Teaching Notes

The volume of a cone for pre-calculus students

What do you do about the formula for the volume of a cone with pre-calculus students? Dehn's solution to Hilbert's 3rd problem means that it is impossible to find an analogue of the dissection proofs for the area of a triangle and that some form of limiting argument is unavoidable. In an interesting recent article, [1], Colin Foster describes a possible approach to the volume formulae for a cone and sphere and there have been several previous notes in the *Gazette* on this theme such as [2, 3, 4]. We offer three arguments below. The first (which I suspect is part of the mathematical folklore) uses a scaling argument similar to [2], but couched in the language of current syllabuses; the second uses a Cavalieri argument. The third uses a dissection method different to that given in [1]: I am very grateful to the referee for suggesting its inclusion.

Scaling argument

We give the argument for the right circular cone shown in Figure 1: the same method works for any pyramid, whatever the shape of its base. We divide the cone of height h and base radius r into n slices (frusta) by planes parallel to the base.

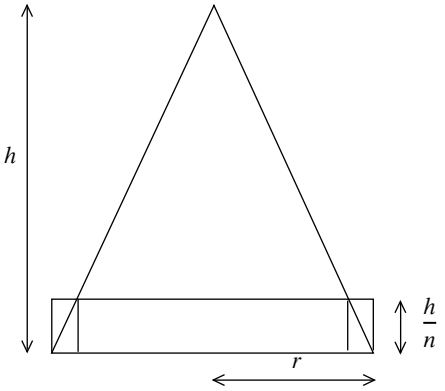


FIGURE 1

If V is the volume of the whole cone and v the volume of the bottom slice, then (using volume scale factors),

$$v = V - \left(1 - \frac{1}{n}\right)^3 V = \left(\frac{3}{n} - \frac{3}{n^2} + \frac{1}{n^3}\right)V.$$

But the volume of the bottom slice lies between that of the two cylinders shown whence

$$\frac{1}{n}\pi r^2 h > v = \left(\frac{3}{n} - \frac{3}{n^2} + \frac{1}{n^3}\right)V > \frac{1}{n}\pi\left(1 - \frac{1}{n}\right)^2 r^2 h.$$

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This gives

$$\frac{\pi r^2 h}{3 - \frac{3}{n} + \frac{1}{n^2}} > V > \frac{\pi\left(1 - \frac{1}{n}\right)^2 r^2 h}{3 - \frac{3}{n} + \frac{1}{n^2}}.$$

Letting $n \to \infty$ (our unavoidable limiting argument!) then shows that $V = \frac{1}{3}\pi r^2 h$.

Exactly the same method (but using area scale factors) may be used to show that the area of a triangle with base b and height h is $\frac{1}{2}bh$.

Cavalieri argument

Again, we give the argument for the right circular cone in Figure 1. If we double the height of the cone, chop the big cone in half and invert the base, we get Figure 2(a).

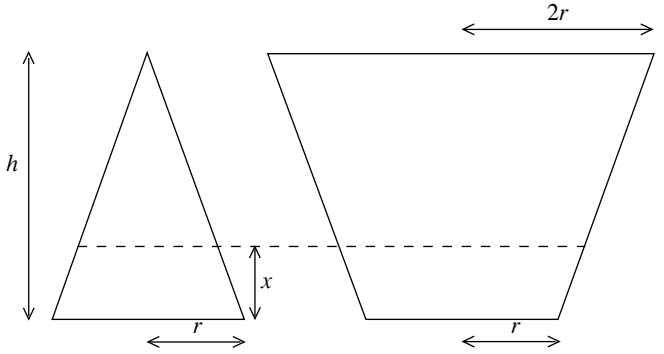


FIGURE 2(a)

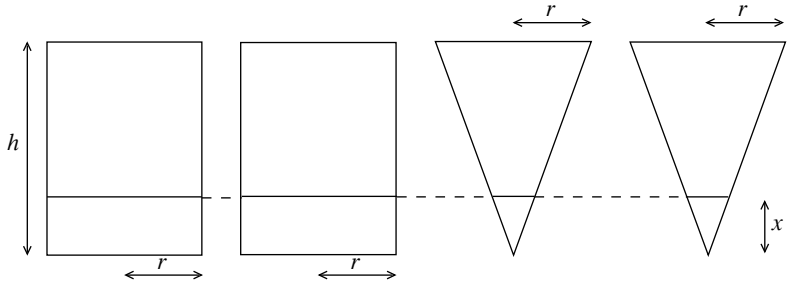


FIGURE 2(b)

The total cross-sectional area of Figure 2(a) at height x is

$$\pi r^2\left[\left(1 - \frac{x}{h}\right)^2 + \left(1 + \frac{x}{h}\right)^2\right].$$

But this simplifies to $2\pi r^2 + \dfrac{2\pi r^2 x^2}{h^2}$ which is the total cross-sectional area at height x of the two cylinders and two cones shown in Figure 2(b). By

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Cavalieri's principle, if the cross-sectional areas are equal at all heights, then the total volumes are equal, so that $2^3V = 2\pi r^2 h + V + V$ and $V = \frac{1}{3}\pi r^2 h$.

Dissection argument

As shown in Figure 3, the triangular-based prism $ABCDEF$ comprising half of the cube may be dissected into three tetrahedra $ADEF$, $ABCF$, $ABEF$. Combining two such dissections shows that a cube can be subdivided into six tetrahedra; three are directly congruent to each other and three indirectly congruent, so all have equal volume. (A nice animation that helps to visualise this is given in [5].) Moreover, each tetrahedron has the special property that it contains three vertex angles of 90°. For example, $ABCF$ has $\angle ABC = \angle BCF = \angle ACF = 90°$: such tetrahedra are sometimes called *orthoschemes*.

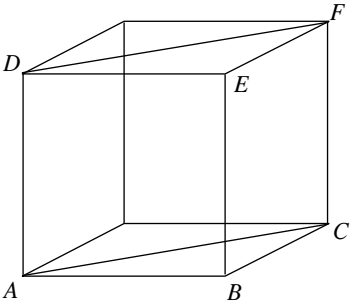


FIGURE 3(a)

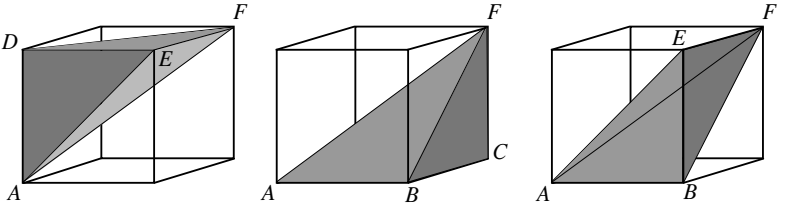


FIGURE 3(b)

If we stretch the cube into a cuboid with edge-lengths a, b, c, these right-angles are preserved and, although the six tetrahedra are no longer all congruent, their volumes are still equal so the volume of each is $\frac{1}{6}abc$: this establishes the volume formula '$\frac{1}{3}$ × base × height' for orthoschemes in the case where the height is an edge of the surrounding cuboid. We can then use orthoschemes to approximate a right circular cone as shown in Figure 4.

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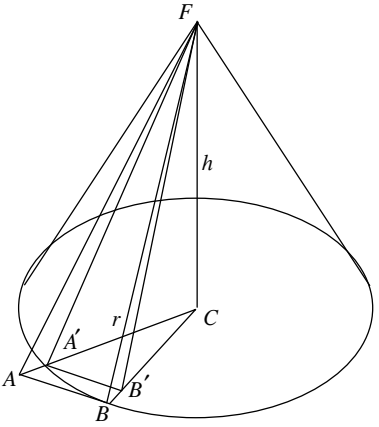


FIGURE 4

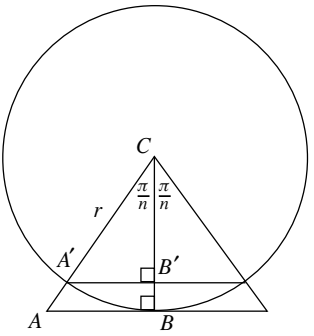


FIGURE 5

Here, C is the centre of the base, radius r, and F the vertex of the cone; AB is the tangent at B and A' (on the circle) is on AC with $A'B'$ parallel to AB. Then the portion of the cone on sector $CA'B$ contains the orthoscheme $A'B'CF$ and is contained in $ABCF$. Choosing $\angle A'CB = \frac{\pi}{n}$ and $2n$ such pieces shows (Figure 5) that $\frac{1}{3}I_n h < V < \frac{1}{3}C_n h$, where V is the volume of the cone, $CF = h$, I_n the area of an inscribed regular n-gon and C_n the area of a circumscribed regular n-gon. Letting $n \to \infty$ then shows that $V = \frac{1}{3}\pi r^2 h$.

As the referee notes, an attractive feature of this proof is that the limiting argument at the end is the same as that needed to establish the formula πr^2 for the area of the circle, so the degree of rigour can easily be adjusted to the maturity of the students.

References

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