$$\rho_n(X) \leqslant C \sum_{\nu=0}^{n-1} \rho_\nu(X) + C\rho(X + b_1). \tag{12}$$

Let $0 < \eta < \min(1, b_1)$. For $\nu \leqslant n-1$,

$$|y^{(\nu)}(x)| = \left| \int_x^\infty y^{(\nu+1)}(t)\, dt \right| \leqslant \rho_{\nu+1}(x),$$

$$\rho_\nu(X) = \rho_\nu(X + \eta) + \int_X^{X+\eta} |y^{(\nu)}(x)|\, dx \leqslant \rho_\nu(X + \eta) + \eta\rho_{\nu+1}(X). \tag{13}$$

Using this repeatedly in (12), we have

$$\rho_n(X) \leqslant C(\eta + \eta^\nu + \ldots + \eta^n)\, \rho_n(X) + C\rho(X + \eta),$$

where C does not depend on η. Hence, if $\eta < C_3$,

$$\rho_n(X) \leqslant C\rho(X + \eta).$$

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Combining this with (13), we have

$$\rho(X) \leqslant C\rho(X+\eta) \quad (X \geqslant x_0,\ 0 < \eta < C_3).$$

If $\rho(x_0) \neq 0$, it follows that $\rho(X) > Ce^{-C_4 X}$ for $X \geqslant x_0$. But this is impossible, since

$$e^{C_4 X}\rho_\nu(X) \leqslant \int_X^\infty |\, y^{(\nu)}(x)\,|\, e^{C_4 x}\, dx \to 0$$

as $X \to \infty$, by (11) and (iv). Hence $\rho(x_0) = 0$, $\rho_n(x_0) = 0$ and so $y^{(n)}(x) = 0$ for almost all $x \geqslant x_0$. Hence $y^{(\nu)}(x) = 0$ for $\nu \leqslant n-1$ and all $x \geqslant x_0$.

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