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Dimension of the generalized 4-corner set and its projections

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Abstract. In the last two decades, considerable attention has been paid to the dimension theory of self-affine sets. In the case of generalized 4-corner sets (see Figure 1), the iterated function systems obtained as the projections of self-affine systems have maps of common fixed points. In this paper, we extend our result [B. Bárány. On the Hausdorff dimension of a family of self-similar sets with complicated overlaps. *Fund. Math.* **206** (2009), 49–59], which introduced a new method of computation of the box and Hausdorff dimensions of self-similar families where some of the maps have common fixed points. The extended version of our method presented in this paper makes it possible to determine the box dimension of the generalized 4-corner set for Lebesgue-typical contracting parameters.

1. *Introduction and statements*

We call a set self-affine if it can be represented as a finite union of its affine copies, that is $\Lambda \subset \mathbb{R}^2$ is self-affine if there exists a finite list of contracting affine maps $\{f_i(x) = A_i x + a_i\}_{i=1}^m$ such that $\Lambda = \bigcup_{i=1}^m f_i(\Lambda)$, where A_i are 2×2 real matrices on the plane. The dimension theory of self-affine sets is far from well-understood, even in the diagonal case, that is when all A_i are diagonal matrices.

We consider the generalized 4-corner set $\Lambda(\underline{\alpha}, \underline{\beta})$, which is the attractor of the self-affine iterated function system (IFS) of Figure 1. (For a precise definition, see §4.) The parameters $\underline{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ and $\underline{\beta} = (\beta_0, \beta_1, \beta_2, \beta_3)$ are chosen such that the rectangles R_0, R_1, R_2, R_3 in Figure 1 are disjoint. One of the main goals of the present paper is to determine the box dimension of this set for Lebesgue-typical parameters.

The most natural upper bound of the box dimension is the subadditive-pressure formula, which is called the Falconer dimension or singularity dimension, and was introduced by Falconer [**4**] and Barreira [**2**]. The Falconer theorem (see [**4**]) states that the Hausdorff and box dimensions of a self-affine attractor coincide for almost every translation parameter, and are equal to the singularity dimension, whenever the norm of all the IFS affine maps is

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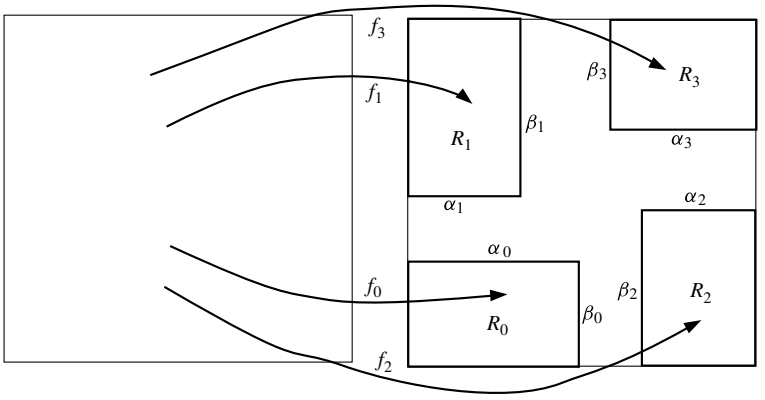


FIGURE 1. Maps of the generalized 4-corner set.

smaller than $1/3$. However, in our case, we work with fixed translations and we modulate the multiplicative part of the affine maps. On the other hand, we do not impose any conditions on the norm as long as the rectangles in Figure 1 are disjoint.

In the case of the generalized 4-corner set, the singularity dimension can be given by (see [**9**], for example)

$$d_{\text{sing}} = \inf\left\{ s : \sum_{n=1}^{\infty} \sum_{i_1 \cdots i_n \in \{0,\ldots,3\}^n} \phi^s(i_1 \cdots i_n) < \infty \right\}, \tag{1.1}$$

where

$$\phi^s(i_1 \cdots i_n) = \begin{cases} a(i_1 \cdots i_n)^s & \text{if } 0 \le s \le 1, \\ a(i_1 \cdots i_n) b(i_1 \cdots i_n)^{s-1} & \text{if } 1 \le s \le 2, \end{cases}$$

where

$$a(i_1 \cdots i_n) = \max\{\alpha_{i_1} \cdots \alpha_{i_n}, \beta_{i_1} \cdots \beta_{i_n}\}$$

and

$$b(i_1 \cdots i_n) = \min\{\alpha_{i_1} \cdots \alpha_{i_n}, \beta_{i_1} \cdots \beta_{i_n}\}.$$

For another method of calculation of the singularity dimension in our case, see [**7**].

We will prove that for Lebesgue-typical parameters $\underline{\alpha}$, $\underline{\beta}$, the Hausdorff dimension and even the box dimension of the generalized 4-corner set are strictly smaller than the singularity dimension (1.1). The reason for this phenomenon is the very special relative geometric positions of the rectangles which generate the generalized 4-corner set. The speciality of the maps is that the fixed points are the corners of the unit square, so they do not move when we change the parameters $\underline{\alpha}$, $\underline{\beta}$. Therefore, the orthogonal projection to the x-axis (and to the y-axis, respectively) is an attractor of a special iterated function system of four similarities, where the similarities derived from the maps having fixed points with the same coordinate y (and with the same coordinate x) have common fixed points.

In [**3**], we considered the IFS $\{\gamma x, \lambda x, \lambda x + 1\}$, $\gamma < \lambda$ on the real line. See Figure 2 for images of the convex hull of the attractor generated by the functions of this IFS. The novelty of the result obtained in [**3**] about the dimension of Λ was that it tackled the

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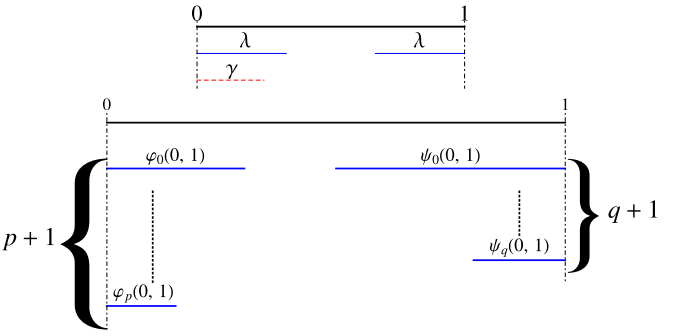


FIGURE 2. The first cylinder sets of the IFSs in [**3**] and of the extended version.

difficulty which comes from the fact that the first two maps have the same fixed point. In this paper, we extend the scope of that result in the following way.

Principal assumptions.

(A1) Let $\mathcal{F}$ be a finite set of linear, real functions such that for every $\varphi \in \mathcal{F}$, $\mathrm{Fix}(\varphi) \in \{0, 1\}$ and $\varphi([0, 1]) \subseteq [0, 1]$.

(A2) For arbitrary $\varphi_i, \varphi_j \in \mathcal{F}$, suppose either $\varphi_i([0, 1]) \cap \varphi_j([0, 1]) = \emptyset$ or $\mathrm{Fix}(\varphi_i) = \mathrm{Fix}(\varphi_j)$.

By Theorem 1.1, we will be able to calculate the Hausdorff and box dimensions of the attractor of iterated function schemes satisfying both of the assumptions (A1) and (A2).

Now we introduce some notation about our iterated function system. Let

$$\varphi_{0,1}(x) = \gamma_{0,1}x,$$
$$\varphi_{0,2}(x) = \gamma_{0,2}x + (1 - \gamma_{0,2})$$

and suppose that $\gamma_{0,1} + \gamma_{0,2} < 1$, which is equivalent to $\varphi_{0,1}([0, 1]) \cap \varphi_{0,2}([0, 1]) = \emptyset$.

Let p, q be positive integers and let

$$\varphi_{i,1}(x) = \gamma_{i,1}x \quad \text{for } i = 1, \ldots, p,$$
$$\varphi_{i,2}(x) = \gamma_{i,2}x + (1 - \gamma_{i,2}) \quad \text{for } i = 1, \ldots, q.$$

Moreover, suppose that $0 < \gamma_{i,1} < \gamma_{0,1}$ for every $i = 1, \ldots, p$ and $0 < \gamma_{i,2} < \gamma_{0,2}$ for every $i = 1, \ldots, q$.

THEOREM 1.1. *Let* $\mathcal{F} = \{\gamma_{i,1}x\}_{i=0}^{p} \cup \{\gamma_{i,2}x + (1 - \gamma_{i,2})\}_{i=0}^{q}$ *such that* $0 < \gamma_{i,1} < \gamma_{0,1} < 1$ *for* $i = 1, \ldots, p$ *and* $0 < \gamma_{j,2} < \gamma_{0,2} < 1$ *for* $j = 1, \ldots, q$ *(see Figure 1). Then*

$$\dim_B \Lambda = \dim_H \Lambda = \min\{1, s\}, \tag{1.2}$$

where s *is the unique solution of*

$$\prod_{i=0}^{p}(1 - \gamma_{i,1}^{s}) + \prod_{i=0}^{q}(1 - \gamma_{i,2}^{s}) = 1 \tag{1.3}$$

for Lebesgue almost every $(\underline{\gamma}_1, \underline{\gamma}_2) \in (0, \gamma_{0,1})^p \times (0, \gamma_{0,2})^q$, *where* $\underline{\gamma}_1 = (\gamma_{1,1}, \ldots, \gamma_{p,1})$ *and, respectively,* $\underline{\gamma}_2 = (\gamma_{1,2}, \ldots, \gamma_{q,2})$.

Moreover, $\mathcal{L}(\Lambda) > 0$ *for Lebesgue almost every* $(\underline{\gamma}_1, \underline{\gamma}_2)$ *if* $s > 1$.

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Note that whenever $\gamma_{0,1} + \gamma_{0,2} \geq 1$, the attractor of $\mathcal{F}$ is an interval, which immediately implies Theorem 1.1. In this way, without loss of generality in the rest of the paper, we may assume that $\gamma_{0,1} + \gamma_{0,2} < 1$.

The following shows that we can calculate the box dimension of the generalized 4-corner set from the dimensions of its orthogonal projections to the axis, which are calculable by Theorem 1.1 as we already mentioned.

THEOREM 1.2. *Let $\Lambda(\underline{\alpha}, \underline{\beta})$ be the attractor of the self-affine IFS of Figure 1. Then*

$$\begin{aligned}\dim_B \Lambda(\underline{\alpha}, \underline{\beta}) = \max\{d_\alpha, d_\beta\}, &\text{ for Lebesgue almost every } (\underline{\alpha}, \underline{\beta}), \text{ such that}\\ &\max\{\alpha_i + \alpha_{i+2}, \beta_i + \beta_{i+2}\} < 1 \quad \text{and}\\ &\min\{\alpha_i + \alpha_{3-i}, \beta_i + \beta_{3-i}\} < 1 \quad \text{for } i = 0, 1, \end{aligned} \tag{1.4}$$

where d_α and d_β are defined in two steps. First, we define two numbers s_α, s_β as the unique solution of the equations

$$\begin{aligned}\alpha_0^{s_\alpha} + \alpha_1^{s_\alpha} + \alpha_2^{s_\alpha} + \alpha_3^{s_\alpha} - \alpha_0^{s_\alpha}\alpha_1^{s_\alpha} - \alpha_2^{s_\alpha}\alpha_3^{s_\alpha} = 1,\\ \beta_0^{s_\beta} + \beta_1^{s_\beta} + \beta_2^{s_\beta} + \beta_3^{s_\beta} - \beta_0^{s_\beta}\beta_2^{s_\beta} - \beta_1^{s_\beta}\beta_3^{s_\beta} = 1.\end{aligned}$$

Then we can define d_α and d_β as the unique real numbers such that

$$\sum_{i=0}^{3} \alpha_i^{\min\{1,s_\alpha\}} \beta_i^{d_\alpha - \min\{1,s_\alpha\}} = 1, \quad \sum_{i=0}^{3} \beta_i^{\min\{1,s_\beta\}} \alpha_i^{d_\beta - \min\{1,s_\beta\}} = 1. \tag{1.5}$$

The condition in (1.4) is equivalent to the rectangles R_0, R_1, R_2, R_3 being pairwise disjoint.

The same formula as in equation (1.5) appeared in Gatzouras and Lalley [**10**], and also in Barański [**1**], for different kinds of self-affine sets. The method of the proof of (1.5) follows the proof of Feng and Wang [**8**, Theorem 1] and Barański [**1**, Theorem B].

Organization of the paper. In §2, we mention some methods for proving the so-called transversality condition. In §3, we prove Theorem 1.1. We divide this section into three parts. In §3.1, we introduce some notation for natural projections. In §3.2, we prove the transversality condition, and in §3.3, we calculate the Hausdorff dimension. Note that our original system does not satisfy transversality (see the precise arguments given in §3.2). The method of the proof is that we consider higher-order iterates of the system, we throw away some of the maps from it and then for this restricted family we apply the transversality condition. By taking higher and higher iterates, we are approximating the original system.

In §4, we apply Theorem 1.1 to prove the formula of the box dimension of a generalized 4-corner set. We obtain a result for almost all types with respect to the contraction coefficients. Furthermore, using the method of Barański [**1**] and Feng and Wang [**8**], we give a general formula (Theorem 4.1) for the box dimension of the self-affine sets on the plane which are constructed with axes-parallel rectangles having disjoint interiors.

2. *Transversality methods*

First, let us introduce the *transversality condition* for self-similar IFSs on the real line with d-dimensional parameter space. The technique of the transversality condition was first introduced in [**11**] to calculate the Hausdorff dimension of λ-expansions with deleted digits.

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The definition corresponds to the definition of Simon *et al* [**13, 14**], which was introduced for much more general IFSs.

Let U be an open, bounded subset of $\mathbb{R}^d$ with a smooth boundary and $\mathcal{I}$ a finite set of symbols. Let $\Psi_{\underline{t}} = \{\psi_i^{\underline{t}}(x) = \lambda_i(\underline{t})x + d_i(\underline{t})\}_{i\in\mathcal{I}}$, where $\lambda_i, d_i \in C^1(\overline{U})$ and $0 < \alpha \le \lambda_i(\underline{t}) \le \beta < 1$ for every $i \in \mathcal{I}$ and $\underline{t} \in \overline{U}$, and for some $\alpha, \beta \in (0, 1)$. Let $\Lambda^{\underline{t}}$ be the attractor of $\Psi_{\underline{t}}$ and $\pi_{\underline{t}}$ the natural projection from the symbolic space $\Sigma = \mathcal{I}^{\mathbb{N}}$ to $\Lambda^{\underline{t}}$. More precisely, for $\mathbf{i} = (i_0 i_1 \cdots) \in \Sigma$, we write

$$\pi_{\underline{t}}(\mathbf{i}) = \lim_{n\to\infty} \psi_{i_0}^{\underline{t}} \circ \psi_{i_1}^{\underline{t}} \circ \cdots \circ \psi_{i_n}^{\underline{t}}(0). \tag{2.1}$$

It is well-known that the limit exists and is independent of the base point 0. Moreover, $\pi_{\underline{t}}$ is a continuous, surjective function from Σ onto $\Lambda^{\underline{t}}$. Denote by σ the left-shift operator on Σ, that is $\sigma : (i_0 i_1 \cdots) \mapsto (i_1 i_2 \cdots)$. It is easy to see that

$$\pi_{\underline{t}}(\mathbf{i}) = \psi_{i_0}^{\underline{t}}(\pi_{\underline{t}}(\sigma\mathbf{i})).$$

Definition 2.1. We say that $\Psi_{\underline{t}}$ satisfies the *transversality condition* on an open, bounded set $U \subset \mathbb{R}^d$, if for any $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_0 \ne j_0$ there exists a constant $C = C(i_0, j_0)$ such that

$$\mathcal{L}_d(\underline{t} \in U : |\pi_{\underline{t}}(\mathbf{i}) - \pi_{\underline{t}}(\mathbf{j})| \le r) \le Cr \quad \text{for every } r > 0,$$

where $\mathcal{L}_d$ is the d-dimensional Lebesgue measure.

In short, we say that there is transversality if the transversality condition holds. This definition is equivalent to the ones given in, for example, [**13, 14**]. As a special case of [**13**, Theorem 3.1] we obtain the following theorem.

THEOREM 2.2. (Simon, Solomyak, Urbański) *Suppose that $\Psi_{\underline{t}}$ satisfies the transversality condition on an open, bounded set $U \subset \mathbb{R}^d$. Then:*

(1) $\dim_H \Lambda^{\underline{t}} = \min\{s(\underline{t}), 1\}$ *for Lebesgue almost every $\underline{t} \in U$; and*

(2) $\mathcal{L}_1(\Lambda^{\underline{t}}) > 0$ *for Lebesgue almost every $\underline{t} \in U$ such that $s(\underline{t}) > 1$,*

where $s(\underline{t})$ is the similarity dimension of $\Psi_{\underline{t}}$. More precisely, $s(\underline{t})$ satisfies the equation

$$\sum_{i\in\mathcal{I}} \lambda_i(\underline{t})^{s(\underline{t})} = 1. \tag{2.2}$$

We can use the following lemma, which follows from [**13**, Lemma 7.3], to prove transversality.

LEMMA 2.3. *Let $U \subset \mathbb{R}^d$ be an open, bounded set with a smooth boundary and $g_{\mathbf{i},\mathbf{j}}(\underline{t}) = \pi_{\underline{t}}(\mathbf{i}) - \pi_{\underline{t}}(\mathbf{j})$. If for every $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_0 \ne j_0$ and for every $\underline{t}_0 \in U$*

$$g_{\mathbf{i},\mathbf{j}}(\underline{t}_0) = 0 \Rightarrow \|\operatorname{grad}_{\underline{t}} g_{\mathbf{i},\mathbf{j}}|_{\underline{t}=\underline{t}_0}\| > 0, \tag{2.3}$$

then there is transversality on any open subset V whose closure is contained in U.

3. *Proof of Theorem 1.1*

3.1. *Natural projection.*

Let p, q be positive integers and let

$$\varphi_{i,1}(x) = \gamma_{i,1}x \quad \text{for } i = 0, \ldots, p,$$
$$\varphi_{i,2}(x) = \gamma_{i,2}x + (1 - \gamma_{i,2}) \quad \text{for } i = 0, \ldots, q.$$

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Then our main assumptions (A1) and (A2) are equivalent to $0 < \gamma_{i,1} < \gamma_{0,1} < 1$ for every $i = 1, \ldots, p$ and $0 < \gamma_{i,2} < \gamma_{0,2} < 1$ for every $i = 1, \ldots, q$. Moreover,

$$\gamma_{0,1} + \gamma_{0,2} < 1.$$

Therefore, without loss of generality, we can assume that

$$\gamma_{i,1} = c_{i,1}\gamma_{0,1},$$
$$\gamma_{i,2} = c_{i,2}\gamma_{0,2},$$

where $0 < c_{i,1}, c_{j,2} < 1$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Then $\mathcal{F}$ can be written in the form

$$\mathcal{F} = \{\gamma_{0,1}x,\ \gamma_{0,2}x + (1 - \gamma_{0,2})\} \cup \{c_{i,1}\gamma_{0,1}x\}_{i=1}^{p} \cup \{c_{i,2}\gamma_{0,2}x + (1 - c_{i,2}\gamma_{0,2})\}_{i=1}^{q}.$$

Let us introduce the vectors of parameters, namely, $\underline{c}_1 = (c_{1,1}, \ldots, c_{p,1}) \in (0, 1)^p$ and $\underline{c}_2 = (c_{1,2}, \ldots, c_{q,2}) \in (0, 1)^q$; moreover, $\underline{c} = (\underline{c}_1, \underline{c}_2)$.

Denote the set of symbols of the functions with fixed point 0 by A_1, and, similarly, denote the set of symbols of the functions with fixed point 1 by A_2. So,

$$A_1 = \{(0, 1), \ldots, (p, 1)\} \quad \text{and} \quad A_2 = \{(0, 2), \ldots, (q, 2)\}.$$

Let Σ be the symbolic space generated by $A_1 \cup A_2$ and Σ^* the set of finite words, that is $\Sigma = (A_1 \cup A_2)^{\mathbb{N}}$ and $\Sigma^* = \bigcup_{n=0}^{\infty}(A_1 \cup A_2)^n$. For any $\underline{i} = ((i_0, \kappa_0)(i_1, \kappa_1) \cdots (i_n, \kappa_n)) \in \Sigma^*$, we use the notation

$$\varphi_{\underline{i}} = \varphi_{i_0,\kappa_0} \circ \varphi_{i_1,\kappa_1} \circ \cdots \circ \varphi_{i_n,\kappa_n} \quad \text{and} \quad \gamma_{\underline{i}} = \gamma_{i_0,\kappa_0} \cdots \gamma_{i_n,\kappa_n}.$$

For an $\mathbf{i} \in \Sigma$, we write $\mathbf{i}(k)$ as the first k elements of $\mathbf{i}$. In particular, $\mathbf{i}(k) = ((i_0, \kappa_0) \cdots (i_{k-1}, \kappa_{k-1}))$ and $\mathbf{i}(0) = \emptyset$. For $j = 1, 2$ and $i = 0, \ldots, p$ or q, we define $\sharp_{i,j}\mathbf{i}(k)$ as the number of (i, j) in $\mathbf{i}(k)$. Moreover, for $j = 1, 2$ we define $\sharp_j\mathbf{i}(k)$ as the number of symbols from A_j in $\mathbf{i}(k)$. Clearly, $\sharp_1\mathbf{i}(k) = \sum_{i=0}^{p} \sharp_{i,1}\mathbf{i}(k)$ and, respectively, $\sharp_2\mathbf{i}(k) = \sum_{i=0}^{q} \sharp_{i,2}\mathbf{i}(k)$. Using this notation and the definition of the natural projection (2.1),

$$\pi_{\underline{c}}(\mathbf{i}) = \sum_{k=0}^{\infty}\left(\sum_{l=0}^{q} \delta_{(i_k,\kappa_k)}^{(l,2)}(1 - \gamma_{l,2})\right)\gamma_{0,1}^{\sharp_1\mathbf{i}(k)}\gamma_{0,2}^{\sharp_2\mathbf{i}(k)} \prod_{i=1}^{p} c_{i,1}^{\sharp_{(i,1)}\mathbf{i}(k)} \prod_{i=1}^{q} c_{i,2}^{\sharp_{(i,2)}\mathbf{i}(k)}, \tag{3.1}$$

where

$$\delta_j^k = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

The set of k's satisfying $(i_k, \kappa_k) \in A_2$ gives us non-zero elements in the infinite sum above. Hence it is useful to define $\beta_i^{\mathbf{i}}$ as the number of $(i, 2)$ in $\mathbf{i}$ and $\beta^{\mathbf{i}}$ as the number of symbols from A_2 in $\mathbf{i}$. Clearly, $\beta_i^{\mathbf{i}} = \lim_{k\to\infty} \sharp_{(i,2)}\mathbf{i}(k)$ and $\beta^{\mathbf{i}} = \sum_{l=0}^{q} \beta_l^{\mathbf{i}}$. Moreover, let $m_k^{\mathbf{i}}$ be the position of the kth symbol from A_2 in $\mathbf{i}$. Applying the notation, $\sharp_2\mathbf{i}(m_k^{\mathbf{i}}) = k - 1$ and

$$\pi_{\underline{c}}(\mathbf{i}) = \sum_{k=1}^{\beta^{\mathbf{i}}}\left(\sum_{l=0}^{q} \delta_{(i_{m_k^{\mathbf{i}}},\kappa_{m_k^{\mathbf{i}}})}^{(l,2)}(1 - \gamma_{l,2})\right)\gamma_{0,2}^{k-1}\gamma_{0,1}^{\sharp_1\mathbf{i}(m_k^{\mathbf{i}})} \prod_{l=1}^{p} c_{l,1}^{\sharp_{(l,1)}\mathbf{i}(m_k^{\mathbf{i}})} \prod_{l=1}^{q} c_{l,2}^{\sharp_{(l,2)}\mathbf{i}(m_k^{\mathbf{i}})}. \tag{3.2}$$

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For every $i = 1, \ldots, p$, we write (3.2) as the power series of $c_{i,1}$. So, we collect all the different exponents of $c_{i,1}$ into the set $P_{\mathbf{i}}^i$. It is easy to see that if $\beta^{\mathbf{i}} = 0$, then $P_{\mathbf{i}}^i = \emptyset$, otherwise

$$P_{\mathbf{i}}^i = \{m \geq 0 : \exists k \geq 1,\ \sharp_{(i,1)}\mathbf{i}(m_k^{\mathbf{i}}) = m\} \quad \text{for } i = 1, \ldots, p.$$

Then we can write the natural projection in the following form:

$$\pi_{\underline{c}}(\mathbf{i}) = \sum_{m \in P_{\mathbf{i}}^i} h_i^m(\mathbf{i}) c_{(i,1)}^m. \tag{3.3}$$

For every $m \in P_{\mathbf{i}}^i$, the coefficient $h_i^m(\mathbf{i})$ of $c_{i,1}^m$ is the sum of those elements of (3.2) divided by $c_{i,1}^m$ whose indexes k satisfy $\sharp_{(i,1)}\mathbf{i}(m_k^{\mathbf{i}}) = m$. Precisely,

$$h_i^m(\mathbf{i}) = \sum_{k=\underline{s}_m^i(\mathbf{i})}^{\overline{s}_m^i(\mathbf{i})} \left(\sum_{l=0}^{q} \delta^{(l,2)}_{(i_{m_k^{\mathbf{i}}}, \kappa_{m_k^{\mathbf{i}}})} (1 - \gamma_{l,2}) \right) \gamma_{0,2}^{k-1} \gamma_{0,1}^{\sharp_1\mathbf{i}(m_k^{\mathbf{i}})} \prod_{\substack{l=1 \\ l \neq i}}^{p} c_{l,1}^{\sharp_{(l,1)}\mathbf{i}(m_k^{\mathbf{i}})} \prod_{l=1}^{q} c_{l,2}^{\sharp_{(l,2)}\mathbf{i}(m_k^{\mathbf{i}})}, \tag{3.4}$$

where

$$\overline{s}_m^i(\mathbf{i}) = \sup\{k : \sharp_{(i,1)}\mathbf{i}(m_k^{\mathbf{i}}) = m\} \quad \text{and} \quad \underline{s}_m^i(\mathbf{i}) = \inf\{k : \sharp_{(i,1)}\mathbf{i}(m_k^{\mathbf{i}}) = m\}.$$

LEMMA 3.1. *Let $\mathbf{i} \in \Sigma$. Then, for every $i = 1, \ldots, p$ and every $m \in P_{\mathbf{i}}^i$,*

$$h_i^m(\mathbf{i}) \leq \gamma_{0,2}^{\underline{s}_m^i(\mathbf{i})-1} \gamma_{0,1}^{\sharp_1\mathbf{i}(m^{\mathbf{i}}_{\underline{s}_m^i(\mathbf{i})})} \prod_{\substack{l=1 \\ l \neq i}}^{p} c_{l,1}^{\sharp_{(l,1)}\mathbf{i}(m^{\mathbf{i}}_{\underline{s}_m^i(\mathbf{i})})}.$$

Moreover, if $0 \in P_{\mathbf{i}}^i$, then

$$h_i^0(\mathbf{i}) \geq \gamma_{0,1}^{m_1^{\mathbf{i}}-1} \prod_{\substack{l=1 \\ l \neq i}}^{p} c_{l,1}^{\sharp_{(l,1)}\mathbf{i}(m_1^{\mathbf{i}})} (1 - \gamma_{0,2}).$$

Proof. Let $\mathbf{i} \in \Sigma$ and for $m \in P_{\mathbf{i}}^i$ let

$$\underline{i}_m = \left(\left(i_{m_{\underline{s}_m^i(\mathbf{i})}}, \kappa_{m_{\underline{s}_m^i(\mathbf{i})}} \right) \cdots \left(i_{m_{\overline{s}_m^i(\mathbf{i})}}, \kappa_{m_{\overline{s}_m^i(\mathbf{i})}} \right) \right).$$

By the definition of $\overline{s}_m^i(\mathbf{i})$ and $\underline{s}_m^i(\mathbf{i})$, $\underline{i}_m$ is the segment of $\mathbf{i}$ corresponding to the coefficient $h_i^m(\mathbf{i})$. By (3.4),

$$h_i^m(\mathbf{i}) = \gamma_{0,2}^{\underline{s}_m^i(\mathbf{i})-1} \gamma_{0,1}^{\sharp_1\mathbf{i}(m^{\mathbf{i}}_{\underline{s}_m^i(\mathbf{i})})} \prod_{\substack{l=1 \\ l \neq i}}^{p} c_{l,1}^{\sharp_{(l,1)}\mathbf{i}(m^{\mathbf{i}}_{\underline{s}_m^i(\mathbf{i})})} \prod_{l=1}^{q} c_{l,2}^{\sharp_{(l,2)}\mathbf{i}(m^{\mathbf{i}}_{\underline{s}_m^i(\mathbf{i})})} \varphi_{\underline{i}_m}(0).$$

By the definition, $\kappa_{m_{\underline{s}_m^i(\mathbf{i})}} = 2$, which implies that

$$1 - \gamma_{0,2} \leq \varphi_{\underline{i}_m}(0) \leq 1$$

for every $m \in P_{\mathbf{i}}^i$.

If $0 \in P_{\mathbf{i}}^i$, then before the first $(i, 1)$, there has to be at least one symbol from A_2. Therefore, $\underline{s}_0^{\mathbf{i}} = 1$. Moreover, before the place of the first symbol from A_2, the number of symbols from A_1 is $m_1^{\mathbf{i}} - 1$. This proves the assertion of the lemma. □

3.2. *Proof of the transversality condition.* For every $\mathbf{i}, \mathbf{j} \in A_\kappa^{\mathbb{N}}$ $(\kappa = 1, 2)$, $\pi_{\underline{c}}(\mathbf{i})$ and $\pi_{\underline{c}}(\mathbf{j})$ are equal as the functions of $\underline{c}$. This implies the IFS $\mathcal{F}$ does not satisfy the

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transversality condition. The goal of this section is to introduce a sequence of iterated function systems which satisfy the transversality and are suitable for approximating the Hausdorff dimension of the attractor of $\mathcal{F}$.

$\varphi_{i_0,\kappa} \circ \varphi_{i_1,\kappa} = \varphi_{i_1,\kappa} \circ \varphi_{i_0,\kappa}$ holds for every (i_0, κ), $(i_1, \kappa) \in A_\kappa$, which is in the way of transversality. To eliminate this problem, we choose a sequence of subsets of Σ^* such that we order the symbols in each word by the first coordinate.

Define

$$\begin{aligned} &\mathcal{P}_0 = \{(0,1);(0,2)\} \quad \text{and} \\ &\mathcal{P}_1 = \{(1,2)(0,1);\cdots;(q,2)(0,1);(1,1)(0,2).\cdots;(p,1)(0,2)\} \end{aligned} \tag{3.5}$$

By induction, for $k \geq 2$,

$$\mathcal{P}_k = \left(\bigcup_{j=1}^{p} \bigcup_{\substack{\underline{i}\in\mathcal{P}_{k-1} \\ \kappa_0\neq 1 \vee j\leq i_0}} \{(j,1)\underline{i}\}\right) \cup \left(\bigcup_{j=1}^{q} \bigcup_{\substack{\underline{i}\in\mathcal{P}_{k-1} \\ \kappa_0\neq 2 \vee j\leq i_0}} \{(j,2)\underline{i}\}\right) \tag{3.6}$$

and

$$\mathcal{U}_k = \bigcup_{l=0}^{k} \mathcal{P}_l. \tag{3.7}$$

Denote $\Sigma_k = \mathcal{U}_k^{\mathbb{N}}$ and the sequence of IFSs

$$\Psi_k = \{\varphi_{\underline{i}}\}_{\underline{i}\in\mathcal{U}_k}. \tag{3.8}$$

PROPOSITION 3.2. *Let $\xi > 0$ be arbitrarily small, then the system Ψ_k satisfies the transversality condition on $\underline{c} \in (\xi, 1-\xi)^{p+q}$ for every $k \geq 1$.*

Proof. Suppose that $\underline{c} \in (\xi/2, 1-\xi/2)^{p+q}$ and let $\mathbf{i}', \mathbf{j}' \in \Sigma_k = \mathcal{U}_k^{\mathbb{N}}$ such that $\underline{i}_0 \neq \underline{j}_0 \in \mathcal{U}_k$. Denote $\mathbf{i}'$ (and $\mathbf{j}'$) as the element of Σ by $\mathbf{i}$ (and $\mathbf{j}$, respectively). To prove transversality by Lemma 2.3, it is enough to show that

$$\pi_{\underline{c}}(\mathbf{i}) = \pi_{\underline{c}}(\mathbf{j}) \implies \operatorname{grad}_{\underline{c}}(\pi_{\underline{c}}(\mathbf{i}) - \pi_{\underline{c}}(\mathbf{j})) \neq 0. \tag{3.9}$$

Suppose that $\pi_{\underline{c}}(\mathbf{i}) = \pi_{\underline{c}}(\mathbf{j})$. Since $\gamma_{0,1} + \gamma_{0,2} < 1$, the first element of $\mathbf{i}$, (i_0, κ_0), and the first element of $\mathbf{j}$, (j_0, τ_0), have to satisfy that $\kappa_0 = \tau_0$. Then $\mathbf{i}$, $\mathbf{j}$ can be written in the form

$$\begin{aligned} \mathbf{i} &= \overbrace{(0,\kappa)\cdots(0,\kappa)}^{r_0}\overbrace{(1,\kappa)\cdots(1,\kappa)}^{r_1}\cdots\overbrace{(s,\kappa)\cdots(s,\kappa)}^{r_s}(l_1, 3-\kappa)\cdots \\ \mathbf{j} &= \overbrace{(0,\kappa)\cdots(0,\kappa)}^{t_0}\overbrace{(1,\kappa)\cdots(1,\kappa)}^{t_1}\cdots\overbrace{(s,\kappa)\cdots(s,\kappa)}^{t_s}(l_2, 3-\kappa)\cdots, \end{aligned}$$

where $r_i, t_i \geq 0$ for $i = 1, \ldots, s$, $s = p$ if $\kappa = 1$ and $s = q$ otherwise.

If $r_i \leq t_i$ for every $i = 0, \ldots, s$, and there exists an $1 \leq i \leq s$ such that $r_i < t_i$, then, by $\gamma_{0,1} + \gamma_{0,2} < 1$, $\pi_{\underline{c}}(\mathbf{i}) \neq \pi_{\underline{c}}(\mathbf{j})$, which is a contradiction. There are therefore two possibilities: there exist $i \neq j$ such that $r_i > t_i$ and $r_j < t_j$ or $r_i = t_i$ for every $i = 0, \ldots, s$. In the second case

$$0 = \pi_{\underline{c}}(\mathbf{i}) - \pi_{\underline{c}}(\mathbf{j}) = \gamma_{0,\kappa}^{\sum_{i=0}^{s} r_i} \prod_{i=1}^{s} c_{i,\kappa}^{r_i} (\pi_{\underline{c}}(\sigma^{\sum_{i=0}^{s} r_i}\mathbf{i}) - \pi_{\underline{c}}(\sigma^{\sum_{i=0}^{s} r_i}\mathbf{j})).$$

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Since $c_{i,\kappa} > \xi/2$ for every $\kappa = 1, 2$ and $i = 1, \dots, p$ or q, and, moreover, $\underline{i}_0 \neq \underline{j}_0$, without loss of generality, we can assume the first case.

First, let us suppose that $\kappa = 1$; then $\mathbf{i}$ and $\mathbf{j}$ are in the form

$$\mathbf{i} = \overbrace{(0,1)\cdots(0,1)}^{r_0}\overbrace{(1,1)\cdots(1,1)}^{r_1}\cdots\overbrace{(s,1)\cdots(s,1)}^{r_p}(l_1,2)\cdots$$
$$\mathbf{j} = \overbrace{(0,1)\cdots(0,1)}^{t_0}\overbrace{(1,1)\cdots(1,1)}^{t_1}\cdots\overbrace{(s,1)\cdots(s,1)}^{t_p}(l_2,2)\cdots,$$

and there exists $1 \le j \le p$ such that $r_j < t_j$. There also exists an $0 \le i \le p$ such that $r_i > t_i$ and $i \neq j$, but we prove the transversality derivation in $c_{j,1}$.

Let

$$\mathbf{i}^* = \overbrace{(0,1)\cdots(0,1)}^{r_0}\cdots\overbrace{(j-1,1)\cdots(j-1,1)}^{r_{j-1}}\overbrace{(j+1,1)\cdots(j+1,1)}^{r_{j+1}}\cdots(l_1,2)\cdots$$

and

$$\mathbf{j}^* = \overbrace{(0,1)\cdots(0,1)}^{t_0}\cdots\overbrace{(j,1)\cdots(j,1)}^{t_j-r_j}\cdots(l_2,2)\cdots.$$

Then

$$\pi_{\underline{c}}(\mathbf{i}) - \pi_{\underline{c}}(\mathbf{j}) = \gamma_{j,1}^{r_j} c_{j,1}^{r_j}(\pi_{\underline{c}}(\mathbf{i}^*) - \pi_{\underline{c}}(\mathbf{j}^*)).$$

Let $a(\underline{c}) = \pi_{\underline{c}}(\mathbf{i}^*) - \pi_{\underline{c}}(\mathbf{j}^*)$. Since $c_{j,1} > \xi/2$, to prove transversality, it is enough to show that

$$a(\underline{c}) = 0 \implies \frac{\partial a}{\partial c_{j,1}}(\underline{c}) \neq 0$$

for every $\underline{c} \in (\xi/2, 1-\xi/2)^{p+q}$, but instead of showing that, we prove

$$\frac{\partial a}{\partial c_{j,1}}(\underline{c}) = 0 \implies a(\underline{c}) > 0 \tag{3.10}$$

for every $\underline{c} \in (\xi/2, 1-\xi/2)^{p+q}$. By (3.3), we have

$$a(\underline{c}) = h_j^0(\mathbf{i}^*) + \sum_{m \in P_{\mathbf{i}^*}^j \setminus \{0\}} h_j^m(\mathbf{i}^*) c_{j,1}^m - \sum_{m \in P_{\mathbf{j}^*}^j} h_j^m(\mathbf{j}^*) c_{j,1}^m.$$

Let $\underline{c} \in (\xi/2, 1-\xi/2)^{p+q}$ such that $(\partial a/\partial c_{j,1})(\underline{c}) = 0$; then

$$0 = c_{j,1}\frac{\partial a}{\partial c_{j,1}}(\underline{c}) = h_j^0(\mathbf{i}^*)\Bigg(\sum_{m \in P_{\mathbf{i}^*}^j \setminus \{0\}} \frac{h_j^m(\mathbf{i}^*)}{h_j^0(\mathbf{i}^*)} m c_{j,1}^m - \sum_{m \in P_{\mathbf{j}^*}^j} \frac{h_j^m(\mathbf{j}^*)}{h_j^0(\mathbf{i}^*)} m c_{j,1}^m\Bigg)$$
$$\le h_j^0(\mathbf{i}^*)\Bigg(\sum_{m \in P_{\mathbf{i}^*}^j \setminus \{0\}} \frac{h_j^m(\mathbf{i}^*)}{h_j^0(\mathbf{i}^*)}(m-1) c_{j,1}^m + \sum_{m \in P_{\mathbf{i}^*}^j \setminus \{0\}} \frac{h_j^m(\mathbf{i}^*)}{h_j^0(\mathbf{i}^*)} c_{j,1}^m - \sum_{m \in P_{\mathbf{j}^*}^j} \frac{h_j^m(\mathbf{j}^*)}{h_j^0(\mathbf{i}^*)} c_{j,1}^m\Bigg).$$

It is enough to prove that

$$\sum_{m \in P_{\mathbf{i}^*}^j \setminus \{0\}} \frac{h_j^m(\mathbf{i}^*)}{h_j^0(\mathbf{i}^*)}(m-1) c_{j,1}^m < 1.$$

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By Lemma 3.1, we have

$$\sum_{m\in P^j_{\mathbf{i}^*}\setminus\{0\}} \frac{h^m_j(\mathbf{i}^*)}{h^0_j(\mathbf{i}^*)}(m-1)c^m_{j,1}$$

$$\le \sum_{m\in P^j_{\mathbf{i}^*}\setminus\{0\}} \frac{\gamma_{0,2}^{\underline{s}^j_m(\mathbf{i}^*)-1}\gamma_{0,1}^{\sharp_1\mathbf{i}^*(m^{\mathbf{i}^*}_{\underline{s}^j_m(\mathbf{i}^*)})}\prod_{\substack{l=1\\ l\neq j}}^{p} c_{l,1}^{\sharp_{(l,1)}\mathbf{i}^*(m^{\mathbf{i}^*}_{\underline{s}^j_m(\mathbf{i}^*)})}}{\gamma_{0,1}^{m^{\mathbf{i}^*}_1-1}\prod_{\substack{l=1\\ l\neq j}}^{p} c_{l,1}^{\sharp_{(l,1)}\mathbf{i}(m^{\mathbf{i}^*}_1)}(1-\gamma_{0,2})}(m-1)c^m_{j,1}. \quad (3.11)$$

Since $\mathbf{i}^*$ does not contain $(j,1)$ before the first element from A_2, $\underline{s}^j_0(\mathbf{i}^*)=1$ and $\sharp_1\mathbf{i}^*(m^{\mathbf{i}^*}_{\underline{s}^j_m(\mathbf{i}^*)}) \ge m^{\mathbf{i}^*}_1 + m - 1$ for every $m \in P^j_{\mathbf{i}^*}\setminus\{0\}$.

Let $q_1 = \min P^j_{\mathbf{i}^*}\setminus\{0\}$ and $q_2 = \min P^j_{\mathbf{i}^*}\setminus\{0, q_1\}$. We define the minimum of the empty set as infinity. Then $\underline{s}^j_{q_1}(\mathbf{i}^*) \ge 2$ and $\underline{s}^j_{q_2}(\mathbf{i}^*) \ge 3$. This implies that the right-hand side of (3.11) is less than or equal to

$$\frac{\gamma_{0,1}^{q_1}\gamma_{0,2}}{1-\gamma_{0,2}}(q_1-1)c^{q_1}_{j,1} + \frac{\gamma_{0,1}^{q_2}\gamma_{0,2}^2}{1-\gamma_{0,2}}(q_2-1)c^{q_2}_{j,1}$$
$$+ \frac{\gamma_{0,2}^3}{1-\gamma_{0,2}} \sum_{m\in P^j_{\mathbf{i}^*}\setminus\{0,q_1,q_2\}} \gamma^m_{0,1}(m-1)c^m_{j,1}. \quad (3.12)$$

Using that $(n-1)\gamma^n_{0,1} \le -\gamma_{0,1}/(e\ln\gamma_{0,1})$ for every $n\in\mathbb{N}$, we get that (3.12) is less than or equal to

$$\frac{-\gamma_{0,1}(\gamma_{0,2}+\gamma_{0,2}^2)}{(1-\gamma_{0,2})e\ln\gamma_{0,1}} + \frac{\gamma_{0,2}^3}{1-\gamma_{0,2}}\sum_{m=3}^{\infty}(m-1)\gamma^m_{0,1}$$
$$= \frac{-\gamma_{0,1}(\gamma_{0,2}+\gamma_{0,2}^2)}{(1-\gamma_{0,2})e\ln\gamma_{0,1}} + \frac{\gamma_{0,2}^3}{1-\gamma_{0,2}}\frac{\gamma_{0,1}^3(2-\gamma_{0,1})}{(1-\gamma_{0,1})^2}.$$

Using the assumption $\gamma_{0,1}+\gamma_{0,2}<1$, by some algebraic manipulation, we get that

$$\frac{-\gamma_{0,1}(\gamma_{0,2}+\gamma_{0,2}^2)}{(1-\gamma_{0,2})e\ln\gamma_{0,1}} + \frac{\gamma_{0,2}^3}{1-\gamma_{0,2}}\frac{\gamma_{0,1}^3(2-\gamma_{0,1})}{(1-\gamma_{0,1})^2} < 1,$$

which implies (3.10).

To prove transversality in the second case, when $\kappa=2$, we introduce the function $\eta(x) = -x+1$. Let us observe that $\eta\circ\eta(x) = x$. Let

$$\widetilde{\varphi}_{i,1}(x) := \eta\circ\varphi_{i,1}\circ\eta(x) = \gamma_{i,1}x + (1-\gamma_{i,1}) \quad \text{for } i=0,\dots,p,$$
$$\widetilde{\varphi}_{i,2}(x) := \eta\circ\varphi_{i,2}\circ\eta(x) = \gamma_{i,2}x \quad \text{for } i=0,\dots,q.$$

The IFS $\widetilde{\mathcal{F}} = \{\widetilde{\varphi}_{i,1}\}_{i=0}^p \cup \{\widetilde{\varphi}_{i,2}\}_{i=0}^q$ and $\mathcal{F}$ are equivalent. More precisely, let $\widetilde{\pi}_{\underline{c}}$ be the natural projection of $\widetilde{\mathcal{F}}$, then $\widetilde{\pi}_{\underline{c}}(\mathbf{i}) = -\pi_{\underline{c}}(\mathbf{i}) + 1$ for every $\mathbf{i}\in\Sigma$. Using this fact, one can prove transversality in the case $\kappa=2$ as in $\kappa=1$.

The proof can be finished by applying Lemma 2.3. □

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3.3. *Hausdorff dimension.* In the first part of the section, we calculate the Hausdorff dimension of the attractor of Ψ_k (see (3.8)) and, in the second part, we will prove that the limit will correspond with the dimension of the attractor of $\mathcal{F}$.

For $k \geq 0$, let

$$d_k(s) = \sum_{\underline{i} \in \mathcal{U}_k} \gamma_{\underline{i}}^s.$$

By the definition of $\mathcal{U}_k$ (see (3.7)), for $k \geq 1$,

$$d_k(s) = \gamma_{0,1}^s + \gamma_{0,2}^s + \gamma_{0,1}^s \sum_{l=1}^{k} \Phi_l + \gamma_{0,2}^s \sum_{l=1}^{k} \Upsilon_l,$$

where

$$\Phi_k = \sum_{\substack{\underline{i} \in \mathcal{P}_k \\ (i_k, h_k) = (0,1)}} \frac{\gamma_{\underline{i}}^s}{\gamma_{0,1}^s}$$

and

$$\Upsilon_k = \sum_{\substack{\underline{i} \in \mathcal{P}_k \\ (i_k, h_k) = (0,2)}} \frac{\gamma_{\underline{i}}^s}{\gamma_{0,2}^s}.$$

LEMMA 3.3. *Let us denote the attractor of Ψ_k by Λ_k. Then*

$$\dim_H \Lambda_k = \min\{1, s_k\} \quad \textit{for Lebesgue-a.e. } \underline{c} \in (0, 1)^{p+q},$$

where s_k is the unique solution of $d_k(s) = 1$.

Proof. By Proposition 3.2, Ψ_k satisfies the transversality condition on $\underline{c} \in (\xi, 1-\xi)^{p+q}$ for every arbitrarily small $\xi > 0$. Since $d_k(s)$ is the sum of the contraction ratios of the functions in the IFS Ψ_k to the power s, Theorem 2.2 implies that the Hausdorff dimension of Λ_k is equal to $\min\{1, s_k\}$, where s_k is the unique solution of

$$d_k(s) = 1 \tag{3.13}$$

for Lebesgue almost every $\underline{c} \in (\xi, 1-\xi)^{p+q}$. Since $\xi > 0$ was arbitrary, the lemma is proved. □

LEMMA 3.4. *Let s_k be the unique solution of $d_k(s) = 1$. Then the limit $\lim_{k\to\infty} s_k = s$ exists and s is the unique solution of*

$$\prod_{i=0}^{p}(1 - \gamma_{i,1}^s) + \prod_{i=0}^{q}(1 - \gamma_{i,2}^s) = 1. \tag{3.14}$$

The proof of formula (3.14) is a sequence of tedious algebraic manipulations, carried out in the following pages.

Proof of Lemma 3.4. Without loss of generality, we can assume that $p \leq q$. Let

$$\Phi_k^{i,\kappa} = \sum_{\substack{\underline{i} \in \mathcal{P}_k \\ (i_k, \kappa_k) = (0,1) \\ (i_1, \kappa_1) = (i, \kappa)}} \frac{\gamma_{\underline{i}}^s}{\gamma_{0,1}^s}, \quad \Upsilon_k^{i,\kappa} = \sum_{\substack{\underline{i} \in \mathcal{P}_k \\ (i_k, \kappa_k) = (0,2) \\ (i_1, \kappa_1) = (i, \kappa)}} \frac{\gamma_{\underline{i}}^s}{\gamma_{0,2}^s}.$$

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Then

$$\Phi_k = \sum_{i=1}^{p} \Phi_k^{i,1} + \sum_{i=1}^{q} \Phi_k^{i,2} \quad \text{and} \quad \Upsilon_k = \sum_{i=1}^{p} \Upsilon_k^{i,1} + \sum_{i=1}^{q} \Upsilon_k^{i,2}.$$

By the definition of $\mathcal{P}_k$ (see (3.5) and (3.6)), we have

$$\begin{aligned} \Phi_1^{i,1} &= 0 \quad \text{for } i = 1, \dots, p, \\ \Phi_1^{i,2} &= \gamma_{i,2}^s \quad \text{for } i = 1, \dots, q, \\ \Upsilon_1^{i,1} &= \gamma_{i,1}^s \quad \text{for } i = 1, \dots, p, \\ \Upsilon_1^{i,2} &= 0 \quad \text{for } i = 1, \dots, q. \end{aligned} \tag{3.15}$$

Moreover, for $k \geq 2$,

$$\begin{aligned} \Phi_k^{i,\kappa} &= \gamma_{i,\kappa}^s \Big(\Phi_{k-1} - \sum_{l=1}^{i-1} \Phi_{k-1}^{l,\kappa} \Big), \\ \Upsilon_k^{i,\kappa} &= \gamma_{i,\kappa}^s \Big(\Upsilon_{k-1} - \sum_{l=1}^{i-1} \Upsilon_{k-1}^{l,\kappa} \Big). \end{aligned} \tag{3.16}$$

Denote

$$\begin{aligned} a_{k,1} &= \sum_{1 \leq j_0 < \cdots < j_{k-1} \leq p} \gamma_{j_0,1}^s \cdots \gamma_{j_{k-1},1}^s \quad \text{for } i = 1, \dots, p, \\ a_{k,2} &= \sum_{1 \leq j_0 < \cdots < j_{k-1} \leq q} \gamma_{j_0,2}^s \cdots \gamma_{j_{k-1},2}^s \quad \text{for } i = 1, \dots, q. \end{aligned} \tag{3.17}$$

Applying (3.16), we have, for $k \geq 2$,

$$\begin{aligned} \Phi_k &= \sum_{i=1}^{p} \Phi_k^{i,1} + \sum_{i=1}^{q} \Phi_k^{i,2} \\ &= \sum_{i=1}^{p} \gamma_{i,1}^s \Big(\Phi_{k-1} - \sum_{l=1}^{i-1} \Phi_{k-1}^{l,1} \Big) + \sum_{i=1}^{q} \gamma_{i,2}^s \Big(\Phi_{k-1} - \sum_{l=1}^{i-1} \Phi_{k-1}^{l,2} \Big) \\ &= a_{1,1} \Phi_{k-1} + a_{1,2} \Phi_{k-1} - \sum_{l=1}^{p-1} \sum_{i=l+1}^{p} \gamma_{i,1}^s \Phi_{k-1}^{l,1} - \sum_{l=1}^{q-1} \sum_{i=l+1}^{q} \gamma_{i,2}^s \Phi_{k-1}^{l,2}, \end{aligned} \tag{3.18}$$

and similarly

$$\Upsilon_k = a_{1,1} \Upsilon_{k-1} + a_{1,2} \Upsilon_{k-1} - \sum_{l=1}^{p-1} \sum_{i=l+1}^{p} \gamma_{i,1}^s \Upsilon_{k-1}^{l,1} - \sum_{l=1}^{q-1} \sum_{i=l+1}^{q} \gamma_{i,2}^s \Upsilon_{k-1}^{l,2}. \tag{3.19}$$

Applying (3.16) to (3.18) and (3.19) n times, where $1 \leq n \leq p-1$ and $k \geq n+1$, we get

$$\begin{aligned} \Phi_k = &\sum_{l=1}^{n} (-1)^{l-1} a_{l,1} \Phi_{k-l} + (-1)^n \sum_{1 \leq j_0 < \cdots < j_n \leq p} \gamma_{j_n,1}^s \cdots \gamma_{j_1,1}^s \Phi_{k-n}^{j_0,1} \\ &+ \sum_{l=1}^{n} (-1)^{l-1} a_{l,2} \Phi_{k-l} + (-1)^n \sum_{1 \leq j_0 < \cdots < j_n \leq q} \gamma_{j_n,2}^s \cdots \gamma_{j_1,2}^s \Phi_{k-n}^{j_0,2} \end{aligned} \tag{3.20}$$

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and

$$\Upsilon_k = \sum_{l=1}^{n}(-1)^{l-1}a_{l,1}\Upsilon_{k-l} + (-1)^n \sum_{1\le j_0<\cdots<j_n\le p} \gamma^s_{j_n,1}\cdots\gamma^s_{j_1,1}\Upsilon^{j_0,1}_{k-n} + \sum_{l=1}^{n}(-1)^{l-1}a_{l,2}\Upsilon_{k-l} + (-1)^n \sum_{1\le j_0<\cdots<j_n\le q} \gamma^s_{j_n,2}\cdots\gamma^s_{j_1,2}\Upsilon^{j_0,2}_{k-n}. \quad (3.21)$$

Then, by (3.15) and choosing $n = k-1$, we get

$$\begin{aligned}\Phi_k &= \sum_{l=1}^{k-1}(-1)^{l-1}a_{l,1}\Phi_{k-l} + \sum_{l=1}^{k-1}(-1)^{l-1}a_{l,2}\Phi_{k-l} + (-1)^{k-1}a_{k,2},\\ \Upsilon_k &= \sum_{l=1}^{k-1}(-1)^{l-1}a_{l,1}\Upsilon_{k-l} + \sum_{l=1}^{k-1}(-1)^{l-1}a_{l,2}\Upsilon_{k-l} + (-1)^{k-1}a_{k,1}\end{aligned} \quad (3.22)$$

for $2 \le k \le p$. If $p < q$, we can apply (3.16) to (3.18) and (3.19) n times, where $p \le n \le q-1$ and $k \ge n+1$, and we have

$$\Phi_k = \sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Phi_{k-l} + \sum_{l=1}^{n}(-1)^{l-1}a_{l,2}\Phi_{k-l} + (-1)^n \sum_{1\le j_0<\cdots<j_n\le q} \gamma^s_{j_n,2}\cdots\gamma^s_{j_1,2}\Phi^{j_0,2}_{k-n} \quad (3.23)$$

and

$$\Upsilon_k = \sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Upsilon_{k-l} + \sum_{l=1}^{n}(-1)^{l-1}a_{l,2}\Upsilon_{k-l} + (-1)^n \sum_{1\le j_0<\cdots<j_n\le q} \gamma^s_{j_n,2}\cdots\gamma^s_{j_1,2}\Upsilon^{j_0,2}_{k-n}. \quad (3.24)$$

By (3.15) and $k = n+1$, we have

$$\begin{aligned}\Phi_k &= \sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Phi_{k-l} + \sum_{l=1}^{k-1}(-1)^{l-1}a_{l,2}\Phi_{k-l} + (-1)^{k-1}a_{k,2},\\ \Upsilon_k &= \sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Upsilon_{k-l} + \sum_{l=1}^{k-1}(-1)^{l-1}a_{l,2}\Upsilon_{k-l}\end{aligned} \quad (3.25)$$

for $p+1 \le k \le q$. By similar methods, we get, for $k \ge q+1$, that

$$\begin{aligned}\Phi_k &= \sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Phi_{k-l} + \sum_{l=1}^{q}(-1)^{l-1}a_{l,2}\Phi_{k-l},\\ \Upsilon_k &= \sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Upsilon_{k-l} + \sum_{l=1}^{q}(-1)^{l-1}a_{l,2}\Upsilon_{k-l}.\end{aligned} \quad (3.26)$$

The convergence of the infinite series $\sum_{l=1}^{\infty}\Phi_l$ and $\sum_{l=1}^{\infty}\Upsilon_l$ depends on the roots of the characteristic polynomial of (3.26). More precisely, $\sum_{l=1}^{\infty}\Phi_l$ and $\sum_{l=1}^{\infty}\Upsilon_l$ are

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convergent if and only if the roots of the characteristic polynomial are strictly less than 1. The characteristic polynomial is

$$x^q = \sum_{l=1}^{p} (-1)^{l-1} a_{l,1} x^{q-l} + \sum_{l=1}^{q} (-1)^{l-1} a_{l,2} x^{q-l}.$$

Since the roots of a polynomial depend continuously on the coefficients of the polynomial, except for the coefficient of x^q, the coefficients tend to zero as s tends to infinity. Therefore, the roots tend to zero as s tends to infinity. So, there exists a $\delta > 0$ such that $\sum_{l=1}^{\infty} \Phi_l$ and $\sum_{l=1}^{\infty} \Upsilon_l$ are convergent for $s \in (\delta, \infty)$. Let δ be the infinum of s such that $\sum_{l=1}^{\infty} \Phi_l$ and $\sum_{l=1}^{\infty} \Upsilon_l$ are convergent. Let

$$d(s) = \gamma_{0,1}^s + \gamma_{0,2}^s + \gamma_{0,1}^s \sum_{l=1}^{\infty} \Phi_l + \gamma_{0,2}^s \sum_{l=1}^{\infty} \Upsilon_l \quad \text{for } s \in (\delta, \infty). \tag{3.27}$$

Then there exists a unique $s^* \in (\delta, \infty)$ such that $d(s^*) = 1$. The sequence s_k (see (3.13)) is monotone increasing and bounded by s^*, therefore it is convergent. It is easy to see that $\lim_{k\to\infty} s_k = \sup_k s_k = s^*$.

Let

$$\Phi = \sum_{k=1}^{\infty} \Phi_k \quad \text{and} \quad \Upsilon = \sum_{k=1}^{\infty} \Upsilon_k.$$

Then, by (3.26),

$$\begin{aligned}
\Phi &= \sum_{k=q+1}^{\infty} \Phi_k + \sum_{k=1}^{q} \Phi_k \\
&= \sum_{k=q+1}^{\infty} \left(\sum_{l=1}^{p} (-1)^{l-1} a_{l,1} \Phi_{k-l} + \sum_{l=1}^{q} (-1)^{l-1} a_{l,2} \Phi_{k-l} \right) + \sum_{k=1}^{q} \Phi_k \\
&= \sum_{l=1}^{p} (-1)^{l-1} a_{l,1} \sum_{k=q+1-l}^{\infty} \Phi_k + \sum_{l=1}^{q} (-1)^{l-1} a_{l,2} \sum_{k=q+1-l}^{\infty} \Phi_k + \sum_{k=1}^{q} \Phi_k \\
&= \sum_{l=1}^{p} (-1)^{l-1} a_{l,1} \left(\Phi - \sum_{k=1}^{q-l} \Phi_k \right) + \sum_{l=1}^{q} (-1)^{l-1} a_{l,2} \left(\Phi - \sum_{k=1}^{q-l} \Phi_k \right) + \sum_{k=1}^{q} \Phi_k.
\end{aligned}$$

Therefore,

$$\Phi = \frac{\sum_{l=1}^{p} (-1)^l a_{l,1} \sum_{k=1}^{q-l} \Phi_k + \sum_{l=1}^{q} (-1)^l a_{l,2} \sum_{k=1}^{q-l} \Phi_k + \sum_{k=1}^{q} \Phi_k}{1 + \sum_{l=1}^{p} (-1)^l a_{l,1} + \sum_{l=1}^{q} (-1)^l a_{l,2}} \tag{3.28}$$

and, similarly,

$$\Upsilon = \frac{\sum_{l=1}^{p} (-1)^l a_{l,1} \sum_{k=1}^{q-l} \Upsilon_k + \sum_{l=1}^{q} (-1)^l a_{l,2} \sum_{k=1}^{q-l} \Upsilon_k + \sum_{k=1}^{q} \Upsilon_k}{1 + \sum_{l=1}^{p} (-1)^l a_{l,1} + \sum_{l=1}^{q} (-1)^l a_{l,2}}. \tag{3.29}$$

Applying (3.15), (3.22) and (3.25), we get

$$\begin{aligned}
\sum_{k=1}^{q} \Phi_k &= \Phi_1 + \sum_{k=2}^{p} \Phi_k + \sum_{k=p+1}^{q} \Phi_k \\
&= a_{1,2} + \sum_{k=2}^{p} \left(\sum_{l=1}^{k-1} (-1)^{l-1} a_{l,1} \Phi_{k-l} + \sum_{l=1}^{k-1} (-1)^{l-1} a_{l,2} \Phi_{k-l} + (-1)^{k-1} a_{k,2} \right)
\end{aligned}$$

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$$+\sum_{k=p+1}^{q}\left(\sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Phi_{k-l}+\sum_{l=1}^{k-1}(-1)^{l-1}a_{l,2}\Phi_{k-l}+(-1)^{k-1}a_{k,2}\right)$$
$$=\sum_{k=1}^{q}(-1)^{k-1}a_{k,2}+\sum_{l=1}^{p}\sum_{k=1}^{q-l}(-1)^{l-1}a_{l,1}\Phi_k+\sum_{l=1}^{q}\sum_{k=1}^{q-l}(-1)^{l-1}a_{l,2}\Phi_k \quad (3.30)$$

and, by similar arguments,

$$\sum_{k=1}^{q}\Upsilon_k=\sum_{k=1}^{p}(-1)^{k-1}a_{k,1}+\sum_{l=1}^{p}\sum_{k=1}^{q-l}(-1)^{l-1}a_{l,1}\Upsilon_k+\sum_{l=1}^{q}\sum_{k=1}^{q-l}(-1)^{l-1}a_{l,2}\Upsilon_k. \quad (3.31)$$

Hence the numerator of (3.28) is $\sum_{k=1}^{q}(-1)^{k-1}a_{k,2}$ and the numerator of (3.29) is $\sum_{k=1}^{p}(-1)^{k-1}a_{k,1}$, which implies that

$$\Phi=\frac{\sum_{k=1}^{q}(-1)^{k-1}a_{k,2}}{1+\sum_{l=1}^{p}(-1)^{l}a_{l,1}+\sum_{l=1}^{q}(-1)^{l}a_{l,2}} \quad \text{and}$$
$$\Upsilon=\frac{\sum_{k=1}^{p}(-1)^{k-1}a_{k,1}}{1+\sum_{l=1}^{p}(-1)^{l}a_{l,1}+\sum_{l=1}^{q}(-1)^{l}a_{l,2}}. \quad (3.32)$$

Then $d(s)=1$ (see (3.27)) is equivalent to

$$\gamma_{0,1}^{s}+\gamma_{0,2}^{s}+\sum_{l=1}^{p}(-1)^{l-1}a_{l,1}+\sum_{l=1}^{q}(-1)^{l-1}a_{l,2}$$
$$+\gamma_{0,1}^{s}\sum_{l=1}^{p}(-1)^{l}a_{l,1}+\gamma_{0,2}^{s}\sum_{l=1}^{q}(-1)^{l}a_{l,2}=1.$$

Let us observe that

$$x^{p+1}-\sum_{m=0}^{p}(-1)^{m}\sum_{0\le j_0<\cdots<j_m\le p}\gamma_{j_0,1}^{s}\cdots\gamma_{j_m,1}^{s}x^{p-m}=\prod_{k=0}^{p}(x-\gamma_{k,1}^{s})$$

and

$$x^{q+1}-\sum_{m=0}^{q}(-1)^{m}\sum_{0\le j_0<\cdots<j_m\le q}\gamma_{j_0,2}^{s}\cdots\gamma_{j_m,2}^{s}x^{q-m}=\prod_{k=0}^{q}(x-\gamma_{k,2}^{s}).$$

Then, by $x=1$, we get that $d(s)=1$ is equivalent to

$$2-\prod_{k=0}^{p}(1-\gamma_{k,1}^{s})-\prod_{k=0}^{q}(1-\gamma_{k,2}^{s})=1,$$

which is (3.14).

The proof will be complete if we show that (3.14) has a unique solution. We have that the left-hand side is equal to 2 if $s=0$ and the derivative is

$$\sum_{l=0}^{p}\gamma_{l,1}^{s}\log\gamma_{l,1}\prod_{\substack{k=0\\k\neq l}}^{p}(1-\gamma_{k,1}^{s})+\sum_{l=0}^{q}\gamma_{l,2}^{s}\log\gamma_{l,2}\prod_{\substack{k=0\\k\neq l}}^{q}(1-\gamma_{k,2}^{s}),$$

which is negative for $s>0$. This completes the proof. □

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Now we show that the unique solution of (3.14) is an upper bound for the Hausdorff dimension. To give a good cover of the attractor, we need to introduce another sequence of subsets of Σ^*. Let

$$\mathcal{C}_0 = \{(0, 1), (0, 2)\} \tag{3.33}$$

and, by induction, let

$$\mathcal{C}_k = \bigcup_{j=0}^{p} \bigcup_{\substack{\underline{i}\in\mathcal{C}_{k-1} \\ \kappa_0\neq 1\vee j\leq i_0}} \{(j, 1)\underline{i}\} \cup \bigcup_{j=0}^{q} \bigcup_{\substack{\underline{i}\in\mathcal{C}_{k-1} \\ \kappa_0\neq 2\vee j\leq i_0}} \{(j, 2)\underline{i}\}. \tag{3.34}$$

LEMMA 3.5. *Let $\widetilde{s}_k$ be the unique solution of*

$$\sum_{\underline{i}\in\mathcal{C}_k} \gamma_{\underline{i}}^{s} = 1,$$

and let $\widetilde{s} = \sup_k \widetilde{s}_k$, then

$$\dim_H \Lambda \leq \min\{1, \widetilde{s}\}.$$

Note that the sequence $\widetilde{s}_k$ is bounded since $\mathcal{C}_k \subseteq (A_1 \cup A_2)^{k+1}$.

Proof. Using that for every (i, κ), $(j, \kappa) \in A_\kappa$

$$\varphi_{(i,\kappa)} \circ \varphi_{(j,\kappa)} \equiv \varphi_{(j,\kappa)} \circ \varphi_{(i,\kappa)},$$

and $\gamma_{j,\kappa}$, $\gamma_{i,\kappa} \leq \gamma_{0,\kappa}$, we have that the set of closed intervals

$$\{\varphi_{\underline{i}}([0, 1])\}_{\underline{i}\in\mathcal{C}_k}$$

gives a cover of Λ with diameter at most $\gamma_{\max}^k$, where $\gamma_{\max} = \max_{i,\kappa}\{\gamma_{i,\kappa}\}$. Then

$$\mathcal{H}^{\widetilde{s}}_{\gamma_{\max}^k}(\Lambda) \leq \sum_{\underline{i}\in\mathcal{C}_k} |\varphi_{\underline{i}}([0, 1])|^{\widetilde{s}} = \sum_{\underline{i}\in\mathcal{C}_k} \gamma_{\underline{i}}^{\widetilde{s}} \leq \sum_{\underline{i}\in\mathcal{C}_k} \gamma_{\underline{i}}^{\widetilde{s}_k} = 1.$$

This proves the lemma. □

Proof of Theorem 1.1. By the definition of $\mathcal{C}_k$, we have that, for every $k \geq 1$,

$$\mathcal{C}_k \subset \bigcup_{l=1}^{k} \mathcal{U}_k^l. \tag{3.35}$$

More precisely, every $\underline{i} \in \mathcal{C}_k$ can be decomposed as a juxtaposition $\underline{i} = \underline{j}_1 \cdots \underline{j}_r$, where each $\underline{j}_l \in \mathcal{U}_k$. By similar arguments as in the proof of Proposition 3.2, one can show that the system $\widetilde{\Psi}_k = \{\varphi_{\underline{i}}\}_{\underline{i}\in\mathcal{C}_k}$ satisfies the transversality condition on $(\xi, 1-\xi)^{p+q}$. Since $\xi > 0$ was arbitrary, by Theorem 2.2, we have

$$\dim_H \widetilde{\Lambda}_k = \min\{1, \widetilde{s}_k\} \quad \text{for } \mathcal{L}\text{-a.e. } \underline{c} \in (0, 1)^{p+q}, \tag{3.36}$$

where $\widetilde{\Lambda}_k$ denotes the attractor of $\{\varphi_{\underline{i}}\}_{\underline{i}\in\mathcal{C}_k}$. Using (3.35), we have $\widetilde{\Lambda}_k \subseteq \Lambda_k \subseteq \Lambda$, which implies

$$\dim_H \widetilde{\Lambda}_k \leq \dim_H \Lambda_k \leq \dim_H \Lambda.$$

Therefore, by Lemmas 3.3 and 3.5, we have

$$\min\{1, \widetilde{s}_k\} \leq \min\{1, s_k\} \leq \min\{1, \widetilde{s}\}.$$

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By Lemma 3.4, s_k is convergent and $\lim_{k\to\infty} s_k = \sup_k s_k = s$. This implies that $\min\{1, s\} = \min\{1, \widetilde{s}\}$; moreover,

$$\dim_H \Lambda = \min\{1, s\}.$$

To complete the proof, we have to prove the measure claim. If $s > 1$, then there exists a $k \geq 2$ such that $s_k > 1$. Therefore, by Theorem 2.2 and Proposition 3.2, $\mathcal{L}(\Lambda) \geq \mathcal{L}(\Lambda_k) > 0$ for almost every $\underline{c} \in (0, 1)^{p+q} \cap \{\underline{c} : s > 1\}$. □

4. *Box dimension of the generalized* 4*-corner set*

In this section, we show an application of the results for two-dimensional, diagonally self-affine iterated function systems. Before we compute the box dimension of the generalized 4-corner set (see Figure 1), we state a general theorem on the box dimension of diagonally self-affine sets.

Let

$$f_i(x, y) = (\alpha_i x + t_i, \beta_i y + u_i) \tag{4.1}$$

for $i = 0, \ldots, m$ such that

$$\begin{aligned} &0 < \alpha_i, \beta_i < 1, \\ &f_i([0, 1]^2) \subseteq [0, 1]^2 \quad \text{for } i = 0, \ldots, m, \\ &f_i((0, 1)^2) \cap f_j((0, 1)^2) = \emptyset \quad \text{for } i \neq j. \end{aligned} \tag{4.2}$$

Denote the attractor of $\Psi = \{f_i(x, y)\}_{i=0}^m$ by Λ and define $\operatorname{proj}_x \Lambda$ (and $\operatorname{proj}_y \Lambda$) as the projection of Λ onto the x-axis (and y-axis, respectively).

THEOREM 4.1. *Let f_i be in the form (4.1) for $i = 0, \ldots, m$ and let us suppose that $\Psi = \{f_i(x, y)\}_{i=0}^m$ satisfies (4.2). Then the attractor Λ of Ψ satisfies*

$$\dim_B \Lambda = \max\{d_\alpha, d_\beta\},$$

where d_α and d_β are the unique solutions of

$$\sum_{i=0}^m \alpha_i^{s_\alpha} \beta_i^{d_\alpha - s_\alpha} = 1 \quad \text{and} \quad \sum_{i=0}^m \beta_i^{s_\beta} \alpha_i^{d_\beta - s_\beta} = 1,$$

where $s_\alpha = \dim_B \operatorname{proj}_x \Lambda$ and $s_\beta = \dim_B \operatorname{proj}_y \Lambda$.

Using this and [**12**, Theorem 2.1], we can compute the box dimension of the attractor, at least for almost all translations such that (4.2) holds.

COROLLARY 4.2. *Let f_i be in the form (4.1) for $i = 0, \ldots, m$ and let $\mathcal{T} \subset \mathbb{R}^{2m+2}$ be the set of translation vectors such that $\Psi = \{f_i(x, y)\}_{i=0}^m$ satisfies (4.2). Then the attractor Λ of Ψ satisfies*

$$\dim_B \Lambda = \max\{d_\alpha, d_\beta\} \text{ for almost every translation in } \mathcal{T} \text{ with respect to the } (2m+2)\text{-dimensional Lebesgue measure},$$

where d_α and d_β are the unique solutions of

$$\sum_{i=0}^m \alpha_i^{\min\{1, s_\alpha\}} \beta_i^{d_\alpha - \min\{1, s_\alpha\}} = 1 \quad \text{and} \quad \sum_{i=0}^m \beta_i^{\min\{1, s_\beta\}} \alpha_i^{d_\beta - \min\{1, s_\beta\}} = 1,$$

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and s_α, s_β are the unique solutions of

$$\sum_{i=0}^{m} \alpha_i^{s_\alpha} = 1 \quad \textit{and} \quad \sum_{i=0}^{m} \beta_i^{s_\beta} = 1.$$

In the proof of Theorem 4.1, we follow the proofs of Feng and Wang [**8**, Theorem 1] and Barański [**1**, Theorem B], with slight manipulations. For the convenience of the reader, we present this lengthy calculation here. The proof is broken into Lemmas 4.3, 4.4 and 4.5.

Before we prove the theorem, let us introduce some notation. Let $\Sigma = \{0, \ldots, m\}^{\mathbb{N}}$ and $\Sigma^* = \bigcup_{n=0}^{\infty} \{0, \ldots, m\}^n$. Denote the right cut on Σ^* by δ. More precisely, let $\delta(\emptyset) = \emptyset$ and

$$\delta(i_0 \cdots i_k) = i_0 \cdots i_{k-1}.$$

For any $\underline{i} \in \Sigma^*$, let $f_{\underline{i}} = f_{i_0} \circ \cdots \circ f_{i_k}$, $\alpha_{\underline{i}} = \alpha_{i_0} \cdots \alpha_{i_k}$ and $\beta_{\underline{i}} = \beta_{i_0} \cdots \beta_{i_k}$. For every $0 < r < 1$, let

$$\Delta_r = \{\underline{i} \in \Sigma^* : \min\{\alpha_{\delta\underline{i}}, \beta_{\delta\underline{i}}\} \geq r, \min\{\alpha_{\underline{i}}, \beta_{\underline{i}}\} < r\}$$

and

$$\Delta_r^\alpha = \{\underline{i} \in \Delta_r : \alpha_{\underline{i}} \geq \beta_{\underline{i}}\} \quad \text{and} \quad \Delta_r^\beta = \{\underline{i} \in \Delta_r : \alpha_{\underline{i}} < \beta_{\underline{i}}\}.$$

It is easy to see that Δ_r is a partition of Σ.

For every $\underline{i} \in \Delta_r^\alpha$, we set $\omega_\alpha(\underline{i}) = [\alpha_{\underline{i}}/\beta_{\underline{i}}]$ and, similarly, for every $\underline{i} \in \Delta_r^\beta$ we set $\omega_\beta(\underline{i}) = [\beta_{\underline{i}}/\alpha_{\underline{i}}]$. For any $\underline{i} \in \Delta_r^\alpha$, we divide $f_{\underline{i}}([0,1]^2)$ into $\omega_\alpha(\underline{i})$ equal rectangles with height $\beta_{\underline{i}}$ and width $\alpha_{\underline{i}}/\omega_\alpha(\underline{i})$, and denote the kth rectangle by $R_k^\alpha(\underline{i})$ for $k = 1, \ldots, \omega_\alpha(\underline{i})$. Similarly, for $\underline{i} \in \Delta_r^\beta$, we divide $f_{\underline{i}}([0,1]^2)$ into $\omega_\beta(\underline{i})$ equal rectangles with width $\alpha_{\underline{i}}$ and height $\beta_{\underline{i}}/\omega_\beta(\underline{i})$, and denote the kth rectangle by $R_k^\beta(\underline{i})$ for $k = 1, \ldots, \omega_\beta(\underline{i})$.

Let

$$C_r^\alpha = \{R_k^\alpha(\underline{i}) : \underline{i} \in \Delta_r^\alpha,\ 1 \leq k \leq \omega_\alpha(\underline{i}),\ R_k^\alpha(\underline{i}) \cap f_{\underline{i}}(\Lambda) \neq \emptyset\},$$
$$C_r^\beta = \{R_k^\beta(\underline{i}) : \underline{i} \in \Delta_r^\beta,\ 1 \leq k \leq \omega_\beta(\underline{i}),\ R_k^\beta(\underline{i}) \cap f_{\underline{i}}(\Lambda) \neq \emptyset\}.$$

Moreover,

$$\eta_r^\alpha(\underline{i}) = \sharp\{R_k^\alpha(\underline{i}) : 1 \leq k \leq \omega_\alpha(\underline{i}),\ R_k^\alpha(\underline{i}) \cap f_{\underline{i}}(\Lambda) \neq \emptyset\} \quad \text{for } \underline{i} \in \Delta_r^\alpha$$

and

$$\eta_r^\alpha(\underline{i}) = \sharp\{R_k^\beta(\underline{i}) : 1 \leq k \leq \omega_\beta(\underline{i}),\ R_k^\beta(\underline{i}) \cap f_{\underline{i}}(\Lambda) \neq \emptyset\} \quad \text{for } \underline{i} \in \Delta_r^\beta.$$

LEMMA 4.3. *Let f_i be in the form (4.1) for $i = 0, \ldots, m$, and let us suppose that $\Psi = \{f_i(x, y)\}_{i=0}^m$ satisfies (4.2). Moreover, let $\widetilde{N}_r = \sharp(C_r^\alpha \cup C_r^\beta)$. Then the attractor Λ of Ψ satisfies*

$$\overline{\dim}_B \Lambda = \limsup_{r \to 0+} \frac{\log \widetilde{N}_r}{-\log r} \quad \textit{and} \quad \underline{\dim}_B \Lambda = \liminf_{r \to 0+} \frac{\log \widetilde{N}_r}{-\log r}.$$

Proof. Let N_r be the minimal number of squares with side length r covering the attractor Λ.

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By definition, $C_r^\alpha \cup C_r^\beta$ covers Λ. For every real number $c \geq 1$ where $\frac{1}{2}c \leq [c] \leq c$, we have that every rectangle in $C_r^\alpha \cup C_r^\beta$ has side length at most $2r$, therefore

$$N_{2r} \leq \widetilde{N}_r.$$

Let $\alpha_{\min} = \min_{i=0,\dots,m} \alpha_i$ and $\beta_{\min} = \min_{i=0,\dots,m} \beta_i$; moreover, let $\rho = \min\{\alpha_{\min}, \beta_{\min}\}$. Then every rectangle in $C_r^\alpha \cup C_r^\beta$ has a side of length at least ρr. Therefore, by condition (4.2), every square with side length $(\rho/2)r$ can intersect at most four rectangles in $C_r^\alpha \cup C_r^\beta$, which implies that

$$4N_{(\rho/2)r} \geq \widetilde{N}_r.$$

One can finish the proof using the definition of the lower and upper box dimension. □

For $\underline{i} \in \Delta_r^\alpha$, by some simple manipulation, we get that

$$\begin{aligned}
\eta_r^\alpha(\underline{i}) &= \sharp\{R_k^\alpha(\underline{i}) : 1 \leq k \leq \omega_\alpha(\underline{i}),\ R_k^\alpha(\underline{i}) \cap f_{\underline{i}}(\Lambda) \neq \emptyset\} \\
&= \sharp\left\{\left[\frac{k-1}{\omega_\alpha(\underline{i})}, \frac{k}{\omega_\alpha(\underline{i})}\right] \times [0,1] : 1 \leq k \leq \omega_\alpha(\underline{i}),\right. \\
&\qquad \left.\left[\frac{k-1}{\omega_\alpha(\underline{i})}, \frac{k}{\omega_\alpha(\underline{i})}\right] \times [0,1] \cap \Lambda \neq \emptyset\right\} \\
&= \sharp\left\{\left[\frac{k-1}{\omega_\alpha(\underline{i})}, \frac{k}{\omega_\alpha(\underline{i})}\right] : 1 \leq k \leq \omega_\alpha(\underline{i}),\right. \\
&\qquad \left.\left[\frac{k-1}{\omega_\alpha(\underline{i})}, \frac{k}{\omega_\alpha(\underline{i})}\right] \cap \operatorname{proj}_x \Lambda \neq \emptyset\right\}
\end{aligned} \tag{4.3}$$

and by similar arguments for $\underline{i} \in \Delta_r^\beta$

$$\eta_r^\beta(\underline{i}) = \sharp\left\{\left[\frac{k-1}{\omega_\beta(\underline{i})}, \frac{k}{\omega_\beta(\underline{i})}\right] : 1 \leq k \leq \omega_\beta(\underline{i}), \left[\frac{k-1}{\omega_\beta(\underline{i})}, \frac{k}{\omega_\beta(\underline{i})}\right] \cap \operatorname{proj}_y \Lambda \neq \emptyset\right\}. \tag{4.4}$$

Let us divide the unit interval into $n \in \mathbb{N}$ equal parts and denote by $N_{1/n}(\operatorname{proj}_x \Lambda)$ (and $N_{1/n}(\operatorname{proj}_y \Lambda)$) the number of intervals with length $1/n$ that intersect the set $\operatorname{proj}_x \Lambda$ (and $\operatorname{proj}_y \Lambda$, respectively). Since $\operatorname{proj}_x \Lambda$ and $\operatorname{proj}_y \Lambda$ are self-similar sets, the box dimensions exist. Therefore, for every $\varepsilon > 0$, there exists a $c = c(\varepsilon) > 0$ such that for every integer $n \geq 1$

$$\begin{aligned}
c^{-1} n^{s_\alpha - \varepsilon} &\leq N_{1/n}(\operatorname{proj}_x \Lambda) \leq c n^{s_\alpha + \varepsilon} \quad \text{and} \\
c^{-1} n^{s_\beta - \varepsilon} &\leq N_{1/n}(\operatorname{proj}_y \Lambda) \leq c n^{s_\beta + \varepsilon},
\end{aligned} \tag{4.5}$$

where $s_\alpha = \dim_B \operatorname{proj}_x \Lambda$ and $s_\beta = \dim_B \operatorname{proj}_y \Lambda$. Using (4.3) and (4.4), we have

$$\begin{aligned}
\widetilde{N}_r = \sum_{\underline{i} \in \Delta_r^\alpha} \eta_r^\alpha(\underline{i}) + \sum_{\underline{i} \in \Delta_r^\beta} \eta_r^\beta(\underline{i}) &\leq c \sum_{\underline{i} \in \Delta_r^\alpha} \omega_\alpha(\underline{i})^{s_\alpha + \varepsilon} + c \sum_{\underline{i} \in \Delta_r^\beta} \omega_\beta(\underline{i})^{s_\beta + \varepsilon} \\
&\leq c \sum_{\underline{i} \in \Delta_r^\alpha} \left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha + \varepsilon} + c \sum_{\underline{i} \in \Delta_r^\beta} \left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta + \varepsilon}
\end{aligned} \tag{4.6}$$

and, similarly,

$$\widetilde{N}_r \geq c^{-1} 2^{-(s_\alpha - \varepsilon)} \sum_{\underline{i} \in \Delta_r^\alpha} \left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha - \varepsilon} + c^{-1} 2^{-(s_\beta - \varepsilon)} \sum_{\underline{i} \in \Delta_r^\beta} \left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta - \varepsilon}. \tag{4.7}$$

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Let $d_\alpha(t)$ and $d_\beta(t)$ be the unique solutions for $t \geq -\min\{s_\alpha, s_\beta\}$ of

$$\sum_{i=0}^{m}\left(\frac{\alpha_i}{\beta_i}\right)^{s_\alpha+t}\beta_i^{d_\alpha(t)} = 1 \quad \text{and} \quad \sum_{i=0}^{m}\left(\frac{\beta_i}{\alpha_i}\right)^{s_\beta+t}\alpha_i^{d_\beta(t)} = 1.$$

We remark that $d_\alpha(0) = d_\alpha$ and $d_\beta(0) = d_\beta$.

LEMMA 4.4. *Let f_i be in the form (4.1) for $i = 0, \ldots, m$ and let us suppose that $\Psi = \{f_i(x, y)\}_{i=0}^m$ satisfies (4.2). Then the attractor Λ of Ψ satisfies $\overline{\dim}_B \Lambda \leq \max\{d_\alpha, d_\beta\}$.*

Proof. Let $\varepsilon > 0$ be arbitrarily small. Then, by (4.6),

$$\begin{aligned}\frac{\log \widetilde{N}_r}{-\log r} &\leq \frac{\log c}{-\log r} + \frac{\log(\sum_{\underline{i}\in\Delta_r^\alpha}(\alpha_{\underline{i}}/\beta_{\underline{i}})^{s_\alpha+\varepsilon} + \sum_{\underline{i}\in\Delta_r^\beta}(\beta_{\underline{i}}/\alpha_{\underline{i}})^{s_\beta+\varepsilon})}{-\log r}\\ &\leq \frac{\log c}{-\log r} + \max\{d_\alpha(\varepsilon), d_\beta(\varepsilon)\}\left(1 + \frac{\log \rho}{\log r}\right)\\ &\quad + \frac{\log(\sum_{\underline{i}\in\Delta_r^\alpha}(\alpha_{\underline{i}}/\beta_{\underline{i}})^{s_\alpha+\varepsilon}\beta_{\underline{i}}^{d_\alpha(\varepsilon)} + \sum_{\underline{i}\in\Delta_r^\beta}(\beta_{\underline{i}}/\alpha_{\underline{i}})^{s_\beta+\varepsilon}\alpha_{\underline{i}}^{d_\beta(\varepsilon)})}{-\log r}.\end{aligned}$$

Since Δ_r is a partition, $\sum_{\underline{i}\in\Delta_r}(\alpha_{\underline{i}}/\beta_{\underline{i}})^{s_\alpha+\varepsilon}\beta_{\underline{i}}^{d_\alpha(\varepsilon)} = 1$ and $\sum_{\underline{i}\in\Delta_r}(\beta_{\underline{i}}/\alpha_{\underline{i}})^{s_\beta+\varepsilon}\alpha_{\underline{i}}^{d_\beta(\varepsilon)} = 1$, which implies that

$$\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha+\varepsilon}\beta_{\underline{i}}^{d_\alpha(\varepsilon)} + \sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta+\varepsilon}\alpha_{\underline{i}}^{d_\beta(\varepsilon)} \leq 2.$$

Therefore

$$\frac{\log \widetilde{N}_r}{-\log r} \leq \frac{\log c}{-\log r} + \max\{d_\alpha(\varepsilon), d_\beta(\varepsilon)\}\left(1 + \frac{\log \rho}{\log r}\right) + \frac{\log 2}{-\log r}.$$

Taking the limit superior as r tends to 0, and by Lemma 4.3,

$$\overline{\dim}_B \Lambda \leq \max\{d_\alpha(\varepsilon), d_\beta(\varepsilon)\}$$

for every $\varepsilon > 0$. Finally, since $\varepsilon > 0$ was arbitrary, we have proved the lemma. □

LEMMA 4.5. *Let f_i be in form (4.1) for $i = 0, \ldots, m$, and let us suppose that $\Psi = \{f_i(x, y)\}_{i=0}^m$ satisfies (4.2), then*

$$\underline{\dim}_B \Lambda \geq \max\{d_\alpha, d_\beta\}.$$

Before we prove the lower bound of the lower box dimension, we have to state another lemma about the dimension of the projections. To state this lemma, we need a sublemma about the partitions of Σ. First, let us introduce some notation. Let $\mathcal{G}$ be a partition of Σ containing only cylinder sets, and denote by $\lceil \mathcal{G} \rceil$ the length of the longest and denote by $\lfloor \mathcal{G} \rfloor$ the length of the shortest cylinder set of $\mathcal{G}$.

SUBLEMMA 4.6. *Let $\mathcal{G}$ be a partition of $\Sigma = \{0, \ldots, m\}^{\mathbb{N}}$ containing only cylinder sets and let γ_i, $i = 0, \ldots, m$ be positive real numbers such that $\sum_{i=0}^m \gamma_i > 1$. Then*

$$\sum_{\underline{i}\in\mathcal{G}} \gamma_{\mathbf{i}} \geq \left(\sum_{i=0}^{m}\gamma_i\right)^{\lfloor \mathcal{G} \rfloor}.$$

Proof. We prove the statement of the sublemma by induction for the length of the longest cylinder set of $\mathcal{G}$.

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For $\lceil \mathcal{G} \rceil = 1$, the statement holds trivially. Let us suppose that for every partition with the length of the longest cylinder set equal to n the statement is true. Let $\mathcal{G}$ be a partition containing only cylinder sets with $\lceil \mathcal{G} \rceil = n + 1$.

If $\lceil \mathcal{G} \rceil = \lfloor \mathcal{G} \rfloor$, then the statement is true since

$$\sum_{\underline{i} \in \mathcal{G}} \gamma_{\underline{i}} = \left(\sum_{i=0}^{m} \gamma_i \right)^{\lfloor \mathcal{G} \rfloor}.$$

Therefore, without loss of generality, we may assume that $\lfloor \mathcal{G} \rfloor < \lceil \mathcal{G} \rceil$. Let $[i_0 \cdots i_n] \in \mathcal{G}$ be one of the longest cylinder sets of $\mathcal{G}$. Since $\mathcal{G}$ is a partition of Σ, $[i_0 \cdots i_{n-1} j] \in \mathcal{G}$ for every $j = 0, \ldots, m$. Using this fact, we can define a partition $\mathcal{G}_2$ such that for every $\underline{i} \in \mathcal{G}$ with length strictly less than $n + 1$, $\underline{i} \in \mathcal{G}_2$ and for every $\underline{i} \in \mathcal{G}$ with length $n + 1$, $\underline{i}|_n \in \mathcal{G}_2$. Then

$$\sum_{\underline{i} \in \mathcal{G}} \gamma_{\underline{i}} \geq \sum_{\underline{i} \in \mathcal{G}_2} \gamma_{\underline{i}} \geq \left(\sum_{i=0}^{m} \gamma_i \right)^{\lfloor \mathcal{G} \rfloor}.$$

In the last inequality, we used the inductional assumption and $\lfloor \mathcal{G} \rfloor = \lfloor \mathcal{G}_2 \rfloor$ by the definition of $\mathcal{G}_2$. □

LEMMA 4.7. *Let f_i be in the form (4.1) for $i = 0, \ldots, m$ and let us suppose that $\Psi = \{f_i(x, y)\}_{i=0}^{m}$ satisfies (4.2). Then*

$$\sum_{i=0}^{m} \alpha_i^{s_\alpha} \beta_i^{s_\beta} \leq 1. \tag{4.8}$$

Proof. We begin the proof of the lemma by dividing the [0, 1] interval on the x- and y-axes into intervals with length r. Let $\varepsilon > 0$ be arbitrary small but fixed. Let us take the intervals which intersect $\mathrm{proj}_x \Lambda$ on the x-axis and $\mathrm{proj}_y \Lambda$ on the y-axis; moreover, take the left and the right neighbor intervals of those intervals. Then, for every sufficiently small r, the number of intervals on the x-axis (and y-axis) is at most $3(1/r)^{s_\alpha + \varepsilon}$ (and $3(1/r)^{s_\beta + \varepsilon}$). Let us take the direct product of these intervals. It is easy to see that the cover constructed in this way covers the approximate squares $C_r^\alpha \cup C_r^\beta$ and this implies that the area of $C_r^\alpha \cup C_r^\beta$ is less than or equal to the area of the squares constructed above, that is

$$\begin{aligned} 9\left(\frac{1}{r}\right)^{s_\alpha+\varepsilon}\left(\frac{1}{r}\right)^{s_\beta+\varepsilon} r^2 &\geq c^{-1} \sum_{\underline{i} \in \Delta_r^\alpha} \beta_{\underline{i}} \frac{\alpha_{\underline{i}}}{\omega_\alpha(\underline{i})} \omega_\alpha(\underline{i})^{s_\alpha - \varepsilon} + c^{-1} \sum_{\underline{i} \in \Delta_r^\beta} \alpha_{\underline{i}} \frac{\beta_{\underline{i}}}{\omega_\beta(\underline{i})} \omega_\beta(\underline{i})^{s_\beta - \varepsilon} \\ &\geq c^{-1} \sum_{\underline{i} \in \Delta_r^\alpha} \beta_{\underline{i}}^2 \omega_\alpha(\underline{i})^{s_\alpha - \varepsilon} + c^{-1} \sum_{\underline{i} \in \Delta_r^\beta} \alpha_{\underline{i}}^2 \omega_\beta(\underline{i})^{s_\beta - \varepsilon}, \end{aligned}$$

where c is a constant depending only on ε, as in (4.5). By simple algebraic manipulations and using the definitions of $\omega_\alpha(\underline{i})$, $\omega_\beta(\underline{i})$ and Δ_r^α, Δ_r^β, we have

$$\sum_{\underline{i} \in \Delta_r^\alpha} \beta_{\underline{i}}^2 \omega_\alpha(\underline{i})^{s_\alpha - \varepsilon} r^{s_\alpha + s_\beta - 2} \geq c_1 r^\varepsilon \sum_{\underline{i} \in \Delta_r^\alpha} \alpha_{\underline{i}}^{s_\alpha} \beta_{\underline{i}}^{s_\beta}$$

and

$$\sum_{\underline{i} \in \Delta_r^\beta} \alpha_{\mathbf{i}}^2 \omega_\beta(\underline{i})^{s_\beta - \varepsilon} r^{s_\alpha + s_\beta - 2} \geq c_1 r^\varepsilon \sum_{\underline{i} \in \Delta_r^\beta} \alpha_{\underline{i}}^{s_\alpha} \beta_{\underline{i}}^{s_\beta},$$

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where c_1 depends only on ε. Then there exists a constant $\widetilde{c}$ depending only on ε such that for every sufficiently small r

$$\widetilde{c}r^{-3\varepsilon} \geq \sum_{\underline{i}\in\Delta_r} \alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta}.$$

Since ε was arbitrary, we have

$$0 \leq \liminf_{r\to 0+} \frac{\log \sum_{\underline{i}\in\Delta_r} \alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta}}{\log r}. \tag{4.9}$$

Now let us suppose indirectly that $\sum_{i=0}^{m} \alpha_i^{s_\alpha}\beta_i^{s_\beta} > 1$. Then, by using Sublemma 4.6, we have

$$\sum_{\underline{i}\in\Delta_r} \alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta} \geq \left(\sum_{i=0}^{m} \alpha_i^{s_\alpha}\beta_i^{s_\beta}\right)^{\lfloor\Delta_r\rfloor}.$$

It is easy to see that $\lfloor\Delta_r\rfloor = \lceil \log r/\log\rho\rceil$, where $\rho = \min_i\{\alpha_i, \beta_i\}$. This implies that

$$\limsup_{r\to 0+} \frac{\log \sum_{\underline{i}\in\Delta_r} \alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta}}{\log r} \leq \frac{\log \sum_{i=0}^{m} \alpha_i^{s_\alpha}\beta_i^{s_\beta}}{\log\rho} < 0,$$

which contradicts (4.9). □

Proof of Lemma 4.5. By Lemma 4.7, we divide the proof into two parts. First, let us assume that

$$\sum_{i=0}^{m} \alpha_i^{s_\alpha}\beta_i^{s_\beta} = 1. \tag{4.10}$$

Let us observe that in this case $d_\alpha = d_\beta = s_\alpha + s_\beta$. Then, by inequality (4.5), we have

$$\begin{aligned}
\frac{\log \widetilde{N}_r}{-\log r} &\geq \frac{\log(\sum_{\underline{i}\in\Delta_r^\alpha} (\alpha_{\underline{i}}/\beta_{\underline{i}})^{s_\alpha-\varepsilon} + \sum_{\underline{i}\in\Delta_r^\beta} (\beta_{\underline{i}}/\alpha_{\underline{i}})^{s_\beta-\varepsilon})}{-\log r} \\
&\geq s_\alpha + s_\beta + \frac{\log(\sum_{\underline{i}\in\Delta_r^\alpha} (\alpha_{\underline{i}}/\beta_{\underline{i}})^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{s_\alpha+s_\beta} + \sum_{\underline{i}\in\Delta_r^\beta} (\beta_{\underline{i}}/\alpha_{\underline{i}})^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{s_\alpha+s_\beta})}{-\log r} \\
&\geq s_\alpha + s_\beta - \varepsilon\frac{\lceil\Delta_r\rceil \log\max_i\{\alpha_i/\beta_i, \beta_i/\alpha_i\}}{-\log r} + \frac{\log(\sum_{\underline{i}\in\Delta_r} \alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta})}{-\log r}.
\end{aligned}$$

It is easy to see that $\lceil\Delta_r\rceil = \log r/(\log\max_i\{\alpha_i, \beta_i\})$. Applying this fact and our assumption (4.10), we get, for every $\varepsilon > 0$, that

$$\liminf_{r\to 0} \frac{\log \widetilde{N}_r}{-\log r} \geq s_\alpha + s_\beta - \varepsilon\frac{1}{-\log\max_i\{\alpha_i, \beta_i\}},$$

and this completes the proof in the first case.

In the second case, let us assume that

$$\sum_{i=0}^{m} \alpha_i^{s_\alpha}\beta_i^{s_\beta} < 1. \tag{4.11}$$

Without loss of generality, we assume that $d_\alpha \geq d_\beta$.

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Then there exists a $\varepsilon^* > 0$ by (4.11) such that, for every $0 < \varepsilon < \varepsilon^*$,

$$\sum_{i=0}^{m} \alpha_i^{s_\alpha - \varepsilon} \beta_i^{s_\beta - \varepsilon} < 1.$$

This implies that

$$d_\beta(-\varepsilon),\, d_\alpha(-\varepsilon) \le s_\alpha + s_\beta - 2\varepsilon. \tag{4.12}$$

Then, for every $\underline{i} \in \Delta_r^\beta$,

$$\frac{\alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon}}{\beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon}} = \left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha+s_\beta-2\varepsilon-d_\alpha(-\varepsilon)} \alpha_{\underline{i}}^{d_\alpha(-\varepsilon)-d_\beta(-\varepsilon)} \le \alpha_{\underline{i}}^{d_\alpha(-\varepsilon)-d_\beta(-\varepsilon)} \tag{4.13}$$

and, for every $\underline{i} \in \Delta_r^\alpha$,

$$\frac{\beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon}}{\alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon}} \le \beta_{\underline{i}}^{d_\beta(-\varepsilon)-d_\alpha(-\varepsilon)}. \tag{4.14}$$

Now we prove the lemma in the case when $d_\alpha > d_\beta$. Then there exists a $\varepsilon^{**} > 0$ such that for every $0 < \varepsilon < \varepsilon^{**}$, $d_\alpha(-\varepsilon) > d_\beta(-\varepsilon)$. Then, by (4.13),

$$\sum_{\underline{i}\in\Delta_r^\beta} \alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\mathbf{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon} \le \frac{1}{2}\sum_{\underline{i}\in\Delta_r^\beta} \beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon} \le \frac{1}{2}$$

holds for sufficiently small $r > 0$. Therefore

$$\sum_{\underline{i}\in\Delta_r^\alpha} \alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon} \ge \frac{1}{2}. \tag{4.15}$$

Using (4.7),

$$\begin{aligned}\frac{\log \widetilde{N}_r}{-\log r} &\ge \frac{\log(c^{-1}2^{-(s_\alpha-\varepsilon)}\sum_{\underline{i}\in\Delta_r^\alpha}(\alpha_{\underline{i}}/\beta_{\underline{i}})^{s_\alpha-\varepsilon} + c^{-1}2^{-(s_\beta-\varepsilon)}\sum_{\underline{i}\in\Delta_r^\beta}(\beta_{\underline{i}}/\alpha_{\underline{i}})^{s_\beta-\varepsilon})}{-\log r} \\ &\ge \frac{\log(c^{-1}2^{-(\max\{s_\alpha,s_\beta\}-\varepsilon)})}{-\log r} + \frac{\log r^{-d_\alpha(-\varepsilon)}\sum_{\underline{i}\in\Delta_r^\alpha}(\alpha_{\underline{i}}/\beta_{\underline{i}})^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)}}{-\log r},\end{aligned}$$

and, by (4.15),

$$\frac{\log \widetilde{N}_r}{-\log r} \ge d_\alpha(-\varepsilon) + \frac{\log(c^{-1}2^{-(\max\{s_\alpha,s_\beta\}-\varepsilon)})}{-\log r} + \frac{\log 2}{\log r}.$$

Taking the limit inferior r to 0 and Lemma 4.3, we get

$$\underline{\dim}_B \Lambda \ge d_\alpha(-\varepsilon).$$

Since $\varepsilon > 0$ was arbitrarily small, we proved the lemma in the case $d_\alpha > d_\beta$.

Now let us consider the case $d_\alpha = d_\beta$. The facts (4.12 and 4.13), (4.14) imply, for every sufficiently small $\varepsilon > 0$, that

$$\sum_{\underline{i}\in\Delta_r^\beta} \alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon} \le \sum_{\underline{i}\in\Delta_r^\beta} \beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon} \quad \text{or}$$

$$\sum_{\underline{i}\in\Delta_r^\alpha} \beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon} \le \sum_{\underline{i}\in\Delta_r^\alpha} \alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon}.$$

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Therefore

$$\sum_{\underline{i}\in\Delta_r^{\alpha}}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)}+\sum_{\underline{i}\in\Delta_r^{\beta}}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)}\geq 1. \tag{4.16}$$

Using (4.7),

$$\frac{\log\widetilde{N}_r}{-\log r}\geq\frac{\log(c^{-1}2^{-(\max\{s_\alpha,s_\beta\}-\varepsilon)})}{-\log r}+\min\{d_\alpha(-\varepsilon),\,d_\beta(-\varepsilon)\}+\frac{\log(\sum_{\underline{i}\in\Delta_r^{\alpha}}(\alpha_{\underline{i}}/\beta_{\underline{i}})^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)}+\sum_{\underline{i}\in\Delta_r^{\beta}}(\beta_{\underline{i}}/\alpha_{\underline{i}})^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)})}{-\log r}$$

and, by (4.16),

$$\frac{\log\widetilde{N}_r}{-\log r}\geq\frac{\log(c^{-1}2^{-(\max\{s_\alpha,s_\beta\}-\varepsilon)})}{-\log r}+\min\{d_\alpha(-\varepsilon),\,d_\beta(-\varepsilon)\}.$$

Taking the limit inferior r to 0 and Lemma 4.3, we have

$$\underline{\dim}_B\Lambda\geq\min\{d_\alpha(-\varepsilon),\,d_\beta(-\varepsilon)\}.$$

Since $\varepsilon>0$ was arbitrarily small and $d_\alpha=d_\beta$, this completes the proof of the lemma. □

Proof of Theorem 4.1. The proof is the combination of Lemmas 4.4 and 4.5. □

Now we consider the generalized 4-corner set. Let $\Psi=\{f_0(\underline{x}),\,f_1(\underline{x}),\,f_2(\underline{x}),\,f_3(\underline{x})\}$ be an iterated function system on the real plane such that

$$\begin{aligned}
f_0(\underline{x})&=\begin{pmatrix}\alpha_0&0\\0&\beta_0\end{pmatrix}\underline{x},\\
f_1(\underline{x})&=\begin{pmatrix}\alpha_1&0\\0&\beta_1\end{pmatrix}\underline{x}+\begin{pmatrix}0\\1-\beta_1\end{pmatrix},\\
f_2(\underline{x})&=\begin{pmatrix}\alpha_2&0\\0&\beta_2\end{pmatrix}\underline{x}+\begin{pmatrix}1-\alpha_2\\0\end{pmatrix},\\
f_3(\underline{x})&=\begin{pmatrix}\alpha_3&0\\0&\beta_3\end{pmatrix}\underline{x}+\begin{pmatrix}1-\alpha_3\\1-\beta_3\end{pmatrix}.
\end{aligned}\tag{4.17}$$

Assume that Ψ satisfies (4.2), see Figure 1. This condition is equivalent to (4.18)–(4.23):

$$\alpha_0+\alpha_2<1 \tag{4.18}$$

$$\alpha_1+\alpha_3<1 \tag{4.19}$$

$$\beta_0+\beta_1<1 \tag{4.20}$$

$$\beta_2+\beta_3<1 \tag{4.21}$$

$$\alpha_0+\alpha_3<1\quad\text{or}\quad\beta_0+\beta_3<1 \tag{4.22}$$

$$\alpha_1+\alpha_2<1\quad\text{or}\quad\beta_1+\beta_2<1. \tag{4.23}$$

Let $\mathcal{R}$ be the set of possible parameters, that is

$$\mathcal{R}=\{(\underline{\alpha},\,\underline{\beta})\in(0,\,1)^8:(\underline{\alpha},\,\underline{\beta})\text{ satisfies (4.18)–(4.23)}\}.$$

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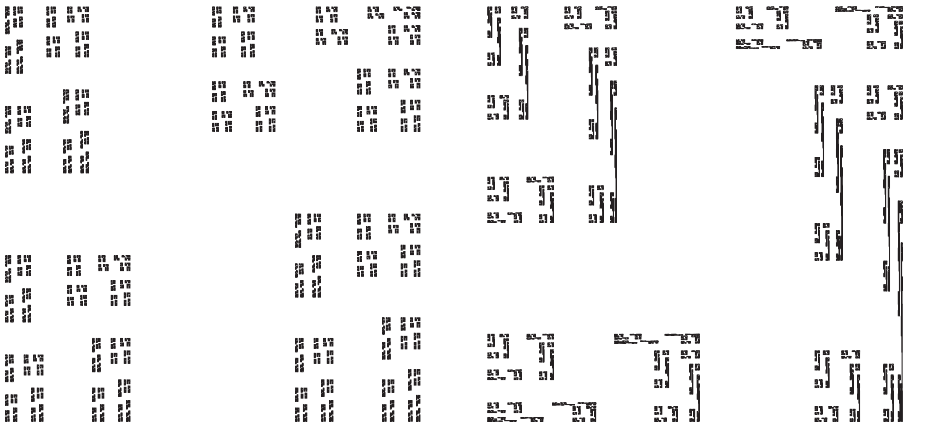


FIGURE 3. Generalized 4-corner sets with box dimension ≈1.394 44 and ≈1.408 19 for sufficiently small perturbation.

THEOREM 4.8. *Let Ψ be as in (4.17). Let Λ be the attractor of Ψ. Then*

$$\dim_B \Lambda = \max\{d_\alpha, d_\beta\} \quad \textit{for } \mathcal{L}\textit{-a.e. } (\underline{\alpha}, \underline{\beta}) \in \mathcal{R},$$

where

$$\sum_{i=0}^{3} \alpha_i^{\min\{1,s_\alpha\}} \beta_i^{d_\alpha - \min\{1,s_\alpha\}} = 1 \quad \textit{and} \quad \sum_{i=0}^{3} \beta_i^{\min\{1,s_\beta\}} \alpha_i^{d_\beta - \min\{1,s_\beta\}} = 1,$$

where s_α is the unique solution of

$$\alpha_0^s + \alpha_1^s + \alpha_2^s + \alpha_3^s - \alpha_0^s\alpha_1^s - \alpha_2^s\alpha_3^s = 1. \tag{4.24}$$

Similarly, s_β is the unique solution of

$$\beta_0^s + \beta_1^s + \beta_2^s + \beta_3^s - \beta_0^s\beta_2^s - \beta_1^s\beta_3^s = 1. \tag{4.25}$$

Proof. The proof is an easy consequence of Theorems 1.1 and 4.1. □

For two examples, see Figure 3.

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