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A new class of rank-one transformations with singular spectrum

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Dedicated to Professor J. De Sam Lazaro

Abstract. We introduce a new tool to study the spectral type of rank-one transformations using the method of the central limit theorem for trigonometric sums. We get some new applications.

1. *Introduction*

The purpose of this paper is to introduce a new tool to the study of the spectral type of rank-one transformations. Rank-one transformations have simple spectrum and Ornstein [**Or**], using a random procedure, produced a family of mixing rank-one transformations. It follows that Ornstein's class of transformations may possibly contain a candidate for Banach's well-known problem on whether there exists a dynamical system $(\Omega, \mathcal{A}, \mu, T)$ with simple Lebesgue spectrum. However, in 1993, Bourgain [**Bo**] proved that almost surely Ornstein's transformations have singular spectrum. Subsequently, using the same method, Klemes [**Kl**] and Klemes and Reinhold [**KR**] obtained that mixing staircase transformations of Adams [**Ad**] and Adams and Friedman [**AF**] have singular spectrum. They conjecture that all rank-one transformations have singular spectrum.

Here we shall exhibit a new class of rank-one transformations with singular spectrum. Our assumptions include a new class of Ornstein transformations and a class of Creutz–Silva rank-one transformations [**CS**]. Our proof is based on techniques introduced by Bourgain [**Bo**] in the context of rank-one transformations and developed by Klemes [**Kl**], Klemes and Reinhold [**KR**], and Dooley and Eigen [**DE**], together with some ideas from the proof of the central limit theorem for trigonometric sums. The fundamental key, as noted by Klemes [**Kl**], is the estimation of the L^1-norm of a certain trigonometric polynomial $(|P_m|^2 - 1)$. We shall use the method of the central limit theorem for trignometric sums to produce an estimate of this L^1-norm.

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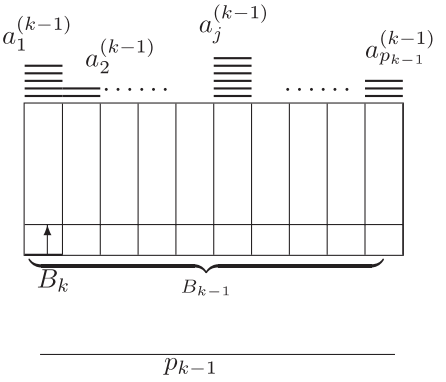


FIGURE 1. The kth tower.

2. *Rank-one transformation by construction*

Using the cutting and stacking method described in [**Fr1**] and [**Fr2**], one can construct inductively a family of measure-preserving transformations, called rank-one transformations, as follows.

Let B_0 be the unit interval equipped with the Lebesgue measure. At stage one we divide B_0 into p_0 equal parts, add spacers and form a stack of height h_1 in the usual fashion. At the kth stage we divide the stack obtained at the $(k-1)$th stage into p_{k-1} equal columns, add spacers and obtain a new stack of height h_k (Figure 1). If during the kth stage of our construction the number of spacers put above the jth column of the $(k-1)$th stack is $a_j^{(k-1)}$, $0 \le a_j^{(k-1)} < \infty$, $1 \le j \le p_k$, then we have

$$h_k = p_{k-1}h_{k-1} + \sum_{j=1}^{p_{k-1}} a_j^{(k-1)}.$$

Proceeding in this way we get a rank-one transformation T on a certain measure space $(X, \mathcal{B}, \nu)$ which may be finite or σ-finite depending on the number of spacers added.

The construction of a rank-one transformation thus needs two parameters, $(p_k)_{k=0}^{\infty}$ (parameter of cutting and stacking), and $((a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}$ (parameter of spacers). Put

$$T \stackrel{\text{def}}{=} T_{(p_k,(a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}}.$$

In [**CN1**] and [**KR**] it is proved that the spectral type of this transformation is given (up to possibly some discrete measure) by

$$d\sigma = W^* \lim \prod_{k=1}^{n} |P_k|^2 \, d\lambda, \tag{1.1}$$

where

$$P_k(z) = \frac{1}{\sqrt{p_k}} \left(\sum_{j=0}^{p_k-1} z^{-\left(jh_k + \sum_{i=1}^{j} a_i^{(k)}\right)} \right),$$

in which λ denotes the normalized Lebesgue measure on the circle group $\mathbb{T}$ and W^* lim denotes weak* limit in the space of bounded Borel measures on $\mathbb{T}$.

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The principal result of this paper is the following.

THEOREM 2.1. *Let* $T = T_{(p_k,(a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}}$ *be a rank-one transformation such that:*

(i) $a_{j+1}^{(k)} \geq 2s_k(j)$, $j = 0, \ldots, p_k - 1$, *with* $s_k(j) = a_1^{(k)} + \cdots + a_j^{(k)}$, $s_k(0) = 0$;

(ii) $s_k(p_k)/h_k < \frac{1}{2}$;

then the spectrum of T *is singular.*

We remark that the spectrum of a rank-one transformation is always singular if the cutting parameter p_k is bounded. In fact, Klemes and Reinhold prove that if $\sum_{k=0}^{\infty}(1/p_k^2) = \infty$ then the associated rank-one transformation has singular spectrum. Henceforth, we assume that the series $\sum_{k=0}^{\infty}(1/p_k^2)$ converges.

We point out also that the condition (ii) of Theorem 2.1 holds in the case of rank-one transformations satisfying a restricted growth condition provided that $\min_{1\leq j\leq p_k}(a_j^{(k)}) = 0$. Following Creutz and Silva [**CS**], we say that a rank-one transformation $T = T_{(p_k,(a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}}$ has restricted growth if

$$\frac{\sum_{i=1}^{p_k}\left(a_i^{(k)} - \min_{1\leq j\leq p_k}(a_j^{(k)})\right)}{h_k + \min_{1\leq j\leq p_k}(a_j^{(k)})} \xrightarrow[k\to+\infty]{} 0.$$

The proof of our main result is based on the method of Bourgain [**Bo**]. We shall recall the main ideas of this method.

PROPOSITION 2.2. *The following are equivalent:*

(i) $\sigma \perp \lambda$;

(ii) $\inf\left\{\int\cdots\int \prod_{l=1}^{k} |P_{n_l}(z)|\, d\lambda,\ k \in \mathbb{N},\ n_1 < n_2 < \cdots < n_k\right\} = 0.$

Now fix some subsequence $\mathcal{N} = \{n_1 < n_2 < \cdots < n_k\}$, $k \in \mathbb{N}$, $m > n_k$ and put

$$Q(z) = \prod_{i=1}^{k} |P_{n_i}(z)|.$$

One can show, using the same arguments as in [**EA**], the following lemma.

LEMMA 2.3. *We have*

$$\int Q(z)|P_m(z)|\, d\lambda(z) \leq \frac{1}{2}\left(\int Q\, d\lambda + \int Q(z)|P_m(z)|^2\, d\lambda(z)\right) - \frac{1}{8}\left(\int Q\left||P_m(z)|^2 - 1\right| d\lambda(z)\right)^2.$$

PROPOSITION 2.4. *We have*

$$\lim_{m\to\infty}\int Q|P_m(z)|^2\, d\lambda(z) = \int Q\, d\lambda(z).$$

Proof. The sequence of probability measures $|P_m(z)|^2\, d\lambda(z)$ converges weakly to the Lebesgue measure. □

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We also have the following proposition.

PROPOSITION 2.5. *There exists a subsequence of the sequence* $(||P_m(z)|-1|)$ *which converges weakly to some non-negative function* ϕ *which satisfies* $\phi \le 2$, *almost surely with respect to the Lebesgue measure.*

Proof. The sequence $||P_m(z)|-1|$ is bounded in L^2. It follows that there exists a subsequence which converges weakly to some non-negative L^2 function ϕ. Let ω be a non-negative continuous function. Then we have

$$\int \omega||P_m(z)|-1|\,d\lambda(z) \le \int \omega|P_m(z)|\,d\lambda(z) + \int \omega\,d\lambda(z)$$
$$\le \left(\int \omega\,d\lambda(z)\right)^{1/2}\left(\int \omega|P_m(z)|^2\,d\lambda(z)\right)^{1/2} + \int \omega\,d\lambda(z).$$

Hence

$$\int \omega\phi\,d\lambda \le 2\int \omega\,d\lambda.$$

Since this holds for all non-negative continuous ω, we have $\phi \le 2$ almost everywhere (a.e.). □

Put

$$\alpha = \phi\,d\lambda.$$

We shall prove the following.

PROPOSITION 2.6. *We have* $\alpha \perp \sigma$.

For the proof of Proposition 2.6 we need the following classical lemma [**KS**].

LEMMA 2.7. *Let* ρ, τ *be two non-negative finite measures on a measurable space* X. *Then the following properties are equivalent:*

(i) $\rho \perp \tau$;

(ii) *given* $\varepsilon > 0$, *there exists a non-negative measurable function* f *on* X *such that* $f > 0$, τ*-a.e. and such that*

$$\left(\int f\,d\rho\right)\left(\int \frac{d\tau}{f}\right) < \varepsilon.$$

Proof of Proposition 2.6. Let

$$\beta_1 = \inf\left\{\int Q\,d\alpha :\ Q = \prod_{i=1}^{k} |P_{n_i}(z)|,\ k \in \mathbb{N},\ n_1 < n_2 < \cdots < n_k\right\}$$

and

$$\beta_2 = \inf\left\{\int Q\,d\lambda :\ Q = \prod_{i=1}^{k} |P_{n_i}(z)|,\ k \in \mathbb{N},\ n_1 < n_2 < \cdots < n_k\right\}.$$

Then, for any Q, we have

$$\int Q\,d\alpha \ge \beta_1, \quad \liminf \int Q|P_m|\,d\lambda \ge \beta_2.$$

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Combine Lemma 2.3, Proposition 2.4 and Proposition 2.5 and take the inf over all Q to obtain

$$\beta_2 \le \beta_2 - \tfrac{1}{8}\beta_1^2.$$

It follows that

$$\beta_1 = 0.$$

We claim that $\beta_1 = 0$ implies

$$\int \prod_{j=1}^{k} |P_j|\, d\alpha \xrightarrow[k\to\infty]{} 0.$$

In fact, let $\mathcal{N} = \{n_1 < n_2 < \cdots < n_k\} \subset \mathbb{N}^*$ and $N > n_k$. Then, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int \prod_{j=1}^{N} |P_j|\, d\alpha &= \int \sqrt{\prod_{j\in\mathcal{N}} |P_j|}\sqrt{\prod_{j\in\mathcal{N}} |P_j|}\prod_{j\notin\mathcal{N}} |P_j|\, d\alpha \\ &\le \Big(\int \prod_{j\in\mathcal{N}} |P_j|\, d\alpha\Big)^{1/2}\Big(\int \prod_{j\in\mathcal{N}} |P_j|\, d\alpha \prod_{j\notin\mathcal{N}} |P_j|^2\, d\alpha\Big)^{1/2}. \qquad (1.2)\end{aligned}$$

But

$$\begin{aligned}\int \prod_{j\in\mathcal{N}} |P_j| \prod_{j\notin\mathcal{N}} |P_j|^2\, d\alpha &\le 2\int \prod_{j\in\mathcal{N}} |P_j| \prod_{j\notin\mathcal{N}} |P_j|^2\, d\lambda \\ &\le 2\int \prod_{j=1}^{N} |P_j| \prod_{j\notin\mathcal{N}} |P_j|\, d\lambda \\ &\le 2\Big(\int \prod_{j=1}^{N} |P_j|^2\, d\lambda\Big)^{1/2}\Big(\int \prod_{j\notin\mathcal{N}} |P_j|^2\, d\lambda\Big)^{1/2} \\ &\le 2. \qquad (1.3)\end{aligned}$$

Combine (1.2) and (1.3) to get the claim. The proof of the proposition follows from Lemma 2.7. □

3. *Estimation of L_1-norm of $(|P_m(z)| - 1)$ and the central limit theorem*

In this section we assume that, for m sufficiently large,

(i) $a_{j+1}^{(m)} \ge 2s_m(j)$, $j = 0, \ldots, p_m - 1$, with $s_m(j) = a_1^{(m)} + \cdots + a_j^{(m)}$, $s_m(0) = 0$,

(ii) $s_m(p_m)/h_m < \frac{1}{2}$.

Under the above assumptions, we shall prove the following.

PROPOSITION 3.1. *We have $\alpha \ge K\lambda$, for some positive constant K.*

The proof of the proposition is based on an estimate of $\int_A ||P_m| - 1|\, d\lambda$, where A is a Borel set with $\lambda(A) > 0$. More precisely we shall study the stochastic behavior of the sequence $|P_m|$. For that our principal strategy is based on the method of the central limit theorem for trigonometric sums. A nice account can be found in [**Kac**]. It is well known

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that the Hadamard lacunary trigonometric series satisfies the central limit theorem [**Zy**]. The central limit theorem for trigonometric sums has been studied by many authors, among whom are Zygmund and Salem [**Zy**], Erdös [**Er**], Kahane [**Kah**], Berkes [**Be**], Murai [**Mu**], Takahashi [**Ta**], and Fukuyama and Takahashi [**FT**]. The same method is used to study the asymptotic behavior of Riesz–Raikov sums [**Pe**].

Here, we shall prove the following result.

PROPOSITION 3.2. *If (i) and (ii) hold then for any Borel subset A of $\mathbb{T}$ with $\lambda(A) > 0$, the distribution of the sequence of random variables*

$$\frac{\sqrt{2}}{\sqrt{p_m}} \sum_{j=0}^{p_m-1} \cos((jh_m + s_m(j))t)$$

converges to the Gauss distribution. That is

$$\frac{1}{\lambda(A)} \left\{ t \in A : \frac{\sqrt{2}}{\sqrt{p_m}} \sum_{j=0}^{p_m-1} \cos((jh_m + s_m(j))t) \le x \right\} \xrightarrow[m\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt \stackrel{\text{def}}{=} \mathcal{N}(]-\infty, x]). \tag{1.4}$$

The proof of Proposition 3.2 is based on the idea of the proof of the martingale central limit theorem due to McLeish [**McL**]. The main ingredient is the following.

LEMMA 3.3. *For $n \ge 1$, let U_n, T_n be random variables such that:*

(1) $U_n \longrightarrow a$ *in probability;*
(2) $\{T_n\}$ *is uniformly integrable;*
(3) $\{|T_n U_n|\}$ *is uniformly integrable;*
(4) $\mathbb{E}(T_n) \longrightarrow 1$.

Then $\mathbb{E}(T_n U_n) \longrightarrow a$.

Proof. Write $T_n U_n = T_n(U_n - a) + aT_n$. As $\mathbb{E}(T_n) \longrightarrow 1$, we need to show that $\mathbb{E}(T_n(U_n - a)) \longrightarrow 0$.

Since $\{T_n\}$ is uniformly integrable, we have $T_n(U_n - a) \longrightarrow 0$ in probability. Also, both $T_n U_n$ and aT_n are uniformly integrable, and so the combination $T_n(U_n - a)$ is uniformly integrable. Hence, $\mathbb{E}(T_n(U_n - a)) \longrightarrow 0$. □

Let us recall the following expansion:

$$\exp(ix) = (1 + ix)\exp\left(-\frac{x^2}{2} + r(x)\right),$$

where $|r(x)| \le |x|^3$, for real x.

THEOREM 3.4. *Let $\{X_{nj} : 1 \le j \le k_n, n \ge 1\}$ be a triangular array of random variables. Let*

$$S_n = \sum_{j=1}^{k_n} X_{nj}, \quad T_n = \prod_{j=1}^{k_n} (1 + itX_{nj}),$$

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and

$$U_n = \exp\left(-\frac{t^2}{2}\sum_j X_{nj}^2 + \sum_j r(tX_{nj})\right).$$

Suppose that:

(1) $\{T_n\}$ *is uniformly integrable;*

(2) $\mathbb{E}(T_n) \longrightarrow 1$*;*

(3) $\sum_j X_{nj}^2 \longrightarrow 1$ *in probability;*

(4) $\max |X_{nj}| \longrightarrow 0$ *in probability.*

Then

$$\mathbb{E}(\exp(itS_n)) \longrightarrow \exp\left(-\frac{t^2}{2}\right).$$

Proof. Let t be fixed. From conditions (3) and (4),

$$\left|\sum_j r(tX_{nj})\right| \le |t|^3 \sum_j |X_{nj}|^3 \le |t|^3 \max_j |X_{nj}| \sum_j X_{nj}^2 \xrightarrow[n\to\infty]{} 0$$

in probability. Then

$$U_n = \exp\left(-\frac{t^2}{2}\sum_j X_{nj}^2 + \sum_j r(tX_{nj})\right) \longrightarrow \exp\left(-\frac{t^2}{2}\right)$$

in probability as $n \longrightarrow +\infty$. This verifies the condition (1) of Lemma 3.3 with $a = \exp(-t^2/2)$. It is easy to check that the conditions (2), (3) and (4) of Lemma 3.3 hold. Thus $E(\exp(itS_n)) = E(T_nU_n) \longrightarrow \exp(-t^2/2)$. □

Let m be a positive integer and put

$$W_m \overset{\text{def}}{=} \left\{\left(\sum_{i\in I}\varepsilon_i p_i\right)h_m + \sum_{i\in I}\varepsilon_i s_m(p_i) : \varepsilon_i \in \{0, -1, 1\},\ I \subset \{0, \dots, p_m - 1\},\right.$$
$$\left. p_i \in \{0, \dots, p_m - 1\}\right\}.$$

The element $w = \left(\sum_{i\in I}\varepsilon_i p_i\right)h_m + \sum_{i\in I}\varepsilon_i s_m(p_i)$ is called a *word*.

We shall need the following combinatorial lemma.

LEMMA 3.5. *Under the assumptions (i) and (ii) of Theorem 2.1, all the words of W_m are distinct.*

Proof. Let $w, w' \in W_m$, and write

$$w = \left(\sum_{i\in I}\varepsilon_i p_i\right)h_m + \sum_{i\in I}\varepsilon_i s_m(p_i),$$
$$w' = \left(\sum_{i\in I'}\varepsilon_i' p_i'\right)h_m + \sum_{i\in I}\varepsilon_i' s_m(p_i').$$

Then $w = w'$ implies that

$$\left\{\left(\sum_{i\in I}\varepsilon_i p_i\right) - \left(\sum_{i\in I'}\varepsilon_i' p_i'\right)\right\}h_m = \sum_{i\in I'}\varepsilon_i' s_m(p_i') - \sum_{i\in I}\varepsilon_i s_m(p_i).$$

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But the modulus of the left-hand side is greater than h_m, and the modulus of the right-hand side is less than $2\sum_{j=0}^{p_m-1} a_j^{(m)}$. It follows that

$$\sum_{i\in I}\varepsilon_i p_i = \left(\sum_{i\in I'}\varepsilon_i' p_i'\right) \quad \text{and} \quad \sum_{i\in I}\varepsilon_i s_m(p_i) = \sum_{i\in I'}\varepsilon_i' s_m(p_i').$$

Since from (i) we have $s_m(p+1) \geq 3s_m(p)$, we get that the representation in the form $\sum_{i\in I}\varepsilon_i s_m(p_i)$ is unique and the proof of the lemma is complete. □

Proof of Proposition 3.2. Let A be a Borel set, $\lambda(A) > 0$. Using the Helly theorem we may assume that the sequence

$$\frac{\sqrt{2}}{\sqrt{p_m}}\sum_{j=0}^{p_m-1}\cos((jh_m + s_m(j))t)$$

converges in distribution. As is well known, it is sufficient to show that, for every real number x,

$$\frac{1}{\lambda(A)}\int_A \exp\left\{-ix\frac{\sqrt{2}}{\sqrt{p_m}}\sum_{j=0}^{p_m-1}\cos((jh_m + s_m(j))t)\right\}dt \xrightarrow[m\to\infty]{} \exp\left(-\frac{x^2}{2}\right).$$

To this end we apply Theorem 3.4 in the following context. The measure space is the given Borel set A of positive Lebesgue measure in the circle with the normalized measure and the random variables are given by

$$X_{mj} = \frac{\sqrt{2}}{\sqrt{p_m}}\cos((jh_m + s_m(j))t) \quad \text{where } 0 \leq j \leq p_m - 1,\ m \in \mathbb{N}.$$

It is easy to check that the variables $\{X_{mj}\}$ satisfy conditions (1) and (4). Further, condition (3) follows from the fact that

$$\int_0^{2\pi}\left|\sum_{j=0}^{p_m-1} X_{mj}^2 - 1\right|^2 dt \xrightarrow[m\to\infty]{} 0.$$

It remains to verify condition (2) of Theorem 3.4. It is sufficient to show that

$$\int_A \prod_{j=0}^{p_m-1}\left(1 - ix\frac{\sqrt{2}}{\sqrt{p_m}}\cos((jh_m + s_m(j))t)\right)dt \xrightarrow[m\to\infty]{} \lambda(A). \tag{1.5}$$

Write

$$\begin{aligned}\Theta_m(x,t) &= \prod_{j=0}^{p_m-1}\left(1 - ix\frac{\sqrt{2}}{\sqrt{p_m}}\cos((jh_m + s_m(j))t)\right)\\ &= 1 + \sum_{w=1}^{N_m}\rho_w{}^{(m)}(x)\cos(wt),\end{aligned}$$

where $\rho_w = 0$ if w is not of the form $\left(\sum_{i\in I}\varepsilon_i p_i\right)h_m + \sum_{i\in I}\varepsilon_i s_m(p_i)$, and

$$N_m = \left\{\frac{p_m(p_m-1)}{2}\right\}h_m + s_m(p_m-1) + s_m(p_m-2) + \cdots + 1.$$

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We claim that it is sufficient to prove the following:

$$\int_0^{2\pi} R(t) \prod_{j=0}^{p_m-1} \left(1 - ix\frac{\sqrt{2}}{\sqrt{p_m}} \cos((jh_m + s_m(j))t)\right) dt \xrightarrow[m\to\infty]{} \int_0^{2\pi} R(t)\, dt, \quad (1.6)$$

for any trigonometric polynomial R. In fact, assume that (1.6) holds and let $\epsilon > 0$. Then, by the density of trigonometric polynomials, one can find a trigonometric polynomial R_ϵ such that

$$\|\chi_A - R_\epsilon\|_1 < \epsilon,$$

where χ_A is the indicator function of A. But

$$\left|\prod_{j=0}^{p_m-1} \left(1 - ix\frac{\sqrt{2}}{\sqrt{p_m}} \cos((jh_m + s_m(j))t)\right)\right| \le \left\{\prod_{j=0}^{p_m-1} \left(1 + \frac{2x^2}{p_m}\right)\right\}^{1/2}.$$

Since $1 + u \le e^u$, we get

$$|\Theta_m(x, t)| \le e^{x^2}. \quad (1.7)$$

Hence, according to (1.6), for m sufficiently large, we have

$$\begin{aligned}
&\left|\int_A \Theta_m(x, t)\, dt - \lambda(A)\right| \\
&\quad = \left|\int_A \Theta_m(x, t)\, dt - \int_0^{2\pi} \Theta_m(x, t) R_\epsilon(t)\, dt + \int_0^{2\pi} \Theta_m(x, t) R_\epsilon(t)\, dt \right. \\
&\qquad \left. - \int_0^{2\pi} R_\epsilon(t)\, dt + \int_0^{2\pi} R_\epsilon(t)\, dt - \lambda(A)\right| \\
&\quad < e^{x^2}\epsilon + 2\epsilon.
\end{aligned}$$

The proof of the claim is complete. It still remains to prove (1.6). Observe that

$$\int_0^{2\pi} \Theta_m(x, t) R(t)\, dt = \int_0^{2\pi} R(t)\, dt + \sum_{w=1}^{N_m} {\rho_w}^{(m)}(x) \int_0^{2\pi} R(t) \cos(wt)\, dt$$

and for

$$w = p_{i_1} h_m + s_m(p_{i_1}) + \sum_{j=1}^{r} \varepsilon_j \{(p_{i_j} h_m) + s_m(p_{i_j})\},$$

we have

$$|\rho_w^{(m)}(x)| \le \frac{2^{1-r}|x|^r}{p_m^{r/2}},$$

hence

$$\max_{w\in W} |\rho_w^{(m)}(x)| \le \frac{|x|}{p_m^{1/2}} \xrightarrow[m\to\infty]{} 0.$$

Since

$$\sum_{w\in\mathbb{Z}} \left|\int_0^{2\pi} e^{-iwt} R(t)\, dt\right|$$

is bounded, we deduce that

$$\left|\sum_{w=1}^{N_m} {\rho_w}^{(m)}(x) \int_0^{2\pi} R(t) \cos(wt)\, dt\right| \le \frac{|x|}{p_m^{1/2}} \sum_{w\in\mathbb{Z}} \left|\int_0^{2\pi} e^{-iwt} R(t)\, dt\right| \xrightarrow[m\to\infty]{} 0.$$

The proof of Proposition 3.2 is complete. □

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Proof of Proposition 3.1. Let A be a Borel subset of $\mathbb{T}$, and $x \in]1, +\infty[$. Then, for any positive integer m, we have

$$\begin{aligned}\int_A ||P_m(\theta)| - 1|\, d\lambda(\theta) &\geq \int_{\{\theta\in A:|P_m(\theta)|>x\}} ||P_m| - 1|\, d\lambda(\theta)\\ &\geq (x-1)\lambda\{\theta \in A : |P_m(\theta)| > x\}\\ &\geq (x-1)\lambda\{\theta \in A : |\Re(P_m(\theta))| > x\}.\end{aligned}$$

Let m go to infinity and use Propositions 2.5 and 3.2 to get

$$\int_A \phi\, d\lambda \geq (x-1)\{1 - \mathcal{N}([-\sqrt{2}x, \sqrt{2}x])\}\lambda(A).$$

Put $K = (x-1)\{1 - \mathcal{N}([-\sqrt{2}x, \sqrt{2}x])\}$. Hence $\alpha(A) \geq K\lambda(A)$, for any Borel subset A of $\mathbb{T}$, which proves the proposition. □

Now, we give the proof of our main result.

Proof of Theorem 2.1. This follows easily from Proposition 2.6 combined with Proposition 3.1. □

Let us mention that the same proof works for the following more general statement.

THEOREM 3.6. *Let $T = T_{(p_k,(a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}}$ be a rank-one transformation such that*

(i) $a_{j+1}^{(k)} \geq 2s_k(j)$, $j = 0, \ldots, p_k - 1$, *with* $s_k(j) = a_1^{(k)} + \cdots + a_j^{(k)}$, $s_k(0) = 0$,

(ii) $(s_k(p_k) - p_k \min_{1\leq j\leq p_k}(a_j^{(k)}))/(h_k + \min_{1\leq j\leq p_k}(a_j^{(k)})) < \frac{1}{2}$,

then the spectrum of T is singular.

Remark. We note that, in [**Bo**] and [**Kl**], the strategy of the authors is to show that the absolutely continuous measure β, obtained as the limit of some subsequence of the sequence $(||P_m|^2 - 1|\, d\lambda)_{m\geq 0}$, is equivalent to Lebesgue measure. In fact the authors prove that

$$\beta \geq K\lambda, \tag{E}$$

for some $K > 0$. In the case of Ornstein transformations, the relation (E) holds almost surely.

4. *Simple proof of Bourgain's theorem*

Bourgain's theorem deals with Ornstein transformations. In Ornstein's construction, the p_k are rapidly increasing, and the number of spacers, $a_i^{(k)}$, $1 \leq i \leq p_k - 1$, are chosen randomly. This may be organized in different ways as pointed out by Bourgain [**Bo**]. Here we suppose that we are given two sequences (t_k), (p_k) of positive integers and a sequence (ξ_k) of probability measure such that the support of each ξ_k is a subset of $X_k = \{-t_k/2, \ldots, t_k/2\}$. We now choose independently, according to ξ_k, the numbers $(x_{k,i})_{i=1}^{p_k-1}$, and x_{k,p_k} is chosen deterministically in $\mathbb{N}$. We put, for $1 \leq i \leq p_k$,

$$a_i^{(k)} = t_k + x_{k,i} - x_{k,i-1} \quad \text{with } x_{k,0} = 0.$$

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We have

$$h_{k+1} = p_k(h_k + t_k) + x_{k,p_k}.$$

So the deterministic sequences of positive integers $(p_k)_{k=0}^{\infty}$, $(t_k)_{k=0}^{\infty}$ and $(x_{k,p_k})_{k=0}^{\infty}$ determine completely the sequence of heights $(h_k)_{k=0}^{\infty}$. The total measure of the resulting measure space is finite if

$$\sum_{k=0}^{\infty} \frac{t_k}{h_k} + \sum_{k=0}^{\infty} \frac{x_{k,p_k}}{p_k h_k} < \infty. \tag{1.9}$$

We will assume that this requirement is satisfied.

We thus have a probability space of Ornstein transformations $\Omega = \prod_{l=0}^{\infty} X_l^{p_l-1}$ equipped with the natural probability measure $\mathbb{P} \stackrel{\text{def}}{=} \bigotimes_{l=1}^{\infty} P_l$, where $P_l \stackrel{\text{def}}{=} \bigotimes_{j=1}^{p_l-1} \xi_j$; ξ_j is the probability measure on X_j. We denote this space by $(\Omega, \mathcal{A}, \mathbb{P})$. So $x_{k,i}$, $1 \le i \le p_k - 1$, is the projection from Ω onto the ith coordinate space of $\Omega_k \stackrel{\text{def}}{=} X_k^{p_k-1}$, $1 \le i \le p_k - 1$. Naturally each point $\omega = (\omega_k = (x_{k,i}(\omega))_{i=1}^{p_k-1})_{k=0}^{\infty}$ in Ω defines the spacers and therefore a rank-one transformation $T_{\omega,x}$, where $x = (x_{k,p_k})$.

This construction is more general than the construction due to Ornstein [**Or**] which corresponds to the case $t_k = h_{k-1}$, ξ_k is the uniform distribution on X_k and $p_k \gg h_{k-1}$.

We recall that Ornstein [**Or**] proved that there exists a sequence $(p_k, x_{k,p_k})_{k\in\mathbb{N}}$ such that $T_{\omega,x}$ is almost surely mixing. Later Prikhod'ko [**Pr**] obtains the same result for some special choice of the sequence of the distribution (ξ_m) and recently, using the idea of Creutz and Silva [**CS**], one can extend this result to a large family of probability measures associated to Ornstein's construction. In our general construction, according to (1.1) the spectral type of each T_ω, up to a discrete measure, is given by

$$\sigma_{T_\omega} = \sigma_{\chi_{B_0}}^{(\omega)} = \sigma^{(\omega)} = W^* \lim \prod_{l=1}^{N} \frac{1}{p_l} \left| \sum_{p=0}^{p_l-1} z^{p(h_l+t_l)+x_{l,p}} \right|^2 d\lambda.$$

With the above notation, we state Bourgain's theorem in the following form.

THEOREM 4.1. [**EAPP**] *For every choice of (p_k), (t_k), (x_{k,p_k}) and for any family of probability measures ξ_k on $X_k = \{-t_k, \dots, t_k\}$ of $\mathbb{Z}$, $k \in \mathbb{N}^*$, for which*

$$\sum_{s\in X_m} \xi(s)^2 \xrightarrow[m\to\infty]{} 0,$$

the associated generalized Ornstein transformation has almost surely singular spectrum, i.e.

$$\mathbb{P}\{\omega : \sigma^{(\omega)} \perp \lambda\} = 1,$$

where $\mathbb{P} \stackrel{\text{def}}{=} \bigotimes_{l=0}^{\infty} \bigotimes_{j=1}^{p_l-1} \xi_l$ is the probability measure on $\Omega = \prod_{l=0}^{\infty} X_l^{p_l-1}$.

In the context of Ornstein's construction, we state Proposition 2.5 in the following form.

PROPOSITION 4.2. *There exists a subsequence of the sequence $(||P_m(z)| - 1|)$ which converges weakly to some non-negative function $\phi(\omega, \theta)$ which satisfies $\phi \le 2$, almost surely with respect to $\mathbb{P} \otimes \lambda$.*

Proof. This is left as an easy exercise. □

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Put, for any $\omega \in \Omega$,

$$\alpha_\omega = \phi(\omega, \theta)\, d\lambda.$$

We shall prove that α_ω is equivalent to the Lebesgue measure for almost all ω. In fact, we have the following proposition.

PROPOSITION 4.3. *There exists an absolutely positive constant K such that for almost all ω we have*

$$\alpha_\omega \geq K\lambda.$$

The proof is based on the following two lemmas.

LEMMA 4.4. *We have*

$$\lim \iint ||P_m| - |P'_m||\, d\theta\, d\mathbb{P} \xrightarrow[n\to\infty]{} 0,$$

where $P'_m(\theta) = P_m(\theta) - \int_\Omega P_m(\theta)\, d\mathbb{P}$.

Proof. For any $m \in \mathbb{N}$, we have

$$\begin{aligned}\iint ||P_m| - |P'_m||\, d\theta\, d\mathbb{P} &\leq \iint |P_m - P'_m|\, d\theta\, d\mathbb{P} \\ &= \int \left| \int P_m\, d\mathbb{P} \right| d\theta \\ &\leq \int \left| \sum_{p=0}^{p_m-1} \frac{1}{\sqrt{p_m}} z^{p(h_m+t_m))} \right| \left| \sum_{s\in X_m} \xi_m(s) z^s \right| dz.\end{aligned}$$

Hence by the Cauchy–Schwarz inequality

$$\iint ||P_m| - |P'_m||\, d\theta\, d\mathbb{P} \leq \sum_{s\in X_m} \xi(s)^2 \xrightarrow[m\to\infty]{} 0.$$

The proof of the lemma is complete. □

Now observe that we have

$$\int_T \left| \sum_{s\in X_m} \xi_m(s) z^s \right|^2 dz = \int_T \left| \sum_{s\in X_m} \xi_m(s) z^{2s} \right|^2 dz = \sum_{s\in X_m} \xi(s)^2 \xrightarrow[m\to\infty]{} 0.$$

So, we may extract a subsequence (m_k) for which, for almost all $t \in [0, 2\pi)$, we have

$$\sum_{s\in X_{m_k}} \xi_{m_k}(s) e^{ist} \xrightarrow[k\to\infty]{} 0 \quad \text{and} \quad \sum_{s\in X_{m_k}} \xi_{m_k}(s) e^{is2t} \xrightarrow[k\to\infty]{} 0.$$

Define

$$\Theta \stackrel{\text{def}}{=} \left\{ \theta : \sum_{s\in X_{m_k}} \xi_{m_k}(s) e^{is\theta} \xrightarrow[k\to\infty]{} 0 \text{ and } \sum_{s\in X_{m_k}} \xi_{m_k}(s) e^{is2\theta} \xrightarrow[k\to\infty]{} 0 \right\}.$$

Choose $m \in \{m_k\}$, $t \in \Theta$ and put, for $j \in \{0, \ldots, p_m - 1\}$,

$$Y_{m,j}(\omega) = \cos((j(h_m + t_m) + x_{m,j}(\omega))t) - \int \cos((j(h_m + t_m) + x_{m,j}(\omega))t)\, d\mathbb{P},$$

$$Z_{m,j}(\omega) = \sqrt{\frac{2}{p_m}} Y_{m,j}(\omega).$$

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LEMMA 4.5. *For any fixed $t \in \Theta$, the distribution of the sequence of random variables $\sum_{j=0}^{p_m-1} Z_{m,j}(\omega)$ converges to the Gauss distribution. That is,*

$$\mathbb{P}\left\{\omega \in \Omega : \sum_{j=0}^{p_m-1} Z_{m,j}(\omega) \le x\right\} \xrightarrow[m\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\, du \stackrel{\text{def}}{=} \mathcal{N}(]-\infty, x]).$$

Proof. Since the random variables are independent, centered and uniformly bounded by $2\sqrt{2}/\sqrt{p_m}$, the conditions (1), (2) and (4) of Theorem 3.4 are satisfied. We also have the following:

$$\mathbb{E}\left\{\left(\sum_{j=1}^{p_m-1} Z_{m,j}^2 - 1\right)^2\right\} = \frac{4(p_m-1)}{p_m^2}\mathbb{E}(Y_{m,1}^4) + \frac{(p_m-1)(p_m-2)}{p_m^2}(\mathbb{E}(2Y_{m,1}^2))^2 - 2\frac{p_m-1}{p_m}\mathbb{E}(2Y_{m,1}^2) + 1.$$

But

$$\mathbb{E}(2Y_{m,1}^2) \xrightarrow[m\to\infty]{} 1,$$

so it follows that the variables $\{Z_{m,j}\}$ satisfy condition (3) of Theorem 3.4. Thus all the conditions of Theorem 3.4 hold and we conclude that the distribution of $\sum_{j=0}^{p_m-1} Z_{m,j}(\omega)$ converges to the normal distribution. □

Proof of Proposition 4.3. Let A be a Borel subset of $\mathbb{T}$, C a cylinder set in Ω, and $x \in]1, +\infty[$. Then, for any positive integer m, we have

$$\begin{aligned}
&\int_{A\times C} ||P_m(\theta)| - 1|\, d\lambda(\theta)\, d\mathbb{P} \\
&\quad \ge \mathbb{P}(C) \int_{A\times\Omega} ||P_m'(\theta)| - 1|\, d\lambda(\theta)\, d\mathbb{P} - \int ||P_m| - |P_m'||\, d\mathbb{P}\, d\lambda \\
&\quad \ge \mathbb{P}(C) \int_{\{|P_m'|>x\}\cap A\times\Omega} ||P_m'| - 1|\, d\lambda(\theta)\, d\mathbb{P} - \int ||P_m| - |P_m'||\, d\mathbb{P}\, d\lambda \\
&\quad \ge \mathbb{P}(C)(x-1) \int_A \mathbb{P}\{|\Re(P_m'(\theta))| > x\}\, d\lambda - \int ||P_m| - |P_m'||\, d\mathbb{P}\, d\lambda.
\end{aligned}$$

Let m go to infinity and combine Lemmas 4.4 and 4.5 to get

$$\int_{A\times C} \phi\, d\lambda\, d\mathbb{P} \ge (x-1)\{1 - \mathcal{N}([-\sqrt{2}x, \sqrt{2}x])\}\lambda(A)\mathbb{P}(C).$$

Put $K = (x-1)\{1 - \mathcal{N}([-\sqrt{2}x, \sqrt{2}x])\}$. Hence, for almost all ω, we have, for any Borel set $A \subset \mathbb{T}$, $\alpha_\omega(A) \ge K\lambda(A)$, and the proof of the proposition is complete. □

Proof of Theorem 4.1. This follows easily from Proposition 4.3 combined with Proposition 2.6. □

Remark. We point out that there exist rank-one mixing transformations on the space with finite measure satisfying the conditions of Theorem 3.6. In fact, following the notation of §4, one may define the spacers in the Ornstein construction by

$$a_j^{(k)} = 3^j t_k + x_{k,j} - x_{k,j-1},$$

and choose the sequence $(t_k)_{k\in\mathbb{N}}$ such that the measure of the dynamical system is finite. Thus the conditions of Theorem 3.6 hold and the class is mixing almost surely.

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