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FOURTH MOMENT STRUCTURE OF THE GARCH(p,q) PROCESS

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In this paper, a necessary and sufficient condition for the existence of the unconditional fourth moment of the GARCH(p,q) process is given and also an expression for the moment itself. Furthermore, the autocorrelation function of the centered and squared observations of this process is derived. The statistical theory is further illustrated by a few special cases such as the GARCH(2,2) process and the ARCH(q) process.

1. INTRODUCTION

The general autoregressive conditional heteroskedasticity (GARCH) model and its many variants are popular in modeling volatility in high-frequency financial series. This is evident from the large number of recent surveys on ARCH and GARCH models and other ways of modeling volatility (see, e.g., Bera and Higgins, 1993; Bollerslev, Engle, and Nelson, 1994; Diebold and Lopez, 1995; Engle, 1995; Gouriéroux, 1997; Guégan, 1994, Ch. 5; Palm, 1996, Shephard, 1996). The statistical properties of the basic GARCH model of Bollerslev (1986) have been discussed in a number of articles. Bollerslev (1986) derived conditions for the existence of unconditional moments of the GARCH(1,1) model under normality. He also derived expressions for these moments as functions of moments of lower order. He and Teräsvirta (1999a) did the same without the normality assumption for a large family of GARCH(1,1) processes. Furthermore, Bollerslev (1988) found the autocorrelation function for the standard GARCH(1,1) process with normal errors, and He and Teräsvirta (1999a) generalized the result to their family of GARCH(1,1) models and non-normal error distributions. As for higher-order processes, Bollerslev (1986, 1988), using Yule–Walker type equations, indicated the shape of the autocorrelation and partial autocorrelation functions of the general GARCH(p,q) process. Milhøj (1985) already derived the autocorrelation function of an ARCH(q) process that is a special case ($p = 0$) of the GARCH(p,q) model.

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Results on the existence of unconditional moments for GARCH models are not only of statistical interest. Practitioners may want to use them to see what kind of moment implications GARCH models they estimate may have. In particular, the existence of the unconditional fourth moment of stochastic processes generating, say, financial return data has interested researchers. Given both the existence conditions and a suitable expression for the fourth moment, the investigator would be able to check what an estimated model implies about this moment. Obtaining estimates of the kurtosis and the autocorrelations of the centered and squared observations from the model would also be useful. That would enable one to see how well the kurtosis and autocorrelation implied by the estimated model match the estimates obtained directly from the data. Such a comparison would give an indication of how well the model fits the data, because high kurtosis and slowly decaying autocorrelations of the centered and squared observations (or residuals) are characteristic features of many financial high-frequency series. We assume fourth order stationarity in this paper; for tests against weak stationarity, see Loretan and Phillips (1994).

The results of Bollerslev (1986, 1988) and He and Teräsvirta (1999a) make model evaluations of that type possible for a family of GARCH(1,1) processes. This paper generalizes some of those results to GARCH(p,q) processes in the standard GARCH framework. The focus will be on the fourth moments and the autocorrelation function for the centered and squared observations.

The plan of the paper is as follows. In Section 2 we derive the necessary and sufficient condition for the existence of the unconditional fourth moment of the GARCH(p,p) process and give an expression for the moment itself. Section 3 presents the autocorrelation function for the squared process. In those sections, the GARCH(2,2) model and the ARCH(q) model are used as illustrations. Section 4 concludes. The proofs are presented in the Appendix.

2. CONDITION FOR EXISTENCE OF THE FOURTH MOMENT

We begin with a necessary and sufficient condition for existence of the unconditional fourth moment in the GARCH(p,p) process. The results derived for the GARCH(p,p) model apply directly to the GARCH(p,q) model with $p \neq q$. This is because we may, without any loss of generality, nest that model in a model with $p = q$ by assuming certain parameters in a GARCH(p,p) model equal to zero. Assume further that $p \geq 2$. This is a technical assumption due to the matrix notation used in the paper (certain matrices are not defined for $p = 1$). The results for the lower-order GARCH processes may be obtained from Corollaries 1 and 2 by setting $\alpha_2 = 0$, $\beta_2 = 0$ or $\alpha_2 = \beta_2 = 0$. The model is defined as

$$u_t = \varepsilon_t \sqrt{h_t}, \tag{1}$$

where $\{\varepsilon_t\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with zero mean. Furthermore,

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$$h_t = \alpha_0 + \sum_{i=1}^{p} \beta_i h_{t-i} + \sum_{i=1}^{p} \alpha_i u_{t-i}^2 \tag{2}$$

(see Bollerslev, 1986). Assume now that this GARCH model has a finite fourth moment and let $\mathrm{E}\varepsilon_t^j = \nu_j, j = 2,4$. Furthermore, let $\alpha_0 > 0$, $\alpha_i \geq 0$, and $\beta_i \geq 0$ for $i = 1,\ldots,p$, in (2). In particular, $\alpha_p > 0$ and $\beta_p > 0$. Let $c_{i,t-i} = \beta_i + \alpha_i \varepsilon_{t-i}^2$ for $i = 1,\ldots,p$, where $\{c_{it}\}$ is a sequence of i.i.d. random variables such that c_{it} is independent of h_t. This allows us to rewrite (2) as

$$h_t = \alpha_0 + \sum_{i=1}^{p} c_{i,t-i} h_{t-i}. \tag{3}$$

In this paper we will make heavy use of (3). From the assumption that the second moments of $\{u_t\}$ exist it follows that $\sum_{i=1}^{p} \gamma_{i1} < 1$, where $\gamma_{i1} = \mathrm{E}c_{it} = \beta_i + \alpha_i \nu_2$, $i = 1,\ldots,p$. From (1) we see that to consider the unconditional fourth moment of $\{u_t\}$ we have to square (3) and then take the unconditional expectations on both sides, which yields

$$\mathrm{E}h_t^2 = \alpha_0^2 + 2\alpha_0\gamma_1 \mathrm{E}h_t + \gamma_2 \mathrm{E}h_t^2 + 2\sum_{l<m} \mathrm{E}(c_{l,t-l} c_{m,t-m} h_{t-l} h_{t-m}), \tag{4}$$

where $\mathrm{E}h_t = \alpha_0/(1-\gamma_1)$, $\gamma_i = \sum_{j=1}^{p} \gamma_{ji}$, $i = 1,2$, and $\gamma_{i2} = \mathrm{E}c_{it}^2 = \beta_i^2 + 2\alpha_i\beta_i\nu_2 + \alpha_i^2\nu_4$ for $i = 1,\ldots,p$. Thus, assuming that it exists, $\mathrm{E}h_t^2$ can be determined through (4) if we are able to express mixed moments $\mathrm{E}(c_{l,t-l} c_{m,t-m} h_{t-l} h_{t-m})$ as functions of $\mathrm{E}h_t$ and $\mathrm{E}h_t^2$. This can be done by recursion that involves repeated use of (3). Redefining h_{t-l} using (3) gives an expression that is a function of h_{t-l-1} but no longer of h_{t-l}. The next step consists of applying (3) to h_{t-l-1}. The appropriate expressions for $h_{t-l}h_{t-m}$ after the kth recursion, $k \geq m - l + 1$, which are functions of lags of h_t and h_t^2, appear in Lemma 1 of the Appendix.

To obtain the desired results we have to define a set of matrices $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_{m-l}, \mathbf{C}_{m-l+1}$ and, finally, $\mathbf{C}_k$ for any $k > m - l + 1$ whose elements are zeros, ones, and of type $c_{i,t-j}$. Their definitions can be found in the Appendix. The corresponding expectations $\mathrm{E}\mathbf{C}_k = \mathbf{\Gamma}_k$ for $k = 1,\ldots,m-l+1$ and $\mathrm{E}\mathbf{C}_k = \mathbf{\Gamma}$ for all $k > m - l + 1$. Note that $\mathbf{\Gamma}$ is a $p^* \times p^*$ matrix, where $p^* = p(p-1)/2$. Let $\lambda_i, i = 1,\ldots,p^*$ be the eigenvalues of $\mathbf{\Gamma}$ and define

$$\lambda(\mathbf{\Gamma}) = \max\{|\lambda_i|\}, \tag{5}$$

the maximum absolute eigenvalue of $\mathbf{\Gamma}$. Although $\mathbf{\Gamma}$ is the expectation of $\mathbf{C}_i$ in (A.4) and thus seems to be a complicated matrix it has a relatively simple structure in that $\det(\mathbf{\Gamma}) = \gamma_{p1}^{p-1}$, $p \geq 2$.

Assume now that the GARCH process started at some finite value infinitely many periods ago. It turns out that the unconditional mean $\mathrm{E}(c_{l,t-l} c_{m,t-m} h_{t-l} h_{t-m})$ exists and is independent of t as $k \to \infty$ if and only if all the eigenvalues of $\mathbf{\Gamma}$ lie inside the unit circle, that is, $\lambda(\mathbf{\Gamma}) < 1$. This condition is only needed for $p \geq 2$ when mixed moments appear in (4). For $p = 2$ it has the simple form $\gamma_{21} = \beta_2 +$

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$\alpha_2\nu_2 < 1$, which is not a very restrictive condition in practice. For $p > 2$, the condition involves the expectations $\gamma_{j1}, j = 1, \ldots, p$. If $\lambda(\mathbf{\Gamma}) < 1$ holds then $\mathbf{\Gamma}^k \to \mathbf{0}$ and $\sum_i^k \mathbf{\Gamma}^i \to (\mathbf{I}_{p^*} - \mathbf{\Gamma})^{-1}$ as $k \to \infty$. This is needed for the mixed moment $\mathsf{E}(c_{l,t-l}c_{m,t-m}h_{t-l}h_{t-m})$ to converge to a finite value that is a linear function of $\mathsf{E}h_t$ and $\mathsf{E}h_t^2$ as $k \to \infty$. This is shown in Lemmas 2–5. This result allows us to determine the expectation $\mathsf{E}h_t^2$ through (4). Once we have this the expression for $\mathsf{E}u_t^4$ follows immediately. Combining the result of Lemma 5 and the assumption $\lambda(\mathbf{\Gamma}) < 1$ we have the following theorem.

THEOREM 1. *A necessary and sufficient condition for the existence of the fourth unconditional moment of* $\{u_t\}$ *in the* GARCH(p,p) *model* (1) *with* (2) *is*

$$\gamma_2 + 2\sum_{l<m} \gamma_{l1} M_2(l,m) < 1, \tag{6}$$

where $M_2(l,m)$ *is a nonlinear function of* γ_{ij} *defined in Lemma* 5 *of the Appendix. The fourth moment is*

$$\mathsf{E}u_t^4 = \frac{\alpha_0^2\nu_4\left[1 + \gamma_1 + 2\sum_{l<m}\gamma_{l1}\gamma_{m1}M_1(l,m)\right]}{(1-\gamma_1)\left[1 - \gamma_2 - 2\sum_{l<m}\gamma_{l1}M_2(l,m)\right]}, \tag{7}$$

where $M_1(l,m)$ *is another nonlinear function of* γ_{ij} *defined in Lemma* 5.

Note that for GARCH(p,p) models with $p \geq 2$, $M_2(l,m) > 0$ in (6).

As seen previously, the basic tool used for obtaining (6) and (7) is repeated recursions on $h_{t-l}h_{t-m}(1 \leq l < m \leq p)$. For comparison, when Bollerslev (1986) derived the general moment $\mathsf{E}u_t^{2m}$, $m \geq 1$, for the standard GARCH(1,1) process under the normality assumption he relied on recursions of $\mathsf{E}(h_t^{2m} | \mathcal{F}_{t-j})$, where $\mathcal{F}_s$ contains the past information about the process up until s.

To illustrate the preceding general theory we consider the GARCH(2,2) process. As mentioned earlier $\mathbf{\Gamma} = \gamma_{21}$ in this case. We have the following corollary.

COROLLARY 1. *For the* GARCH(2,2) *model satisfying* $\gamma_{21} < 1$ *a necessary and sufficient condition for the existence of the unconditional fourth moment of* $\{u_t\}$ *is that*

$$(\gamma_{12} + \gamma_{22})(1 - \gamma_{21}) + 2\gamma_{11}\tilde{\gamma}_{12} + \gamma_{21} < 1, \tag{8}$$

where $\tilde{\gamma}_{12} = \mathsf{E}(c_{1t}c_{2t})$. *When* (8) *holds,*

$$\mathsf{E}u_t^4 = \frac{\alpha_0^2\nu_4[(1 + \gamma_{11} + \gamma_{21})(1 - \gamma_{21}) + 2\gamma_{11}\gamma_{21}]}{(1 - \gamma_{11} - \gamma_{21})[(1 - \gamma_{21})(1 - \gamma_{12} - \gamma_{22}) - 2\gamma_{11}\tilde{\gamma}_{12}]}. \tag{9}$$

Proof. See Appendix.

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Although we assumed $p \geq 2$ while deriving the general result, the existence condition for the fourth unconditional moment of the GARCH(1,1) process may be obtained by setting $\beta_2 = \alpha_2 = 0$ in (8) and has the form

$$\gamma_{12} = \beta_1^2 + 2\beta_1\alpha_1\nu_2 + \alpha_1^2\nu_4 < 1 \tag{10}$$

(see also He and Teräsvirta, 1999a). Setting $\nu_2 = 1$ and $\nu_4 = 3$ in (10) yields the condition that Bollerslev (1986) obtained for standard normal errors.

Finally, it may be mentioned that Milhøj (1985) already presented a necessary and sufficient condition for the existence of the unconditional fourth moment of the ARCH(p) process under normality. It has the form

$$3\boldsymbol{\alpha}'(\mathbf{I}_p - \boldsymbol{\Psi})^{-1}\boldsymbol{\alpha} < 1, \tag{11}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)'$ and the $p \times p$ matrix $\boldsymbol{\Psi} = (\psi_{ij})$ is defined by $\psi_{ij} = \alpha_{i+j} + \alpha_{i-j}$, making use of the fact that $\alpha_k = 0$ for $k \leq 0$ and $k > p$ (note that in Milhøj's notation, α_0 is not the intercept). Of course, (6) under normality and (11) are equivalent, although this may not be easy to see immediately. As an illustration, we demonstrate this equivalence in the ARCH(2) case. Setting $\gamma_{i1} = \alpha_i$, $\gamma_{i2} = 3\alpha_i^2$, $i = 1,2$, and $\tilde{\gamma}_{12} = 3\alpha_1\alpha_2$ in (8) yields

$$3\alpha_1^2 + 3\alpha_2^2 + 3\alpha_1^2\alpha_2 - 3\alpha_2^3 + \alpha_2 < 1. \tag{12}$$

On the other hand, setting $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)'$ and

$$\boldsymbol{\Psi} = \begin{pmatrix} \alpha_2 & 0 \\ \alpha_1 & 0 \end{pmatrix}$$

in (11) also yields (12).

3. THE AUTOCORRELATION FUNCTION FOR THE SQUARED PROCESS

Having obtained an expression for the fourth moment we can derive the autocorrelation function for the squared observations. Milhøj (1985) proved that the squared process $\{u_t^2\}$ has an autocorrelation function similar to that of a standard autoregressive process of order p when $\{u_t\}$ follows an ARCH(p) process with normal errors. He also defined the first p autocorrelations of $\{u_t^2\}$ as

$$\boldsymbol{\rho} = (\mathbf{I}_p - \boldsymbol{\Psi})^{-1}\boldsymbol{\alpha}, \tag{13}$$

where $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_p)'$ and $\boldsymbol{\alpha}$ and $\boldsymbol{\Psi}$ are as in (11). For $n > p$,

$$\rho_n = \sum_{i=1}^{p} \alpha_i \rho_{n-i}. \tag{14}$$

Bollerslev (1986, 1988) applied this approach to the GARCH(p,p) case and obtained the analogue of the Yule–Walker equations for the autocorrelation function of $\{u_t^2\}$. He was able to show that for any $n > p$ and assuming normal errors, the autocorrelations have the form

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$$\rho_n = \sum_{i=1}^{p} (\beta_i + \alpha_i)\rho_{n-i}. \tag{15}$$

However, he did not yet provide the first p autocorrelations $\rho_1, \ldots, \rho_p$. We shall do that here.

To fix notation, write the nth order autocorrelation of $\{u_t^2\}$ as

$$\rho_n = \rho(u_t^2, u_{t-n}^2) = \frac{\mathsf{E}(u_t^2 u_{t-n}^2) - (\mathsf{E}u_t^2)^2}{\mathsf{E}u_t^4 - (\mathsf{E}u_t^2)^2} \tag{16}$$

for $n \geq 1$. To obtain ρ_n, we must find an expression for $\mathsf{E}(u_t^2 u_{t-n}^2)$ as a function of $\mathsf{E}h_t^2$ and $\mathsf{E}h_t$. This problem resembles the one in the previous section where we needed an expression for $\mathsf{E}(h_t h_{t-n})$, $n \leq p$, in the terms of $\mathsf{E}h_t^2$ and $\mathsf{E}h_t$. In this case, the solution will be slightly different because expressing $\mathsf{E}(u_t^2 u_{t-n}^2)$ as functions of $\mathsf{E}h_t^2$ and $\mathsf{E}h_t$ for any $n \geq 1$ is required. The final result can be formulated as follows.

THEOREM 2. *Assume that condition* (6) *holds. For the* GARCH(p,p) *model* (1) *with* (2), *the autocorrelation function for* $\{u_t^2\}$ *is, for any* $n \geq 1$,

$$\rho_n = \frac{\nu_2 \gamma_{S_1}(1-\gamma_1)M_2(n) - \nu_2^2 \gamma_{S_2}[1-(1-\gamma_1)M_1(n)]}{\nu_4 \gamma_{S_1}(1-\gamma_1) - \nu_2^2 \gamma_{S_2}}, \tag{17}$$

where $M_1(n), M_2(n), \gamma_{S_1}$, *and* γ_{S_2} *are nonlinear functions of* γ_{ij} *defined in Lemma* 9 *and* (*A*.69) *and* (*A*.70).

Proof. See Appendix.

It may be pointed out that condition (6) may also be expressed using γ_{S_2} in (17) as follows:

$$0 < \gamma_{S_2} < 1. \tag{18}$$

Values of γ_{S_2} close to zero correspond to the left-hand side of (6) being close to unity.

Properties of the autocorrelation function $\{\rho_n\}$, $n = 1,2,\ldots$, can be established through (17). Some of them are listed here.

1. The first p autocorrelations are positive if the parameter restrictions $\alpha_0 > 0$, $\alpha_i \geq 0$ and $\beta_i \geq 0$, $i = 1,\ldots,p$ hold. These restrictions are sufficient for positivity but not necessary (see Nelson and Cao, 1992; He and Teräsvirta, 1999b).
2. The autocorrelation function satisfies the difference equation $\rho_n = \sum_{i=1}^{p} \gamma_{i1}\rho_{n-i}$ at lags $n > p$, and $\rho_n > 0$.
3. The autocorrelation function is dominated by an exponential decay and $\lim_{n\to\infty} \rho_n = 0$.
4. When γ_{S_2} is sufficiently close to zero the autocorrelation function is persistent; otherwise the autocorrelations decay quickly with increasing n.

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To illustrate the general result we again consider the GARCH(2,2) process. In the GARCH(2,2) model, $\gamma_i = \gamma_{1i} + \gamma_{2i}$, $i = 1,2$, $\gamma_{S_1} = (1+\gamma_1)(1-\gamma_{21}) + 2\gamma_{11}\gamma_{21}$ and $\gamma_{S_2} = (1-\gamma_2)(1-\gamma_{21}) - 2\gamma_{11}\tilde{\gamma}_{12}$. By Theorem 2 we have the following corollary.

COROLLARY 2. *For the* GARCH(2,2) *process, assume that condition* (8) *holds. The autocorrelations of* $\{u_t^2\}$ *then equal*

$$\rho_n = \frac{\nu_2\gamma_{S_1}(1-\gamma_1)M_2(n) - \nu_2^2\gamma_{S_2}[1-(1-\gamma_1)M_1(n)]}{\nu_4\gamma_{S_1}(1-\gamma_1) - \nu_2^2\gamma_{S_2}}, \quad n \geq 1. \tag{19}$$

In (19), *for* $n \geq 1$

$$M_1(n) = \boldsymbol{\gamma}_3'\boldsymbol{\Gamma}_*^{n-1}(\alpha_0^{-1} \quad 0 \quad (1-\gamma_{21})^{-1})' \tag{20}$$

and for $n \geq 3$

$$M_2(n) = \boldsymbol{\gamma}_3'\boldsymbol{\Gamma}_*^{n-3}(0 \quad M_2(2) \quad M_2(1))' \tag{21}$$

with $\boldsymbol{\gamma}_3 = (\alpha_0 \quad \gamma_{11} \quad \gamma_{21})'$ *and* $\boldsymbol{\Gamma}_* = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_0 & \gamma_{11} & \gamma_{21} \\ 0 & 1 & 0 \end{pmatrix}$, *whereas*

$$M_2(1) = (1-\gamma_{21})^{-1}[\bar{\gamma}_{11}(1-\gamma_{21}) + \nu_2\tilde{\gamma}_{12}] \tag{22}$$

and

$$M_2(2) = (1-\gamma_{21})^{-1}[(\gamma_{11}\bar{\gamma}_{11} + \bar{\gamma}_{21})(1-\gamma_{21}) + \nu_2\gamma_{11}\tilde{\gamma}_{12}] \tag{23}$$

with $\bar{\gamma}_{11} = \mathsf{E}(\varepsilon_t^2 c_{1t})$ *and* $\bar{\gamma}_{21} = \mathsf{E}(\varepsilon_t^2 c_{2t})$.

Proof. See Appendix.

From Corollary 2 it follows that the first two autocorrelations are

$$\rho_1 = \frac{\nu_2\gamma_{S_1}[\bar{\gamma}_{11}(1-\gamma_{21}) + \nu_2\tilde{\gamma}_{12}](1-\gamma_1) - \nu_2^2\gamma_{11}\gamma_{S_2}}{(1-\gamma_{21})[\nu_4\gamma_{S_1}(1-\gamma_1) - \nu_2^2\gamma_{S_2}]} \tag{24}$$

and

$$\rho_2 = \frac{\nu_2\gamma_{S_1}[(\gamma_{11}\bar{\gamma}_{11} + \bar{\gamma}_{21})(1-\gamma_{21}) + \nu_2\gamma_{11}\tilde{\gamma}_{12}](1-\gamma_1) - \nu_2^2\gamma_{S_2}(\gamma_{11}^2 + \gamma_{21} - \gamma_{21}^2)}{(1-\gamma_{21})[\nu_4\gamma_{S_1}(1-\gamma_1) - \nu_2^2\gamma_{S_2}]}. \tag{25}$$

For higher orders, ρ_n is determined by the powers of matrix $\boldsymbol{\Gamma}_*$ and also $M_2(1)$ and $M_2(2)$ through equation (19). In other words, for $n \geq 3$, ρ_n depends on the first two autocorrelations, which is also clear from the corresponding Yule–Walker equations (15). The autocorrelation function may have its maximum at either the first or the second lag, and it has four basic shapes in all (for a discussion, see He and Teräsvirta, 1999b).

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The corresponding results for the GARCH(1,1) model are obtained by setting $\gamma_{21} = \tilde{\gamma}_{12} = \bar{\gamma}_{21} = 0$ in (19). The autocorrelation function thus has the form

$$\rho_n = \frac{\nu_2(1-\gamma_{11}^2)M_2(n) - \nu_2^2(1-\gamma_{12})[1-(1-\gamma_{11})M_1(n)]}{\nu_4(1-\gamma_{11}^2) - \nu_2^2(1-\gamma_{12})}, \qquad n \geq 1. \tag{26}$$

The terms $M_1(n)$ and $M_2(n)$ can be determined as follows. We have

$$\mathbf{\Gamma}_* = \begin{pmatrix} 1 & 0 \\ \alpha_0 & \gamma_{11} \end{pmatrix}.$$

Thus, for $n \geq 1$,

$$M_1(n) = (\alpha_0 \quad \gamma_{11})\mathbf{\Gamma}_*^{n-1}\begin{pmatrix} \alpha_0^{-1} \\ 0 \end{pmatrix} = (\alpha_0 \quad \gamma_{11})\begin{pmatrix} \alpha_0^{-1} \\ \sum_{i=0}^{n-2}\left(\prod_{j=0}^{i}\gamma_{11}^j\right) \end{pmatrix}$$

$$= \frac{1-\gamma_{11}^n}{1-\gamma_{11}}. \tag{27}$$

Furthermore, for $n \geq 3$,

$$M_2(n) = (\alpha_0 \quad \gamma_{11})\mathbf{\Gamma}_*^{n-3}\begin{pmatrix} 0 \\ \gamma_{11}\bar{\gamma}_{11} \end{pmatrix} = (\alpha_0 \quad \gamma_{11})\begin{pmatrix} 0 \\ \gamma_{11}^{n-2}\bar{\gamma}_{11} \end{pmatrix}$$

$$= \gamma_{11}^{n-1}\bar{\gamma}_{11} \tag{28}$$

with $M_2(1) = \bar{\gamma}_{11}$ and $M_2(2) = \gamma_{11}\bar{\gamma}_{11}$. In fact, $M_2(n) = \gamma_{11}^{n-1}\bar{\gamma}_{11}$ for $n \geq 1$. Inserting (27) and (28) into (26) yields

$$\rho_n = \frac{\nu_2\gamma_{11}^{n-1}[\bar{\gamma}_{11}(1-\gamma_{11}^2) - \nu_2\gamma_{11}(1-\gamma_{12})]}{\nu_4(1-\gamma_{11}^2) - \nu_2^2(1-\gamma_{12})}$$

$$= \frac{\alpha_1\nu_2\gamma_{11}^{n-1}(1-\beta_1^2-\alpha_1\beta_1\nu_2)}{1-\beta_1^2-2\alpha_1\beta_1\nu_2} = \rho_1\gamma_{11}^{n-1} \tag{29}$$

(see He and Teräsvirta, 1999a). Setting $\nu_2 = 1$ and $\nu_4 = 3$ in (29) yields the corresponding result for normal errors given in Bollerslev (1988). The decay rate of these autocorrelations is exponential from the first lag.

As a further illustration, we reproduce the autocorrelation function of $\{u_t^2\}$ in an ARCH(p) model and compare our result with that of Milhøj (1985) (see (13) and (14)). In an ARCH(p) model, $\beta_i = 0$, $i = 1, \ldots, p$. The autocorrelations thus have the form

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$$\rho_n = \frac{\nu_2\gamma_{S_1}\left(1-\nu_2\sum_{i=1}^{p}\alpha_i\right)M_2(n) - \nu_2^2\gamma_{S_2}\left[1-\left(1-\nu_2\sum_{i=1}^{p}\alpha_i\right)M_1(n)\right]}{\nu_4\gamma_{S_1}\left(1-\nu_2\sum_{i=1}^{p}\alpha_i\right) - \nu_2^2\gamma_{S_2}} \tag{30}$$

for $n \geq 1$. Further simplification of (30) gives the following corollary.

COROLLARY 3. *For the* ARCH(p) *model, the autocorrelation function of* $\{u_t^2\}$ *has the form*

$$\rho_n = 1 - \left(1-\nu_2\sum_{i=1}^{p}\alpha_i\right)M_1(n), \qquad n \geq 1. \tag{31}$$

Proof. See Appendix.

If the errors are assumed normal, then (31) gives the corresponding result, which is the one Milhøj (1985) obtained. As an example, for the ARCH(2) model it follows from (31), or (13) and (14), that $\rho_1 = \alpha_1(1-\alpha_2)^{-1}$, $\rho_2 = \alpha_2 + \alpha_1^2(1-\alpha_2)^{-1}$, and $\rho_n = \alpha_1\rho_{n-1} + \alpha_2\rho_{n-2}$ for $n \geq 2$. Alternatively, a general expression for ρ_n, $n \geq 2$ is obtained from Corollary 3.

4. CONCLUSIONS

We have obtained a complete characterization of the fourth moment structure of a general GARCH(p,q) process. With our results, an investigator is able to see what an estimated GARCH model implies about the second and the fourth moments, kurtosis, and the autocorrelation function of the centered and squared observations. Such considerations have previously been possible in the GARCH(1,1) case. These results can be extended to other GARCH processes that are generalizations of the original GARCH(p,q) process. For example, some GARCH processes allowing for asymmetric effects to shocks belong to this category. Those generalizations are a topic of further work.

We have not considered moments of higher than fourth order. Deriving those using the present techniques would no doubt be tedious. On the other hand, as underlined earlier, the fourth moments are probably more interesting in practice than any higher-order ones.

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APPENDIX

This Appendix contains notation and the proofs of Theorem 2, nine supporting lemmas, and three corollaries.

Notation. To derive the results of the paper we have to introduce some notation. Let $\mathbf{e}_r = (1,\ldots,1,0,\ldots,0)'$ be a $p \times 1$ vector with the first r components equal to 1. In particular, $\mathbf{e}_1 = (1,0,\ldots,0)'$. Furthermore, $\mathbf{c}_P = (c_{1,t-1},\ldots,c_{p,t-p})'$ is a $p \times 1$ vector with the index set $P = \{1,\ldots,p\}$, whereas $\mathbf{c}_{P\setminus\{n\}} = (c_{1,t-1},\ldots,c_{n-1,t-n+1},c_{n+1,t-n-1},\ldots,c_{p,t-p})'$ is a $(p-1) \times 1$ subvector of $\mathbf{c}_P$ obtained by excluding the element $c_{n,t-n}$ from $\mathbf{c}_P$. Matrix $\mathbf{I}_n$ is an $n \times n$ identity matrix, $\mathbf{0}_{m\times n}$ is an $m \times n$ null matrix, and $\mathbf{0}_m$ is an $m \times 1$ null vector. We also define $(p-1) \times (p-1)$ matrices $\mathbf{C}_1 = \mathbf{I}_{p-1}$ and

$$\mathbf{C}_j = \begin{pmatrix} \mathbf{c}'_{(P-1)\setminus\{n-j+1\}} & c_{p,t-j-p+1} \\ \mathbf{I}_{p-2} & \mathbf{0}_{p-2} \end{pmatrix} \tag{A.1}$$

for $j = 2,3,\ldots,n$, where $\mathbf{c}'_{(P-1)\setminus\{n-j+1\}} = (c_{1,t-j},\ldots,c_{n-1,t-j-n+2},c_{n+1,t-j-n},\ldots,c_{p-1,t-j-p+2})$. Let $\mathbf{G}_{p-j}[c_{j+1,t-(i+j)+1}]$ be a $(p-j) \times (p-j)$ matrix generated by the "seed" element $c_{j+1,t-(i+j)+1}$. It has the form

$$G_{p-j}[c_{j+1,t-(i+j)+1}] = \begin{pmatrix} \mathbf{c}'_{(P-1)\setminus\{1,\ldots,j\}} & c_{p,t-i-p+2} \\ c_{j,t-i-j+2}\mathbf{I}_{p-j-1} & \mathbf{0}_{p-j-1} \end{pmatrix}, \tag{A.2}$$

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where $\mathbf{c}_{(P-1)\backslash\{1,\ldots,j\}} = (c_{j+1,t-(i+j-1)}, c_{j+2,t-(i+j)}, \ldots, c_{p-1,t-(i+p-3)})'$ is a $(p-j-1)\times 1$ vector obtained by excluding the first j elements from $\mathbf{c}_{P-1}$. Let $p^* = p(p-1)/2$. Using (A.2), we define a $(p-1)\times p^*$ matrix $\mathbf{C}_{n+1}$ as

$$\mathbf{C}_{n+1} = \begin{pmatrix} \mathbf{G}_{p-1}[c_{2,t-n-2}] & \vdots & \mathbf{0}'_{p-2} & \vdots & \cdots & \vdots & \mathbf{0}_{(p-3)\times 2} & \vdots & \mathbf{0}_{p-2} \\ & \vdots & \mathbf{G}_{p-2}[c_{3,t-n-3}] & \vdots & \cdots & \vdots & \mathbf{G}_2[c_{p-1,t-n-p+1}] & \vdots & \mathbf{G}_1[c_{p,t-n-p}] \end{pmatrix}. \tag{A.3}$$

Note that in (A.3) the number of rows of zero elements in each block of columns grows from zero to $p-2$ whereas the number of columns decreases from $p-1$ to one. Moreover, for $i > n+1$,

$$\mathbf{C}_i = \begin{pmatrix} \mathbf{G}_{p-1}[c_{2,t-i-1}] & \vdots & \mathbf{0}'_{p-2} & \vdots & \cdots & \vdots & \mathbf{0}_{(p-3)\times 2} & \vdots & \mathbf{0}_{p-2} \\ & \vdots & \mathbf{G}_{p-2}[c_{3,t-i-2}] & \vdots & \cdots & \vdots & \mathbf{G}_2[c_{p-1,t-i-p+2}] & \vdots & \mathbf{G}_1[c_{p,t-i-p+1}] \\ \cdots\cdots & & \cdots\cdots & & \cdots & & \cdots\cdots & & \cdots\cdots \\ \mathbf{I}_{p-2}:\mathbf{0}_{p-2} & \vdots & \mathbf{0}_{(p-2)\times(p-2)} & \vdots & \cdots & \vdots & 0 & 0 & 0 \\ \mathbf{0}_{(p-3)\times(p-1)} & \vdots & \mathbf{I}_{p-3}:\mathbf{0}_{p-3} & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 & \vdots & \vdots \\ \mathbf{0}'_{p-1} & \vdots & \mathbf{0}'_{p-2} & \vdots & & \vdots & 1 & 0 & 0 \end{pmatrix}, \tag{A.4}$$

which is a $p^* \times p^*$ matrix.

This notation applies to $1 \le n \le p$. When $n > p$, we need more notation as follows. Let $\mathbf{c}^*_{p+1} = (\alpha_0, c_{1,t-1}, \ldots, c_{p,t-p})'$ be a $(p+1)\times 1$ vector. We define the $(p+1)\times(p+1)$ matrices $\mathbf{C}^*_1 = \mathbf{I}_{p+1}$ and

$$\mathbf{C}^*_j = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ \alpha_0 & c_{1,t-j} & c_{2,t-j-1} & \cdots & c_{p-1,t-j-p+2} & c_{p,t-j-p+1} \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \tag{A.5}$$

for $j = 2, \ldots, n$. Furthermore, using (A.2), we define the $(p+1)\times p^*$ matrix $\mathbf{C}^*_{n+1}$ as

$$\mathbf{C}^*_{n+1} = \begin{pmatrix} \mathbf{0}_{2\times(p-1)} & \vdots & \mathbf{0}_{3\times(p-2)} & \vdots & \cdots & \vdots & \mathbf{0}_{(p-1)\times 2} & \vdots & \mathbf{0}_{p\times 1} \\ \mathbf{G}_{p-1}[c_{2,t-n-2}] & \vdots & \mathbf{G}_{p-2}[c_{3,t-n-3}] & \vdots & \cdots & \vdots & \mathbf{G}_2[c_{p-1,t-n-p+1}] & \vdots & \mathbf{G}_1[c_{p,t-n-p}] \end{pmatrix}. \tag{A.6}$$

Finally, $\mathbf{C}^*_i = \mathbf{C}_i$ for $i > n+1$.

Correspondingly, let $\mathsf{E}\mathbf{c}_P = \boldsymbol{\gamma}_P = (\gamma_{11}, \ldots, \gamma_{p1})'$, $\mathsf{E}\mathbf{c}_{P\backslash\{m-l\}} = \boldsymbol{\gamma}_{P\backslash\{m-l\}} = (\gamma_{11}, \ldots, \gamma_{m-l-1}, \gamma_{m-l+1}, \ldots, \gamma_{p1})'$, $\mathsf{E}\mathbf{c}^*_{P+1} = \boldsymbol{\gamma}_{p+1} = (\alpha_0, \gamma_{11}, \ldots, \gamma_{p1})'$, $\mathsf{E}\mathbf{C}^*_i = \boldsymbol{\Gamma}_*$ for $i = 2, \ldots, n$, $\mathsf{E}\mathbf{C}^*_{n+1} = \boldsymbol{\Gamma}^*_{n+1}$ and $\mathsf{E}\mathbf{C}^*_i = \boldsymbol{\Gamma}$ for all $i > n+1$. We observe that here $\mathsf{E}\mathbf{C}^*_i (i = 2, \ldots, n+1)$ are quite different from $\mathsf{E}\mathbf{C}_j (j = 2, \ldots, m-l+1)$ defined in Section 2. However, when $i >$

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$n+1$ and $j > m-l+1$, $\mathsf{E}\mathbf{C}_i^* = \mathsf{E}\mathbf{C}_j = \mathbf{\Gamma}$. Thus the condition $\lambda(\mathbf{\Gamma}) < 1$ is required for expressing $\mathsf{E}(u_t^2 u_{t-n}^2)$ as a function of $\mathsf{E}h_t^2$ and $\mathsf{E}h_t$.

Finally, define $\tilde{\gamma}_{ij} = \mathsf{E}(c_{it}c_{jt})$ for $i < j$ and $\bar{\gamma}_{i1} = \mathsf{E}(\varepsilon_t^2 c_{it})$ for $i = 1, \ldots, p$.

LEMMA 1. *Let $n \geq 1$. For any $k \geq n+1$, $h_t h_{t-n}$ can be expressed by combinations of the terms of h_{t-i}, h_{t-j}^2, and $h_{t-i}h_{t-j}$ such that*

$$h_t h_{t-n} = S_{10} + \sum_{i=n+1}^{k-1} S_{1i} + S_{20} + \sum_{i=n+1}^{k-1} S_{2i} + S_k, \tag{A.7}$$

where, for $1 \leq n \leq p$,

$$S_{10} = \alpha_0 \left[\left(1 + \mathbf{c}'_{P\setminus\{n\}} \sum_{i=1}^{n-1} \left(\prod_{j=1}^{i} \mathbf{C}_j \right) \mathbf{e}_1 \right) h_{t-n} + \mathbf{c}'_{P\setminus\{n\}} \left(\prod_{i=1}^{n} \mathbf{C}_i \right) \mathbf{h}_{10t} \right],$$

$$S_{1i} = \alpha_0 \mathbf{c}'_{P\setminus\{n\}} \left(\prod_{j=1}^{i} \mathbf{C}_j \right) \mathbf{h}_{1it},$$

$$S_{20} = \left(c_{n,t-n} + \mathbf{c}'_{P\setminus\{n\}} \sum_{i=1}^{n-1} \left(\prod_{j=1}^{i} \mathbf{C}_j \right) \mathbf{e}_1 c_{n-i,t-n} \right) h_{t-n}^2 + \mathbf{c}'_{P\setminus\{n\}} \left(\prod_{i=1}^{n} \mathbf{C}_i \right) \mathbf{h}_{20t},$$

$$S_{2i} = \mathbf{c}'_{P\setminus\{n\}} \left(\prod_{j=1}^{i} \mathbf{C}_j \right) \mathbf{h}_{2it}, \qquad S_k = \mathbf{c}'_{P\setminus\{n\}} \left(\prod_{j=1}^{k} \mathbf{C}_j \right) \mathbf{h}_{kt}, \tag{A.8}$$

and for $n > p$,

$$S_{10} = \mathbf{c}_{p+1}^{*\prime} \left(\prod_{i=1}^{n} \mathbf{C}_i^* \right) \mathbf{h}_{10t}^*, \qquad S_{1i} = \alpha_0 \mathbf{c}_{p+1}^{*\prime} \left(\prod_{j=1}^{i} \mathbf{C}_j^* \right) \mathbf{h}_{1it},$$

$$S_{20} = \mathbf{c}_{p+1}^{*\prime} \left(\prod_{i=1}^{n} \mathbf{C}_i^* \right) \mathbf{h}_{20t}^*, \qquad S_{2i} = \mathbf{c}_{p+1}^{*\prime} \left(\prod_{j=1}^{i} \mathbf{C}_j^* \right) \mathbf{h}_{2it},$$

$$S_k = \mathbf{c}_{p+1}^{*\prime} \left(\prod_{j=1}^{k} \mathbf{C}_j^* \right) \mathbf{h}_{kt}. \tag{A.9}$$

In (A.8), $\mathbf{h}_{10t}$ *and* $\mathbf{h}_{20t}$ *are* $(p-1) \times 1$ *vectors given by*

$$\mathbf{h}_{10t} = (h_{t-(n+1)}, \ldots, h_{t-(n+p-1)})',$$

$$\mathbf{h}_{20t} = (c_{1,t-(n+1)} h_{t-(n+1)}^2, \ldots, c_{p-1,t-(n+p-1)} h_{t-(n+p-1)}^2)', \tag{A.10}$$

whereas in (A.9), $\mathbf{h}_{10t}^*$ *and* $\mathbf{h}_{20t}^*$ *are* $(p+1) \times 1$ *vectors given by*

$$\mathbf{h}_{10t}^* = (h_{t-n}, 0, \alpha_0 h_{t-(n+1)}, \ldots, \alpha_0 h_{t-(n+p-1)})',$$

$$\mathbf{h}_{20t}^* = (0, h_{t-n}^2, c_{1,t-(n+1)} h_{t-(n+1)}^2, \ldots, c_{p-1,t-(n+p-1)} h_{t-(n+p-1)}^2)'. \tag{A.11}$$

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Finally, $\mathbf{h}_{1it}$, $\mathbf{h}_{2it}$, *and* $\mathbf{h}_{kt}$ *are* $p^* \times 1$ *vectors given by*

$$\mathbf{h}_{1it} = (h_{t-i-1},\ldots,h_{t-(i+p-1)},0,\ldots,0)',$$

$$\mathbf{h}_{2it} = (c_{1,t-i-1}h_{t-i-1}^2,\ldots,c_{p-1,t-(i+p-1)}h_{t-(i+p-1)}^2,0,\ldots,0)',$$

$$\begin{aligned}\mathbf{h}_{kt} = (&h_{t-k}h_{t-(k+1)},\ldots,h_{t-k}h_{t-(k+p-1)},h_{t-(k+1)}h_{t-(k+2)},\ldots,h_{t-(k+1)}\\ &\times h_{t-(k+p-1)},\ldots,h_{t-(k+p-2)}h_{t-(k+p-1)})'. \end{aligned} \tag{A.12}$$

Proof. Let $1 \le n \le p$. Applying (3) to h_t in $h_t h_{t-n}$ yields

$$h_t h_{t-n} = \alpha_0 h_{t-n} + c_{n,t-n}h_{t-n}^2 + \mathbf{c}'_{P\setminus\{n\}}\mathbf{h}_{1t}h_{t-n}, \tag{A.13}$$

where $\mathbf{h}_{1t} = (h_{t-1},\ldots,h_{t-n+1},h_{t-n-1},\ldots,h_{t-p})'$. Applying (3) to h_{t-1} on the right-hand side of (A.13) and continuing the iteration until the appearance of the matrix $\mathbf{C}_{n+1}$ defined by (A.3) gives

$$\begin{aligned} h_t h_{t-n} = {}& \alpha_0 h_{t-n} + c_{n,t-n}h_{t-n}^2 + \alpha_0 \mathbf{c}'_{P\setminus\{n\}}\sum_{i=1}^{n-1}\left(\prod_{j=1}^{i}\mathbf{C}_j\right)\mathbf{e}_1 h_{t-n}\\ &+ \mathbf{c}'_{P\setminus\{n\}}\sum_{i=1}^{n-1}\left(\prod_{j=1}^{i}\mathbf{C}_j\right)\mathbf{e}_1 c_{n-i,t-n}h_{t-n}^2 + \alpha_0\mathbf{c}'_{P\setminus\{n\}}\prod_{i=1}^{n}\mathbf{C}_i\mathbf{h}_{10t}\\ &+ \mathbf{c}'_{P\setminus\{n\}}\prod_{i=1}^{n}\mathbf{C}_i\mathbf{h}_{20t} + \mathbf{c}'_{P\setminus\{n\}}\prod_{i=1}^{n+1}\mathbf{C}_i\mathbf{h}_{n+1,t}, \end{aligned} \tag{A.14}$$

where $\mathbf{C}_1,\ldots,\mathbf{C}_{n+1}$ are given by (A.1)–(A.3) and $\mathbf{h}_{n+1,t}$ has the form

$$\begin{aligned}\mathbf{h}_{n+1,t} = (&h_{t-n-1}h_{t-n-2},\ldots,h_{t-n-1}h_{t-n-p},h_{t-n-2}h_{t-n-3},\ldots,h_{t-n-2}\\ &\times h_{t-n-p},\ldots,h_{t-n-p+1}h_{t-n-p})'. \end{aligned} \tag{A.15}$$

In particular, $\mathbf{C}_{n+1}$ is a $(p-1)\times p^*$ matrix corresponding to the p^*-component column vector given in (A.15). When $i > n+1$, applying (3) to h_{t-i} in vector $\mathbf{h}_{it}$ on the left-hand side of (A.16) yields

$$\mathbf{h}_{it} = \alpha_0\mathbf{h}_{1,i+1,t} + \mathbf{h}_{2,i+1,t} + \mathbf{C}_{i+1}\mathbf{h}_{i+1,t}, \tag{A.16}$$

where $\mathbf{h}_{it}$, $\mathbf{h}_{i+1,t}$, $\mathbf{h}_{1,i+1,t}$, and $\mathbf{h}_{2,i+1,t}$ are defined in (A.12). Further recursions of (A.13) by applying (A.16) lead to equation (A.7).

Similarly, it can be shown that (A.7) and (A.9) hold when $n > p$. ■

LEMMA 2. *For* $1 \le l < m \le p$,

$$\lim_{k\to\infty}\mathsf{E}(c_{l,t-l}c_{m,t-m}S_k) = 0 \tag{A.17}$$

if and only if $\lambda(\mathbf{\Gamma}) < 1$.

Proof. Substituting $t-l$ for t and $m-l$ for n in Lemma 1 we obtain a recursion formula for $h_{t-l}h_{t-m}$, $l < m$. This technique will also be used in Lemmas 3–5. Let $k \ge p+m-l$. From (A.8),

$$\begin{aligned}\mathsf{E}(c_{l,t-l}c_{m,t-m}S_k) = {}&\gamma_{l1}\gamma_{m1}\mathsf{E}(\mathbf{c}'_{P\setminus\{m-l\}}\mathbf{C}_1\cdots\mathbf{C}_{m-l+1})\\ &\times\mathsf{E}(\mathbf{C}_{m-l+2}\cdots\mathbf{C}_{k-p+1})\mathsf{E}(\mathbf{C}_{k-p+2}\cdots\mathbf{C}_k\mathbf{h}_{kt}). \end{aligned} \tag{A.18}$$

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First, when $k \geq p + m - l$, the expectation of the product $\mathbf{C}_{k-p+2} \cdots \mathbf{C}_k \mathbf{h}_{kt}$ is not a function of k. Second, for $i, j > m - l + 1$, $i \neq j$, we have $\mathsf{E}(\mathbf{C}_i \mathbf{C}_j) = \mathsf{E}\mathbf{C}_i \mathsf{E}\mathbf{C}_j$. Thus, for any $k \geq p + m - l$,

$$\mathsf{E}(\mathbf{C}_{m-l+2} \cdots \mathbf{C}_{k-p+1}) = \mathbf{\Gamma}^{k-(p+m-l)}. \tag{A.19}$$

Equations (A.18) and (A.19) together imply that $\mathsf{E}(c_{l,t-l} c_{m,t-m} S_k) \to 0$ as $k \to \infty$, if and only if $\lambda(\mathbf{\Gamma}) < 1$. ■

LEMMA 3. *For* $1 \leq l < m \leq p$,

$$\mathsf{E}\left(c_{l,t-l} c_{m,t-m} \sum_{i=m-l+1}^{\infty} S_{1i} \right) = \alpha_0 \gamma_{l1} \gamma_{m1} \boldsymbol{\gamma}'_{P\backslash\{m-l\}} \left(\prod_{i=1}^{m-l+1} \mathbf{\Gamma}_i \right) (\mathbf{I}_{p^*} - \mathbf{\Gamma})^{-1} \mathbf{e}_{p-1} \mathsf{E}h_t \tag{A.20}$$

if and only if $\lambda(\mathbf{\Gamma}) < 1$, *where* $\mathbf{e}_{p-1} = (1, \ldots, 1, 0, \ldots, 0)'$ *is a* $p^* \times 1$ *vector with the first* $p - 1$ *elements equal to* 1.

Proof. By (A.8),

$$\begin{aligned}
&\mathsf{E}\left(c_{l,t-l} c_{m,t-m} \sum_{i=m-l+1}^{k-1} S_{1i} \right) \\
&\quad = \alpha_0 \gamma_{l1} \gamma_{m1} \boldsymbol{\gamma}'_{P\backslash\{m-l\}} \mathsf{E}(\mathbf{C}_1 \cdots \mathbf{C}_{m-l+1}) \left[\sum_{i=m-l+1}^{k-1} \mathsf{E}(\mathbf{C}_{m-l+2} \cdots \mathbf{C}_i) \right] \mathsf{E}\mathbf{h}_{1it} \\
&\quad = \alpha_0 \gamma_{l1} \gamma_{m1} \boldsymbol{\gamma}'_{P\backslash\{m-l\}} \mathbf{\Gamma}_1 \cdots \mathbf{\Gamma}_{m-l+1} \left(\sum_{i=m-l+1}^{k-1} \mathbf{\Gamma}^{i-(m-l+1)} \right) \mathbf{e}_{p-1} \mathsf{E}h_t.
\end{aligned} \tag{A.21}$$

Note that $\sum_{i=m-l+1}^{k-1} \mathbf{\Gamma}^{i-(m-l+1)} \to (\mathbf{I}_{p^*} - \mathbf{\Gamma})^{-1}$ as $k \to \infty$, if and only if $\lambda(\mathbf{\Gamma}) < 1$. Thus (A.20) is valid. ■

LEMMA 4. *For* $1 \leq l < m \leq p$,

$$\begin{aligned}
&\mathsf{E}\left(c_{l,t-l} c_{m,t-m} \sum_{i=m-l+1}^{\infty} S_{2i} \right) \\
&\quad = \gamma_{l1} \gamma_{m1} \sum_{i=1}^{p-1} \gamma(P\backslash\{m-l\}, 2, m-l+i-1) \mathsf{E}h_t^2 + \gamma_{l1} \gamma_{m1} \boldsymbol{\gamma}'_{P\backslash\{m-l\}} \left(\prod_{i=1}^{m-l+1} \mathbf{\Gamma}_i \right) \\
&\qquad \times (\mathbf{I}_{p^*} - \mathbf{\Gamma})^{-1} \boldsymbol{\gamma}(m-l+p+1, m-l+2p-1) \mathsf{E}h_t^2
\end{aligned} \tag{A.22}$$

if and only if $\lambda(\mathbf{\Gamma}) < 1$, *where for* $l < i \leq p$,

$$\gamma(P\backslash\{m-l\}, i-l-1, i) = \mathsf{E}(\mathbf{c}'_{P\backslash\{m-l\}} \mathbf{C}_{i-(l+1)} \cdots \mathbf{C}_i \mathbf{c}_{2it}) \tag{A.23}$$

with $\mathbf{c}_{2it} = (c_{1,t-(i+l)}, \ldots, c_{p-1,t-(i+l+p-2)}, 0, \ldots, 0)'$ *and*

$$\boldsymbol{\gamma}(i-j, i) = \mathsf{E}(\mathbf{C}_{i-j} \mathbf{C}_{i-(j-1)} \cdots \mathbf{C}_i \mathbf{c}_{2it}), \qquad j \leq p-2, \qquad i > m-l+2p-1. \tag{A.24}$$

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Proof. Note that $\mathbf{h}_{2it}$ is not stochastically independent of the matrix product $\mathbf{C}_{i-(p-2)} \cdots \mathbf{C}_i$. However, for each nonzero element of $\mathbf{h}_{2it}$, $c_{j,t-(i+j)}$ and $h^2_{t-(i+j)}$ are stochastically independent. From (A.24) we see that $\boldsymbol{\gamma}(i-j,i)$ is a $p^* \times 1$ column vector that is determined by the matrix product $\mathbf{C}_{i-j} \cdots \mathbf{C}_i$ and whose elements consist of $\tilde{\gamma}_{ij}$ for $i < j$.

Suppose that $k > m - l + p$. By (A.8),

$$\mathsf{E}\left(c_{l,t-l}c_{m,t-m}\sum_{i=m-l+1}^{k-1} S_{2i}\right)$$

$$= \gamma_{l1}\gamma_{m1}\sum_{i=m-l+1}^{m-l+p-1}\mathsf{E}S_{2i} + \gamma_{l1}\gamma_{m1}\boldsymbol{\gamma}'_{P\setminus\{m-l\}}\prod_{j=1}^{m-l+1}\boldsymbol{\Gamma}_j$$

$$\times\left[\sum_{i=m-l+p}^{k-1}\mathsf{E}(\mathbf{C}_{m-l+2}\cdots\mathbf{C}_{i-p+1})\mathsf{E}(\mathbf{C}_{i-p+2}\cdots\mathbf{C}_i\mathbf{h}_{2it})\right]. \tag{A.25}$$

Arguing as in Lemma 2, $\mathsf{E}(\mathbf{C}_{i-p+2}\cdots\mathbf{C}_i\mathbf{h}_{2it})$ does not depend on i for $i > m-l+2p-2$. Setting $i = m-l+p$ in (A.24) gives

$$\mathsf{E}(\mathbf{C}_{m-l+p+1}\cdots\mathbf{C}_{m-l+2p-1}\mathbf{c}_{2,m-l+p,t}) = \boldsymbol{\gamma}(m-l+p+1, m-l+2p-1).$$

Then, for any $i > m-l+2p-2$,

$$\mathsf{E}(\mathbf{C}_{i-p+2}\cdots\mathbf{C}_i\mathbf{c}_{2it}) = \boldsymbol{\gamma}(m-l+p+1, m-l+2p-1). \tag{A.26}$$

On the other hand, by (A.19),

$$\sum_{i=m-l+p}^{k-1}\mathsf{E}(\mathbf{C}_{m-l+2}\cdots\mathbf{C}_{i-p+1}) = \sum_{i=m-l+p}^{k-1}\boldsymbol{\Gamma}^{i-(m-l+p)},$$

which implies

$$\lim_{k\to\infty}\sum_{i=m-l+p}^{k-1}\boldsymbol{\Gamma}^{i-(m-l+p)} = (\mathbf{I}_{p_*} - \boldsymbol{\Gamma})^{-1} \tag{A.27}$$

if and only if $\lambda(\boldsymbol{\Gamma}) < 1$. Under this condition and applying (A.26), (A.27), and (A.23) to (A.25) we see that equation (A.22) holds. ■

LEMMA 5. *Assume that* $\lambda(\boldsymbol{\Gamma}) < 1$. *Under this condition*,

$$\mathsf{E}(c_{l,t-l}c_{m,t-m}h_{t-l}h_{t-m}) = \alpha_0\gamma_{l1}\gamma_{m1}M_1(l,m)\mathsf{E}h_t + \gamma_{l1}M_2(l,m)\mathsf{E}h_t^2, \tag{A.28}$$

where, *for* $m - l > 1$,

$$M_1(l,m) = 1 + \boldsymbol{\gamma}'_{P\setminus\{m-l\}}\left[\sum_{i=1}^{m-l-1}\left(\prod_{j=1}^{i}\boldsymbol{\Gamma}_j\right)\mathbf{e}_1 + \prod_{i=1}^{m-l}\boldsymbol{\Gamma}_i(\mathbf{j}_{p-1} + \boldsymbol{\Gamma}_{m-l+1}(\mathbf{I}_{p^*} - \boldsymbol{\Gamma})^{-1}\mathbf{e}_{p-1})\right] \tag{A.29}$$

and, *in particular*,

$$M_1(m-1,m) = 1 + \boldsymbol{\gamma}'_{P\setminus\{1\}}[\,\mathbf{j}_{p-1} + \boldsymbol{\Gamma}_2(\mathbf{I}_{p^*} - \boldsymbol{\Gamma})^{-1}\mathbf{e}_{p-1}], \tag{A.30}$$

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where $\mathbf{j}_{p-1} = (1,1,\ldots,1)'$ *is a* $(p-1)\times 1$ *vector. Furthermore,*

$$M_2(l,m) = M_{21}(l,m) + \gamma_{m1}\sum_{i=2}^{4} M_{2i}(l,m) \tag{A.31}$$

such that for $m - l > 1$

$$M_{21}(l,m) = \tilde{\gamma}_{m-l,m} + \boldsymbol{\gamma}'_{P\setminus\{m-l\}}\left[\sum_{i=1}^{m-l-1}\left(\prod_{j=1}^{i}\boldsymbol{\Gamma}_j\right)\mathbf{e}_1\tilde{\gamma}_{m-l-i,m}\right], \tag{A.32}$$

$$M_{22}(l,m) = \sum_{i=1}^{m-l-1}\gamma_{i1}M_{22}(m-l-i) + \sum_{j=m-l+1}^{p}\tilde{\gamma}_{j-m+l,j}, \tag{A.33}$$

and, for $m - l = 1$, $M_{21}(m-1,m) = \tilde{\gamma}_{1m}$ *and* $M_{22}(m-1,m) = \sum_{j=2}^{p}\tilde{\gamma}_{j-1,j}$. *In addition, in* (A.31), *for any* $l < m$,

$$M_{23}(l,m) = \sum_{i=m-l+1}^{m-l+p-1}\gamma(P\setminus\{m-l\},2,i-1), \tag{A.34}$$

$$M_{24}(l,m) = \boldsymbol{\gamma}'_{P\setminus\{m-l\}}\left(\sum_{j=1}^{m-l+1}\boldsymbol{\Gamma}_j\right)(\mathbf{I}_{p^*} - \boldsymbol{\Gamma})^{-1}\boldsymbol{\gamma}(m-l+p+1, m-l+2p-1). \tag{A.35}$$

Proof. From (A.8) we obtain that

$$\mathsf{E}(c_{l,t-l}c_{m,t-m}S_{10}) = \alpha_0\gamma_{l1}\gamma_{m1}\left[1 + \boldsymbol{\gamma}'_{P\setminus\{m-l\}}\sum_{i=1}^{m-l-1}\left(\prod_{j=1}^{i}\boldsymbol{\Gamma}_j\right)\mathbf{e}_1 + \prod_{i=1}^{m-l}\boldsymbol{\Gamma}_i\mathbf{j}_{p-1}\right]\mathsf{E}h_t, \tag{A.36}$$

provided $\mathbf{C}_{m-l}$ is not an identity matrix. Following Lemma 3 and equation (A.36) we may define the coefficients of $\mathsf{E}h_t$ for the sum of $\mathsf{E}(c_{l,t-l}c_{m,t-m}S_{10})$ and $\mathsf{E}(c_{l,t-l}c_{m,t-m}\sum_{i=m-l+1}^{\infty}S_{1i})$ as follows. When $m - l > 1$, $M_1(l,m)$ is defined by (A.29), and when $m - l = 1$, (A.29) simplifies to (A.30) because $\boldsymbol{\Gamma}_1 = \mathbf{I}_{p-1}$. Equation (A.29) is thus valid.

To evaluate $\mathsf{E}(c_{l,t-l}c_{m,t-m}S_{20})$, consider first

$$\mathsf{E}(c_{l,t-l}c_{m,t-m}c_{m-l,t-m}h_{t-m}^2) = \gamma_{l1}\tilde{\gamma}_{m-l,m}\mathsf{E}h_t^2. \tag{A.37}$$

Second, when $m - l > 1$,

$$\mathsf{E}\left[c_{l,t-l}c_{m,t-m}\mathbf{c}'_{P\setminus\{m-l\}}\sum_{i=1}^{m-l-1}\left(\prod_{j=1}^{i}\mathbf{C}_j\right)\mathbf{e}_1c_{m-l-i,t-m}h_{t-m}^2\right]$$
$$= \left[\gamma_{l1}\boldsymbol{\gamma}'_{P\setminus\{m-l\}}\sum_{i=2}^{m-l}\left(\prod_{j=2}^{i}\boldsymbol{\Gamma}_j\right)\mathbf{e}_1\tilde{\gamma}_{m-l-i,m}\right]\mathsf{E}h_t^2. \tag{A.38}$$

Combining (A.37) and (A.38) and observing (A.28) yields (A.32). Finally, if $m - l = 1$, then

$$\mathsf{E}(c_{l,t-l}c_{m,t-m}\mathbf{c}'_{P\setminus\{1\}}\mathbf{h}_{20t}) = \gamma_{l1}\gamma_{m1}\left(\sum_{j=2}^{p}\tilde{\gamma}_{j-1,j}\right)\mathsf{E}h_t^2. \tag{A.39}$$

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For $m - l = 2$,

$$\mathsf{E}(c_{l,t-l}c_{m,t-m}\mathbf{c}'_{P\backslash\{2\}}\mathbf{C}_1\mathbf{C}_2\mathbf{h}_{20t}) = \gamma_{l1}\gamma_{m1}\left(\gamma_{11}\sum_{j=2}^{p}\tilde{\gamma}_{j-1,j} + \sum_{j=3}^{p}\tilde{\gamma}_{j-2,j}\right)\mathsf{E}h_t^2. \tag{A.40}$$

Let $M_{22}(1) = M_{22}(m-1,m) = \sum_{j=2}^{p}\tilde{\gamma}_{j-1,j}$. We may then write the coefficients of $\mathsf{E}h_t^2$ in equation (A.40) as

$$M_{22}(2) = M_{22}(m-2,m) = \gamma_{11}M_{22}(1) + \sum_{j=3}^{p}\tilde{\gamma}_{j-2,j}. \tag{A.41}$$

Continuing the recursion for $m - l = 3,\ldots,p-1$ proves (A.33).

Next, (A.34) and (A.35) follow from Lemma 4. Letting $k \to \infty$ in

$$\begin{aligned}\mathsf{E}(c_{l,t-l}c_{t-m}h_{t-l}h_{t-m}) &= \mathsf{E}(c_{l,t-l}c_{m,t-m}S_{10}) + \mathsf{E}(c_{l,t-l}c_{m,t-m}S_{20}) + \sum_{i=m-l+1}^{k-1}\mathsf{E}(c_{l,t-l}c_{m,t-m}S_{1i}) \\ &\quad + \sum_{i=m-l+1}^{k-1}\mathsf{E}(c_{l,t-l}c_{m,t-m}S_{2i}) + \mathsf{E}(c_{l,t-l}c_{m,t-m}S_k)\end{aligned} \tag{A.42}$$

gives

$$\begin{aligned}\mathsf{E}(c_{l,t-l}c_{m,t-m}h_{t-l}h_{t-m}) &= \alpha_0\gamma_{l1}\gamma_{m1}M_1(l,m)\mathsf{E}h_t \\ &\quad + \gamma_{l1}\left[M_{21}(l,m) + \gamma_{m1}\sum_{i=2}^{4}M_{2i}(l,m)\right]\mathsf{E}h_t^2,\end{aligned} \tag{A.43}$$

which, given definition (A.31), equals (A.28). This completes the proof. ■

Proof of Corollary 1. In the GARCH(2,2) model, $\mathbf{\Gamma} = \gamma_{21}$. Let $\gamma_{21} < 1$. Thus, applying Lemma 5,

$$\mathsf{E}(c_{1,t-1}c_{2,t-2}h_{t-1}h_{t-2}) = \alpha_0\gamma_{11}\gamma_{21}M_1(1,2)\mathsf{E}h_t + \gamma_{11}M_2(1,2)\mathsf{E}h_t^2$$

holds, where $M_1(1,2) = (1-\gamma_{21})^{-1}$ and $M_2(1,2) = \tilde{\gamma}_{12}(1-\gamma_{21})^{-1}$, whereas $\tilde{\gamma}_{12} = \mathsf{E}(c_{1t}c_{2t})$. Note that $m - l = 1$ and $\mathbf{\Gamma}_2 = \gamma_{21}$. By (A.29),

$$M_1(1,2) = 1 + \gamma_{21}[1 + \gamma_{21}(1-\gamma_{21})^{-1}] = (1-\gamma_{21})^{-1}.$$

Furthermore, by (A.32)–(A.35), $M_{21}(1,2) = M_{22}(1,2) = \tilde{\gamma}_{12}$, $M_{23}(1,2) = \gamma(P\backslash\{1\},2) = \mathsf{E}(c_{2,t-3}c_{2,t-4}c_{1,t-4}) = \gamma_{21}\tilde{\gamma}_{12}$, and $M_{24}(1,2) = \gamma_{21}\mathbf{\Gamma}_1\mathbf{\Gamma}_2(\mathbf{I}_1 - \mathbf{\Gamma})^{-1}\tilde{\gamma}_{12} = \gamma_{21}^2\tilde{\gamma}_{12}(1-\gamma_{21})^{-1}$. Finally, from (A.31),

$$M_2(1,2) = \tilde{\gamma}_{12}(1-\gamma_{21})^{-1}.$$

It follows from Theorem 1 that (8) and (9) hold for the GARCH(2,2) model. ■

LEMMA 6. *Let* $1 \le n \le p$. *Then*,

$$\mathsf{E}(u_t^2u_{t-n}^2) = \alpha_0\nu_2^2M_1(n)\mathsf{E}h_t + \nu_2M_2(n)\mathsf{E}h_t^2, \tag{A.44}$$

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where

$$M_2(n) = M_{21}(n) + \nu_2 \sum_{i=2}^{4} M_{2i}(n) \tag{A.45}$$

such that for $n = 2,\dots,p$

$$M_{21}(n) = \bar{\gamma}_{n1} + \boldsymbol{\gamma}'_{P\setminus\{n\}}\left[\sum_{i=1}^{n-1}\left(\prod_{j=1}^{i}\boldsymbol{\Gamma}_j\right)\mathbf{e}_1\bar{\gamma}_{n-i,1}\right], \tag{A.46}$$

$$M_{22}(n) = \sum_{i=1}^{n-1}\gamma_{i1}M_{22}(n-i) + \sum_{j=n+1}^{p}\tilde{\gamma}_{j-n,j}. \tag{A.47}$$

Furthermore, for $n = 1$, $M_{21}(1) = \bar{\gamma}_{11}$ *and* $M_{22}(1) = \sum_{j=2}^{p}\tilde{\gamma}_{j-1,j}$. *In addition, for* $n = 1,\dots,p$,

$$M_{23}(n) = \sum_{i=n+1}^{n+p-1}\gamma(P\setminus\{n\},2,i-1) \tag{A.48}$$

and

$$M_{24}(n) = \boldsymbol{\gamma}'_{P\setminus\{n\}}\left[\left(\prod_{j=1}^{n+1}\boldsymbol{\Gamma}_j\right)(\mathbf{I}_{p_*} - \boldsymbol{\Gamma})^{-1}\right]\boldsymbol{\gamma}(n+p+1, n+2p-1). \tag{A.49}$$

Proof. Let $M_i(n) = M_i(0,n)$, $i = 1,2$. Then (A.44)–(A.49) follow from results in Lemma 5 for $1 \le n \le p$. ■

LEMMA 7. *For* $n > p$,

$$\begin{aligned}&\mathsf{E}(\varepsilon_{t-n}^2\mathbf{C}_{n-(p-1)}^*\mathbf{C}_{n-(p-2)}^*\cdots\mathbf{C}_n^*\mathbf{c}_{20t}^*)\\ &\quad= (0, M_{21}(p) + \nu_2 M_{22}(p), M_{21}(p-1) + \nu_2 M_{22}(p-1),\dots, M_{21}(1) + \nu_2 M_{22}(1))',\end{aligned} \tag{A.50}$$

where $\mathbf{c}_{20t}^* = (0,1,c_{1,t-(n+1)},\dots,c_{p-1,t-(n+p-1)})'$.

Proof. Note that whereas $c_{i,t-(n+j)}$ and $h_{t-(n+j)}^2$ are stochastically independent, the product $\mathbf{C}_{n-(p-1)}^*\mathbf{C}_{n-(p-2)}^*\cdots\mathbf{C}_n^*$ is not stochastically independent of $\mathbf{h}_{20t}^*$. This is because there is an element such as $c_{i,t-n}$ that is a function of ε_{t-n}^2 for some n on the second row of each matrix, say, $\mathbf{C}_{n-j}^*$, $j = 0,1,\dots,p-1$. Define the $(p+1)\times 1$ vector

$$\mathbf{h}(n) = (0,\mathsf{E}(\varepsilon_{t-n}^2 h_2(n)),\dots,\mathsf{E}(\varepsilon_{t-n}^2 h_{p+1}(n)))'.$$

First, we show that for $n > p$,

$$\mathsf{E}(\varepsilon_{t-n}^2\mathbf{C}_{n-(p-1)}^*\mathbf{C}_{n-(p-2)}^*\cdots\mathbf{C}_n^*\mathbf{c}_{20t}^*) = \mathbf{h}(n), \tag{A.51}$$

where, for $i = 2,\dots,p+1$,

$$h_i(n) = \sum_{j=2}^{p+1}\phi_{ij}(n)c_{j-2,t-(n+j-2)}. \tag{A.52}$$

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In (A.52), $c_{0,t-n}=1$ and $\phi_{ij}(n)$ are determined recursively with respect to integers $m=n-(p-1), n-(p-2),\ldots,n$, such that

$$\phi_{ij}(m)=\begin{cases}\phi_{i2}(m-1)c_{j-1,t-(m-j+2)}+\phi_{i,j+1}(m-1) & \text{for } j=2,\ldots,p\\ \phi_{i2}(m-1)c_{p,t-(m+p-1)} & \text{for } j=p+1\end{cases} \tag{A.53}$$

with the initial values $\phi_{ij}(n-(p-1))=c_{ij}$ for any i and j, where c_{ij} is the (i,j)th element of matrix $\mathbf{C}^*_{n-(p-1)}$.

Let $\Pi^n_{m=n-(p-1)}\mathbf{C}^*_m=\mathbf{\Phi}_n$, where $\mathbf{C}^*_m$ are given by (A.5). Because each $\mathbf{C}^*_m$ has the same first row $(1,0,\ldots,0)$, we can define

$$\mathbf{\Phi}_n=\begin{pmatrix}1 & 0 & \ldots & 0\\ \phi_{21}(n) & \phi_{22}(n) & \ldots & \phi_{2,p+1}(n)\\ \vdots & \vdots & & \vdots\\ \phi_{p+1,1}(n) & \phi_{p+1,2}(n) & \ldots & \phi_{p+1,p+1}(n)\end{pmatrix} \tag{A.54}$$

and rewrite (A.54) as

$$\mathbf{\Phi}_n=\mathbf{\Phi}_{n-1}\mathbf{C}^*_n. \tag{A.55}$$

Equalities between the elements of the matrices on both sides of (A.55) appear in (A.53). We see that $\mathsf{E}(\varepsilon^2_{t-n}\mathbf{\Phi}_n\mathbf{c}^*_{20t})$ can be expressed in terms of $h_i(n)$ defined by (A.52) and (A.53).

Next, let $n=p+1$. Then by (A.51),

$$\mathsf{E}(\varepsilon^2_{t-(p+1)}\mathbf{C}^*_2\mathbf{C}^*_3\cdots\mathbf{C}^*_{p+1}\mathbf{c}^*_{20t})=\mathbf{h}(p+1), \tag{A.56}$$

where $\mathbf{c}^*_{20t}=(0,1,c_{1,t-(p+2)},\ldots,c_{p-1,t-2p})'$. For any $n\geq p+1$, it follows from (A.51) with (A.52) and (A.53) that

$$\mathsf{E}(\varepsilon^2_{t-n}\mathbf{C}^*_{n-(p-1)}\mathbf{C}^*_{n-(p-2)}\cdots\mathbf{C}^*_n\mathbf{c}^*_{20t})=\mathbf{h}(p+1). \tag{A.57}$$

Finally, without loss of generality, we merely show that (A.50) is true for $p=2$. From (A.56),

$$\mathsf{E}\left(\varepsilon^2_{t-3}\mathbf{C}^*_2\mathbf{C}^*_3\begin{pmatrix}0\\1\\c_{1,t-4}\end{pmatrix}\right)=\begin{pmatrix}0\\ \gamma_{11}(\bar{\gamma}_{11}+\nu_2\tilde{\gamma}_{12})+\bar{\gamma}_{21}\\ \bar{\gamma}_{11}+\nu_2\tilde{\gamma}_{12}\end{pmatrix}, \tag{A.58}$$

where $\bar{\gamma}_{11}=\mathsf{E}(\varepsilon^2_t c_{1t})$, $\bar{\gamma}_{21}=\mathsf{E}(\varepsilon^2_t c_{2t})$, and $\tilde{\gamma}_{12}=\mathsf{E}(c_{1t}c_{2t})$. On the other hand, by equations (A.46) and (A.47),

$$M_{21}(1)+\nu_2 M_{22}(1)=\bar{\gamma}_{11}+\nu_2\tilde{\gamma}_{12},$$

$$M_{21}(2)+\nu_2 M_{22}(2)=\bar{\gamma}_{21}+\gamma_{11}(\bar{\gamma}_{11}+\nu_2\tilde{\gamma}_{12}).$$

Lemma 7 implies that for $n>p$, $\mathsf{E}(\varepsilon^2_{t-n}\mathbf{C}^*_{n-(p-1)}\mathbf{C}^*_{n-(p-2)}\cdots\mathbf{C}^*_n\mathbf{h}^*_{20t})$ does not depend on n. ■

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LEMMA 8. *Define the mixed moment* $\mathsf{E}(u_t^2 u_{t-n}^2)$ *as in* (*A*.44) *and let* $n > p$. *Then, for* $n \geq p+1$,

$$M_2(n) = \boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^{n-(p+1)} \mathbf{m}_2, \tag{A.59}$$

where $\mathbf{m}_2 = (0, M_2(p), \ldots, M_2(1))'$ *is a* $(p+1)\times 1$ *vector and* $M_2(i)$, $i = 1, \ldots, p$ *are given by* (*A*.45)–(*A*.49).

Proof. First, for any $n > p$, we show that the coefficient of $\mathsf{E}h_t^2$ in (A.44) is a function of n. By (A.9),

$$\begin{aligned}\sum_{i=n+1}^{\infty} \mathsf{E}(\varepsilon_t^2 \varepsilon_{t-n}^2 S_{2i}) &= \nu_2^2 \boldsymbol{\gamma}_{p+1}' \sum_{i=n+1}^{n+p-1} \boldsymbol{\gamma}(2,i)\mathsf{E}h_t^2 \\ &\quad + \nu_2^2 \boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^{n-1} \boldsymbol{\Gamma}_{n+1}^* (\mathbf{I}_{p^*} - \boldsymbol{\Gamma})^{-1} \boldsymbol{\gamma}(n+p, n+2p-2)\mathsf{E}h_t^2.\end{aligned} \tag{A.60}$$

Set $n = p+1$. Then (A.60) simplifies to

$$\begin{aligned}\sum_{i=p+2}^{\infty} \mathsf{E}(\varepsilon_t^2 \varepsilon_{t-(p+1)}^2 S_{2i}) &= \nu_2^2 \boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^2 \sum_{i=1}^{p-1} \boldsymbol{\Gamma}_*^{i-1} \boldsymbol{\gamma}(3+i, p+1+i)\mathsf{E}h_t^2 \\ &\quad + \nu_2^2 \boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^{n-1} \boldsymbol{\Gamma}_{n+1}^* (\mathbf{I}_{p^*} - \boldsymbol{\Gamma})^{-1} \boldsymbol{\gamma}(2p+1, 3p-1)\mathsf{E}h_t^2.\end{aligned} \tag{A.61}$$

For any $n > p$, $\boldsymbol{\gamma}(2p+1, 3p-1)$ in (A.61) remains unchanged. Thus

$$\begin{aligned}\sum_{i=n+1}^{\infty} \mathsf{E}(\varepsilon_t^2 \varepsilon_{t-n}^2 S_{2i}) &= \nu_2^2 \boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^{n-(p-1)} \sum_{i=1}^{p-1} \boldsymbol{\Gamma}_*^{i-1} \boldsymbol{\gamma}(3+i, p+1+i)\mathsf{E}h_t^2 \\ &\quad + \nu_2^2 \boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^{n-1} \boldsymbol{\Gamma}_{n+1}^* (\mathbf{I}_{p^*} - \boldsymbol{\Gamma})^{-1} \boldsymbol{\gamma}(2p+1, 3p-1)\mathsf{E}h_t^2 \\ &= \nu_2^2 M_{23}(n) + \nu_2^2 M_{24}(n), \qquad n > p.\end{aligned} \tag{A.62}$$

Furthermore, by (A.9) and Lemma 7 we can write, for any $n > p$,

$$\mathsf{E}(\varepsilon_{t-n}^2 S_{20}) = (\boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^{n-(p+1)} \mathbf{m}_1)\mathsf{E}h_t^2 = (M_{21}(n) + \nu_2 M_{22}(n))\mathsf{E}h_t^2, \tag{A.63}$$

where $\mathbf{m}_1 = (0, M_{21}(p) + \nu_2 M_{22}(p), \ldots, M_{21}(1) + \nu_2 M_{22}(1))'$. It follows from (A.62) and (A.63) that $M_2(n) = M_{21}(n) + \nu_2 \sum_{i=2}^4 M_{2i}(n)$ holds for any $n > p$.

It remains to show that (A.59) holds for any $n > p$. Without loss of generality, we prove that it holds for $p = 2$. For proof, see Proof of Corollary 2. ■

LEMMA 9. *The mixed moment* $\mathsf{E}(u_t^2 u_{t-n}^2)$ *has the form*

$$\mathsf{E}(u_t^2 u_{t-n}^2) = \alpha_0 \nu_2^2 M_1(n)\mathsf{E}h_t + \nu_2 M_2(n)\mathsf{E}h_t^2, \tag{A.64}$$

where for $n \geq 1$,

$$M_1(n) = \boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^{n-1} [\mathbf{e}_{\alpha_0} + \boldsymbol{\Gamma}_{n+1}^* (\mathbf{I}_{p^*} - \boldsymbol{\Gamma})^{-1} \mathbf{e}_{p-1}] \tag{A.65}$$

with $\mathbf{e}_{\alpha_0} = (\alpha_0^{-1}, 0, 1, \ldots, 1)'$ *is a* $(p+1)\times 1$ *vector. The term* $M_2(n)$ *in* (*A*.64) *is defined by Lemma* 6 *for* $n \leq p$; *otherwise* $M_2(n)$ *is given by Lemma* 8.

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Proof. It now follows from Lemma 1 that for any n,

$$\mathrm{E}(u_t^2 u_{t-n}^2) = \mathrm{E}(\varepsilon_t^2 \varepsilon_{t-n}^2 S_{10}) + \mathrm{E}(\varepsilon_t^2 \varepsilon_{t-n}^2 S_{20}) + \sum_{i=n+1}^{k-1} \mathrm{E}(\varepsilon_t^2 \varepsilon_{t-n}^2 S_{1i}) + \sum_{i=n+1}^{k-1} \mathrm{E}(\varepsilon_t^2 \varepsilon_{t-n}^2 S_{2i}) + \mathrm{E}(\varepsilon_t^2 \varepsilon_{t-n}^2 S_k). \tag{A.66}$$

Arguing as in Section 2, the limit of (A.66) as $k \to \infty$ exists and is independent of t if and only if all the eigenvalues of $\mathbf{\Gamma}$ lie inside the unit circle. As for the first two terms, when $n \leq p$ we may apply Lemma 6; otherwise we apply Lemmas 7 and 8.

Note that only the second column of $\mathbf{\Phi}_n = \Pi_{i=n-(p-1)}^{n} \mathbf{C}_i^*$ depends on variables ε_{t-n}^2. Thus,

$$\mathrm{E}(\varepsilon_t^2 \varepsilon_{t-n}^2 \mathbf{c}_{p+1}^{*\prime} \mathbf{C}_1^* \cdots \mathbf{C}_n^* \mathbf{h}_{10t}^*) = (\alpha_0 \nu_2^2 \boldsymbol{\gamma}_{p+1}^{\prime} \mathbf{\Gamma}_*^{n-1} \mathbf{e}_{\alpha_0}) \mathrm{E} h_t \tag{A.67}$$

and

$$\sum_{i=n+1}^{\infty} \mathrm{E}(\varepsilon_t^2 \varepsilon_{t-n}^2 S_{1i}) = (\alpha_0 \nu_2^2 \boldsymbol{\gamma}_{p+1}^{\prime} \mathbf{\Gamma}_*^{n-1} \mathbf{\Gamma}_{n+1}^* (\mathbf{I}_{p^*} - \mathbf{\Gamma})^{-1} \mathbf{e}_{p-1}) \mathrm{E} h_t. \tag{A.68}$$

Expression (A.65) follows directly from (A.67) and (A.68). ■

Proof of Theorem 2. To derive the autocorrelation function of $\{u_t^2\}$ we need some notation. Let $\gamma_{M_1}(l,m) = (1-\Delta) M_1(l,m)$ and $\gamma_{M_2}(l,m) = (1-\Delta) M_2(l,m)$, where $1 - \Delta = |\mathbf{I}_{p^*} - \mathbf{\Gamma}|$ is the determinant of $(\mathbf{I}_{p^*} - \mathbf{\Gamma})$ and $M_i(l,m)$, $i = 1,2$ are defined as in Lemma 5. Furthermore, let

$$\gamma_{S_1} = (1+\gamma_1)(1-\Delta) + 2 \sum_{l<m} \gamma_{l1} \gamma_{m1} \gamma_{M_1}(l,m), \tag{A.69}$$

$$\gamma_{S_2} = (1-\gamma_2)(1-\Delta) - 2 \sum_{l<m} \gamma_{l1} \gamma_{M_2}(l,m). \tag{A.70}$$

By applying Theorem 1 and Lemmas 6–9 to equation (16) a straightforward calculation yields (17). ■

Proof of Corollary 2. Here we only illustrate how to calculate $M_1(n)$ and $M_2(n)$ for the GARCH(2,2) model. Write

$$\mathbf{\Gamma}_* = \mathrm{E}\mathbf{C}_i^* = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_0 & \gamma_{11} & \gamma_{12} \\ 0 & 1 & 0 \end{pmatrix}$$

for $i = 2, \ldots, n$, where $\mathbf{C}_i^*$ are defined by (A.5). By (A.65),

$$\begin{aligned} M_1(n) &= \boldsymbol{\gamma}_3^{\prime} \mathbf{\Gamma}_*^{n-1} [(\alpha_0^{-1} \quad 0 \quad 1)^{\prime} + (1-\gamma_{21})^{-1} (0 \quad 0 \quad \gamma_{21})^{\prime}] \\ &= \boldsymbol{\gamma}_3^{\prime} \mathbf{\Gamma}_*^{n-1} (\alpha_0^{-1} \quad 0 \quad (1-\gamma_{21})^{-1})^{\prime} \end{aligned} \tag{A.71}$$

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for $n = 1,2,\ldots$. By (A.46)–(A.49),

$$M_{21}(1) = \bar{\gamma}_{11}, \qquad M_{22}(1) = \tilde{\gamma}_{12},$$

$$M_{23}(1) = \gamma(P\backslash\{1\}1,2) = \mathsf{E}(c_{2,t-3}c_{2,t-4}c_{1,t-4}) = \gamma_{21}\tilde{\gamma}_{12},$$

$$M_{24}(1) = \gamma_{21}\boldsymbol{\Gamma}_2(1-\gamma_{21})^{-1}\tilde{\gamma}_{12} = \gamma_{21}^2\tilde{\gamma}_{12}(1-\gamma_{21})^{-1}.$$

Thus, from (A.45),

$$M_2(1) = (1-\gamma_{21})^{-1}[\bar{\gamma}_{11}(1-\gamma_{21}) + \nu_2\tilde{\gamma}_{12}]. \tag{A.72}$$

From (A.71) it follows that $M_1(1) = (1-\gamma_{21})^{-1}$. Similarly,

$$M_{21}(2) = \gamma_{11}\bar{\gamma}_{11} + \bar{\gamma}_{21},$$

$$M_{22}(2) = \gamma_{11}\tilde{\gamma}_{12},$$

$$M_{23}(2) = \gamma(P\backslash\{2\},2,3) = \mathsf{E}(c_{1,t-1}c_{2,t-3}c_{2,t-4}c_{1,t-4}) = \gamma_{11}\gamma_{21}\tilde{\gamma}_{12},$$

$$M_{24}(2) = \gamma_{11}\boldsymbol{\Gamma}_2\boldsymbol{\Gamma}_3(1-\gamma_{21})^{-1}\tilde{\gamma}_{12} = \gamma_{11}\gamma_{21}^2\tilde{\gamma}_{12}(1-\gamma_{21})^{-1}.$$

Thus, by (A.45),

$$M_2(2) = (1-\gamma_{21})^{-1}[(\gamma_{11}\bar{\gamma}_{11} + \bar{\gamma}_{21})(1-\gamma_{21}) + \nu_2\gamma_{11}\tilde{\gamma}_{12}] \tag{A.73}$$

and, from (A.71),

$$M_1(2) = (1+\gamma_{11}-\gamma_{21})(1-\gamma_{21})^{-1}.$$

For any $n \geq 3$, by Lemma 7,

$$M_{21}(n) + \nu_2 M_{22}(n) = \boldsymbol{\gamma}_3'\boldsymbol{\Gamma}_*^{n-3}\begin{pmatrix} 0 \\ M_{21}(2) + \nu_2 M_{22}(2) \\ M_{21}(1) + \nu_2 M_{22}(1) \end{pmatrix}$$

and by (A.62),

$$M_{23}(n) = \boldsymbol{\gamma}_3'\boldsymbol{\Gamma}_*^{n-1}\boldsymbol{\gamma}(4) = \boldsymbol{\gamma}_3'\boldsymbol{\Gamma}_*^{n-1}\mathsf{E}\begin{pmatrix} 0 \\ 0 \\ c_{2,t-5} \end{pmatrix}c_{1,t-5}$$

$$= \boldsymbol{\gamma}_3'\boldsymbol{\Gamma}_*^{n-1}(0, \quad 0, \quad \tilde{\gamma}_{12})'$$

$$M_{24}(n) = \boldsymbol{\gamma}_3'\boldsymbol{\Gamma}_*^{n-1}(0, \quad 0, \quad \gamma_{21})'\left(\frac{\tilde{\gamma}_{12}}{1-\gamma_{21}}\right).$$

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Therefore, for $n \geq 3$, applying (A.62) and (A.63) yields

$$M_2(n) = \boldsymbol{\gamma}_3' \boldsymbol{\Gamma}_*^{n-3} \left[\begin{pmatrix} 0 \\ M_{21}(2) + \nu_2 M_{22}(2) \\ M_{21}(1) + \nu_2 M_{22}(1) \end{pmatrix} + \nu_2 \boldsymbol{\Gamma}_*^2 \begin{pmatrix} 0 \\ 0 \\ \dfrac{\tilde{\gamma}_{12}}{1-\gamma_{21}} \end{pmatrix} \right]$$

$$= \boldsymbol{\gamma}_3' \boldsymbol{\Gamma}_*^{n-3} \begin{pmatrix} 0 \\ \dfrac{\gamma_{11}[\bar{\gamma}_{11}(1-\gamma_{21}) + \nu_2 \tilde{\gamma}_{12}] + \bar{\gamma}_{21}(1-\gamma_{21})}{1-\gamma_{21}} \\ \dfrac{\bar{\gamma}_{11}(1-\gamma_{21}) + \nu_2 \tilde{\gamma}_{12}}{1-\gamma_{21}} \end{pmatrix}. \tag{A.74}$$

Finally, by (A.72) and (A.73),

$$M_2(n) = \boldsymbol{\gamma}_3' \boldsymbol{\Gamma}_*^{n-3} (0, \quad M_2(2), \quad M_2(1))', \qquad n \geq 3. \tag{A.75}$$

■

Proof of Corollary 3. By Theorem 2, the autocorrelation function ρ_n for the squared process has the form (30) for $n \geq 1$. Showing that Corollary 3 is true is equivalent to demonstrating that

$$M_2(n) = (\nu_4/\nu_2)\left(1 - \left(1 - \nu_2 \sum_{i=1}^{p} \alpha_i\right) M_1(n)\right)$$

holds. Then we can show the validity of (31) by (30).

Without loss of generality, we only investigate the case $p = 2$. It follows from Corollary 2 that

$$M_2(i) = (\nu_4/\nu_2)(1 - (1 - \nu_2\alpha_1 - \nu_2\alpha_2) M_1(i)) \tag{A.76}$$

for $i = 1,2$. We have to prove that (A.76) holds for any $n \geq 3$. From (20) and (21) we see that this is equivalent to showing that

$$(\alpha_0 \quad \nu_2\alpha_1 \quad \nu_2\alpha_2) \boldsymbol{\Gamma}_*^k \boldsymbol{\alpha}_* = 1 \tag{A.77}$$

for any k, where $\boldsymbol{\alpha}_* = (\alpha_0^{-1}(1 - \nu_2\alpha_1 - \nu_2\alpha_2), 1, 1)'$. That (A.77) holds follows from the fact that

$$(\nu_4/\nu_2) \begin{pmatrix} 0 \\ M_2(2) \\ M_2(1) \end{pmatrix} + \boldsymbol{\Gamma}_*^2 \begin{pmatrix} \alpha_0^{-1}(1 - \nu_2\alpha_1 - \nu_2\alpha_2) \\ 0 \\ (1 - \nu_2\alpha_1 - \nu_2\alpha_2)(1 - \nu_2\alpha_2)^{-1} \end{pmatrix} = \boldsymbol{\alpha}_*.$$

Note that $\boldsymbol{\Gamma}_*^k \boldsymbol{\alpha}_* = \boldsymbol{\alpha}_*$ for any k and $(\alpha_0 \quad \nu_2\alpha_1 \quad \nu_2\alpha_2)\boldsymbol{\alpha}_* = 1$. This completes the proof. ■

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