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ON WEIGHTED K-OUT-OF-N SYSTEMS WITH STATISTICALLY DEPENDENT COMPONENT LIFETIMES

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For the weighted k-out-of-n system with statistically dependent component lifetimes, we have a discussion on the system performance through investigating the ordering properties of the total system weight with respect to component weight vector. Applications of the present ordering results to signature of coherent systems are presented as well.

1. INTRODUCTION

A k-out-of-n system functions if and only if at least k of its n components are operational. Such a fault-tolerant structure is very popular in reliability engineering, for a comprehensive exposition of reliability of k-out-of-n systems readers may refer to Kuo and Zuo [17]. During the past two decades, several researchers successfully generalized the traditional binary k-out-of-n structure to the weighted k-out-of-n system, in which the ith component contributes its own weight ω_i to the entire system while functioning ($i = 1, \ldots, n$) and the system itself functions if and only if the total weight due to operational components is above some predetermined threshold k. For more details on the weighted k-out-of-n systems see, for example, Rushdi [28], Ball, Hagstrom, and Provan [1] and Xie and Pham [34]. Note that ω_i's and k are both designed as positive integers in the pioneering research due to Wu and Cheng [32,33] and component lifetimes are usually assumed to be mutually independent in almost all follow-up studies. In the present context, we consider a more general structure in the sense that: (i) ω_i's and k both can take any positive value, and (ii) component lifetimes are not necessarily independent. Such reliability systems with weighted components are not uncommon in many capacity related engineering applications. Here are some simple examples, for more practical examples one may refer to Samaniego and Shaked [30].

- For a traffic system of several transportation channels (say, Lincoln tunnel, Holland tunnel and Washington Bridge between New Jersey and New York City), each has

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its own capacity of traffic flow, then the system gets the total capacity of traffic flow as a linear combination of channel capacities with respect to channels' binary status, and its performance is determined by the probability behavior of the total capacity of traffic flow. Naturally, the system fails if the total capacity goes below the threshold k, the required traffic flow in a certain time interval.

- A power generation system of several plants (for example, a wind power set, a fuel electric plant, a water power station, and a nuclear power plant) can also be modeled by a weighted k-out-of-n system. In this context, the weight of each plant is just its generation capacity, and the threshold k is exactly the market demand for power.
- For a mechanical system with several components suffering from a common stress, every component loses its supporting capacity due to failing to suffer the stress, and the entire system breaks down when the total supporting capacity gets below the threshold k, some designed level.

As is pointed out in Samaniego and Shaked [30], although in the literature there are a bunch of research articles studying various aspects of weighted k-out-of-n systems, while most research focused on reliability computation and had only algorithmic flavor, other few of them had a discussion on some indices of system performance. For example, Wu and Chen [32,33], Xie and Pham [34], Chen and Yang [6], Li and Zuo [20], and Eryilmaz [10,11] etc. Some of these references devoted to study more general versions such as multi-state weighted k-out-of-n system and k-out-of-n system with components having random weights. For example, Li and Zuo [21], Ding et al. [9], Levitin [18], Eryilmaz and Bozbulut [13], and Faghih-Roohi et al. [14]. Among numerous related references, it is remarkable to note that Samaniego and Shaked [30] performed an insightful investigation on lifetime-based analysis of reliability systems with weighted components. Recently, Eryilmaz [12] had a study on capacity loss and residual capacity for binary weighted k-out-of-n systems. To the best of our knowledge, stochastic properties of weighted k-out-of-n systems largely remain to be investigated. Also it should be remarked here that almost all concerned research in the literature is developed in the context of statistically independent component lifetimes. Since the statistical mutual independence among component lifetimes are evidently unsuitable in some practical situations, for example, the mechanical system with components suffering a common stress, it is of great interest to integrate statistical dependence among component lifetimes into the study of reliability weighted k-out-of-n systems. Due to the above two motivations we aim to investigate the ordering property and hence reliability performance of k-out-of-n systems with mutually dependent components in this study. In particular, we build the stochastic order for system lifetime the weighted k-out-of-n system with stochastic arrangement increasing component lifetimes and the increasing convex (concave) order for lifetime of system with right (left) tail weakly stochastic arrangement increasing component lifetimes. For the formal definitions of stochastic arrangement increasing and other related concepts see Section 2.

The rest of this paper is rolled out as follows: In Section 2, we recall some important concerned concepts, including majorization, stochastic orders, stochastic arrangement increasing properties, and several related lemmas as technical preliminaries. Section 3 presents stochastic orders on the total system weight in the context of component lifetimes being stochastic arrangement increasing, right tail weakly stochastic arrangement increasing and left tail weakly stochastic arrangement increasing, respectively. Also we present applications of the ordering results in Section 3 to the signature of coherent system in Section 4.

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2. PRELIMINARIES

For ease of reference, in this section we recall several concepts concerned with main results and also present several lemmas useful in developing the main theoretical results in the sequel.

The notion of majorization is quite useful in economics, reliability and risk management, here it will be employed to describe the difference in vectors of component weights. For a comprehensive exposition one may refer to the well-known monograph Marshall, Olkin, and Arnold [23].

DEFINITION 2.1: *Let $a_{1:n} \le \cdots \le a_{n:n}$ be the increasing arrangement of elements of $\boldsymbol{a} = (a_1, \ldots, a_n)$. Another real vector $\boldsymbol{b} = (b_1, \ldots, b_n)$ is said to be*

(i) majorized *by* $\boldsymbol{a}$ *(denoted as* $\boldsymbol{b} \preceq_{\mathrm{m}} \boldsymbol{a}$*) if* $\sum_{i=1}^{n} b_i = \sum_{i=1}^{n} a_i$ *and* $\sum_{i=1}^{k} b_{i:n} \ge \sum_{i=1}^{k} a_{i:n}$ *for* $k = 1, \ldots, n-1$*;*

(ii) *weakly submajorized by* $\boldsymbol{a}$ *(denoted as* $\boldsymbol{b} \preceq_{w} \boldsymbol{a}$*) if* $\sum_{i=k}^{n} b_{i:n} \le \sum_{i=k}^{n} a_{i:n}$ *for* $k = 1, \ldots, n$*;*

(iii) *weakly supermajorized by* $\boldsymbol{a}$ *(denoted as* $\boldsymbol{b} \preceq^{\mathrm{w}} \boldsymbol{a}$*) if* $\sum_{i=1}^{k} b_{i:n} \ge \sum_{i=1}^{k} a_{i:n}$ *for* $k = 1, \ldots, n$*.*

It is clear that $\boldsymbol{b} \preceq_{\mathrm{m}} \boldsymbol{a}$ implies both $\boldsymbol{b} \preceq_{\mathrm{w}} \boldsymbol{a}$ and $\boldsymbol{b} \preceq^{\mathrm{w}} \boldsymbol{a}$. Also, as is proved by the following lemma, the two weak versions of majorizations can be characterized by the majorization.

LEMMA 2.2: Marshall et al. [23], Theorem 5.A.9 *If $\boldsymbol{a} \preceq_{w} (\preceq^{\mathrm{w}}) \boldsymbol{b}$, then there exists a vector $\boldsymbol{c}$ such that $\boldsymbol{c} \preceq_{\mathrm{m}} \boldsymbol{b}$ and $\boldsymbol{a} \le (\ge) \boldsymbol{c}$ in the sense of $a_i \le c_i$ for all $i = 1, \ldots, n$.*

The next lemma presents the monotonicity of scalar product function due to majorization of their arguments.

LEMMA 2.3: *Suppose that $u(x)$ is increasing and concave. If $(b_1, b_2) \preceq_{\mathrm{m}} (a_1, a_2)$ with $a_1 \ge a_2$ then $u(b_1x_1 + b_2x_2) - u(a_1x_1 + a_2x_2)$ is decreasing in $x_1 \le x_2$ and*

$$u(b_1x_1 + b_2x_2) + u(b_2x_1 + b_1x_2) \ge u(a_1x_1 + a_2x_2) + u(a_2x_1 + a_1x_2). \tag{1}$$

PROOF: Since $(b_1, b_2) \preceq_{\mathrm{m}} (a_1, a_2)$ and $a_1 \ge a_2$, it holds that $a_1 \ge \max\{b_1, b_2\} \ge b_1$ and $a_1 - b_1 = b_2 - a_2$, and thus,

$$\ell(x_1) = (b_1x_1 + b_2x_2) - (a_1x_1 + a_2x_2) = (a_1 - b_1)(x_2 - x_1) \tag{2}$$

is non-negative and decreasing in $x_1 \le x_2$. As a consequence, we have $0 \le \ell(y_1) \le \ell(x_1)$ for $x_1 \le y_1 \le x_2$, and hence

$$u(a_1x_1 + a_2x_2 + \ell(x_1)) \ge u(a_1x_1 + a_2x_2 + \ell(y_1)), \tag{3}$$

for $x_1 \le y_1 \le x_2$ and any increasing function u.

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On the other hand, due to the concavity of $u(x)$, it holds for $x_1 \le y_1 \le x_2$ that

$$u(a_1x_1 + a_2x_2 + \ell(y_1)) - u(a_1x_1 + a_2x_2) \ge u(a_1y_1 + a_2x_2 + \ell(y_1)) - u(a_1y_1 + a_2x_2). \quad \textbf{(4)}$$

In combination with (3) and (4), we get, for $x_1 \le y_1 \le x_2$,

$$u(a_1x_1 + a_2x_2 + \ell(x_1)) - u(a_1x_1 + a_2x_2) \ge u(a_1y_1 + a_2x_2 + \ell(y_1)) - u(a_1y_1 + a_2x_2).$$

By (2) we also have

$$\begin{aligned} u(b_1x_1 + b_2x_2) &= u(a_1x_1 + a_2x_2 + \ell(x_1)), \\ u(b_1y_1 + b_2x_2) &= u(a_1y_1 + a_2x_2 + \ell(y_1)), \end{aligned}$$

and thus the previous inequality can be rephrased as, for $x_1 \le y_1 \le x_2$,

$$u(b_1x_1 + b_2x_2) - u(a_1x_1 + a_2x_2) \ge u(b_1y_1 + b_2x_2) - u(a_1y_1 + a_2x_2).$$

That is, $u(b_1x_1 + b_2x_2) - u(a_1x_1 + a_2x_2)$ is decreasing in $x_1 \le x_2$.

In view of $a_1 \ge \max\{b_1, b_2\}$ and $a_1 + a_2 = b_1 + b_2$, we have, for $x_1 \le x_2$,

$$\begin{aligned} a_2x_1 + a_1x_2 - (b_1x_1 + b_2x_2) &= [(b_1 + b_2) - a_1]x_1 + a_1x_2 - (b_1x_1 + b_2x_2) \\ &= (a_1 - b_2)(x_2 - x_1) \\ &\ge 0, \\ a_2x_1 + a_1x_2 - (b_2x_1 + b_1x_2) &= [(b_1 + b_2) - a_1]x_1 + a_1x_2 - (b_2x_1 + b_1x_2) \\ &= b_1x_1 + a_1x_2 - (a_1x_1 + b_1x_2) \\ &= (a_1 - b_1)(x_2 - x_1) \\ &\ge 0. \end{aligned}$$

Then, by the concavity of u we have, for $x_1 \le x_2$,

$$u(a_2x_1 + a_1x_2) - u(b_2x_1 + b_1x_2) \le u(b_1x_1 + b_2x_2) - u(a_1x_1 + a_2x_2),$$

and this is just the desired result in (1). ■

Stochastic orders play a fundamental role in many probability related areas, such as engineering reliability, finance and actuarial risk management, statistics, and so on. In this study, we will employ stochastic orders to describe the oscillation of the total weight of system due to component weight vector. Standard references for stochastic orders include Kaas, Van Heerwaarden, and Goovaerts [15], Denuit et al. [8], Shaked and Shanthikumar [31], and Li and Li [19]. One may refer these monographs for more details.

DEFINITION 2.4: *A random variable X is said to be smaller than the other one Y in the*

(i) usual stochastic *order (denoted by $X \le_{\rm st} Y$) if $\mathsf{E}[\phi(X)] \le \mathsf{E}[\phi(Y)]$ for any increasing ϕ;*

(ii) increasing convex *order (denoted by $X \le_{\rm icx} Y$) if $\mathsf{E}[\phi(X)] \le \mathsf{E}[\phi(Y)]$ for any increasing and convex ϕ;*

(iii) increasing concave *order (denoted by $X \le_{\rm icv} Y$) if $\mathsf{E}[\phi(X)] \le \mathsf{E}[\phi(Y)]$ for any increasing and concave ϕ.*

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To incorporate statistical dependence among component lifetimes, in this study we consider stochastic arrangement increasing component lifetimes and their weak versions, such multivariate non-parametric dependence notions due to Cai and Wei [4,5] are defined through the following classes of functions.

For any pair (i,j) such that $1 \le i < j \le n$, let

$$\mathcal{A}_{i,j}(n) = \big\{g(\boldsymbol{x}) :\ g(\boldsymbol{x}) \ge g(\tau_{ij}(\boldsymbol{x})) \text{ for any } (x_i, x_j) \text{ such that } x_i \le x_j\big\},$$

$$\mathcal{A}^{+}_{i,j}(n) = \big\{g(\boldsymbol{x}) :\ g(\boldsymbol{x}) - g(\tau_{ij}(\boldsymbol{x})) \text{ is increasing in } x_j \ge x_i \text{ for any } x_i\big\},$$

$$\mathcal{A}^{-}_{i,j}(n) = \big\{g(\boldsymbol{x}) :\ g(\boldsymbol{x}) - g(\tau_{ij}(\boldsymbol{x})) \text{ is decreasing in } x_i \le x_j \text{ for any } x_j\big\},$$

where, for any $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the permutation

$$\tau_{ij}(\boldsymbol{x}) = (x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n).$$

A n-dimensional function belonging to $\mathcal{S}^{i,j}_{\mathrm{ai}}(n)$ is said to be *arrangement increasing*.

DEFINITION 2.5: *A random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ or its distribution is said to be*

(i) *stochastic arrangement increasing (SAI) if* $\mathsf{E}[g(\boldsymbol{X})] \ge \mathsf{E}\big[g\big(\tau_{ij}(\boldsymbol{X})\big)\big]$ *for any* $g \in \mathcal{A}_{i,j}(n)$ *and* $\{i,j\}$ *such that* $1 \le i < j \le n$;

(ii) *right tail weakly stochastic arrangement increasing (RWSAI) if* $\mathsf{E}[g(\boldsymbol{X})] \ge \mathsf{E}\big[g\big(\tau_{ij}(\boldsymbol{X})\big)\big]$ *for any* $g \in \mathcal{A}^{+}_{i,j}(n)$ *and* $\{i,j\}$ *such that* $1 \le i < j \le n$;

(iii) *left tail weakly stochastic arrangement increasing (LWSAI) if* $\mathsf{E}[g(\boldsymbol{X})] \ge \mathsf{E}\big[g\big(\tau_{ij}(\boldsymbol{X})\big)\big]$ *for any* $g \in \mathcal{A}^{+}_{i,j}(n)$ *and* $\{i,j\}$ *such that* $1 \le i < j \le n$.

For more on stochastic properties of the above three non-parametric dependence notions and their applications in financial and actuarial risk management, one may refer to Cai and Wei [4,5], You and Li [35,36] and Li and You [22].

In what follows, let us recall some important properties of SAI, RWSAI and LWSAI, which are of great use in the proof of the main results.

LEMMA 2.6: Cai and Wei [4], Lemma 5.9 *For any increasing function $h(x)$, the random vector $\big(h(X_1), \ldots, h(X_n)\big)$ is SAI (RWSAI, LWSAI) whenever $(X_1, \ldots, X_n)$ is SAI (RWSAI, LWSAI).*

For any $1 \le i < j \le n$, denote

$$\boldsymbol{X}_{\{i,j\}^c} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n)$$

the sub-vector with the ith and jth coordinates being removed.

LEMMA 2.7: Cai and Wei [4], Proposition 3.2 *A random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ is LWSAI if and only if $(X_i, X_j) \mid \boldsymbol{X}_{\{i,j\}^c} = \boldsymbol{x}_{\{i,j\}^c}$ is LWSAI for any $1 \le i < j \le n$ and $\boldsymbol{x}_{\{i,j\}^c}$ in the support of $\boldsymbol{X}_{\{i,j\}^c}$.*

LEMMA 2.8: You and Li [36], Proposition 4.5(i) *Suppose that $g(x,y)$ is supermodular*[1]*, increasing in x and convex with respect to y for any x. If $\boldsymbol{X}$ is RWSAI, then $\boldsymbol{b} \preceq_{\mathrm{m}} \boldsymbol{a}$ implies*

$$\sum_{i=1}^{n} g\big(X_i, a_{i:n}\big) \ge_{\mathrm{icx}} \sum_{i=1}^{n} g(X_i, b_i).$$

[1] A bivariate function g is said to be *supermodular* if $g(x_1,y_1) + g(x_2,y_2) \ge g(x_1,y_2) + g(x_2,y_1)$ for any $x_1 \le x_2$ and $y_1 \le y_2$.

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LEMMA 2.9: You and Li [35], Theorem 3.3 *A random vector* (X_1, X_2) *is LWSAI if and only if*

$$\mathsf{E}\big[g_2(X_1, X_2)\big] \geq \mathsf{E}\big[g_1(X_1, X_2)\big], \quad \textit{for all } g_1 \textit{ and } g_2 \textit{ such that}$$

(i) $g_2(x_1, x_2) - g_1(x_1, x_2)$ *is decreasing in* $x_1 \leq x_2$ *for any* x_2, *and*
(ii) $g_2(x_1, x_2) + g_2(x_2, x_1) \geq g_1(x_1, x_2) + g_1(x_2, x_1)$ *for any* $x_2 \geq x_1$.

3. STOCHASTIC ORDERS ON THE TOTAL SYSTEM WEIGHT

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be the vector of component lifetimes and $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ the vector of component weights with $\omega_i > 0$ being the weight assigned to the ith component, $i = 1, \ldots, n$. Denote $I(A)$ the *indicator* function of event A. Then the total system weight at time $t \geq 0$ can be described by the stochastic process

$$W(t; \boldsymbol{\omega}, \boldsymbol{X}) = \sum_{i=1}^{n} \omega_i I(X_i > t),$$

and the failure time of the system with a threshold $k > 0$ is

$$T(k; \boldsymbol{\omega}, \boldsymbol{X}) = \inf\big\{t : W(t; \boldsymbol{\omega}) < k\big\}.$$

It is well known that the system reliability is

$$\mathrm{P}\big(T(k; \boldsymbol{\omega}, \boldsymbol{X}) > t\big) = \mathrm{P}\big(W(t; \boldsymbol{\omega}, \boldsymbol{X}) \geq k\big), \quad \text{for any } t \geq 0. \tag{5}$$

First, let us present the usual stochastic order on the total weight of system with components having SAI lifetimes.

THEOREM 3.1: *Suppose component lifetime vector* $\boldsymbol{X}$ *is SAI. Then, for any* $1 \leq i < j \leq n$, $\omega_i \leq \omega_j$ *implies* $W(t; \boldsymbol{\omega}, \boldsymbol{X}) \geq_{\mathrm{st}} W(t; \tau_{ij}(\boldsymbol{\omega}), \boldsymbol{X})$ *for any* $t \geq 0$.

PROOF: Note that $I(x > t)$ is increasing for any fixed $t \geq 0$. By Lemma 2.6, the SAI property of $\boldsymbol{X}$ implies that of $(I(X_1 > t), \ldots, I(X_n > t))$.

For any increasing function u, denote $g(\boldsymbol{x}) = u\big(\sum_{i=1}^{n} \omega_i x_i\big)$. It is easy to check that $g(\boldsymbol{x}) \in \mathcal{A}_{i,j}(n)$ for any $1 \leq i < j \leq n$ and $\omega_i \leq \omega_j$. Thus, we have

$$\begin{aligned}
\mathsf{E}\big[u\big(W(t; \boldsymbol{\omega}), \boldsymbol{X}\big)\big] &= \mathsf{E}\big[g\big(I(X_1 > t), \ldots, I(X_n > t)\big)\big] \\
&\geq \mathsf{E}\big[g\big(\tau_{ij}(I(X_1 > t), \ldots, I(X_n > t))\big)\big] \\
&= \mathsf{E}\big[u\big(W(t; \tau_{ij}(\boldsymbol{\omega})), \boldsymbol{X}\big)\big],
\end{aligned}$$

for $\omega_i \leq \omega_j$ and any $1 \leq i < j \leq n$. Due to the arbitrariness of the increasing u, we reach the desired result immediately. ■

According to Theorem 3.1, *the system assigning much weight to more reliable component acquires stochastically more weight at any time point* $t \geq 0$ *whenever component lifetimes are SAI.* By (5) and Theorem 3.1, we directly obtain the following corollary, which reiterates that *the system with increasing weight corresponding to SAI component lifetimes gets stochastically longer lifetime.*

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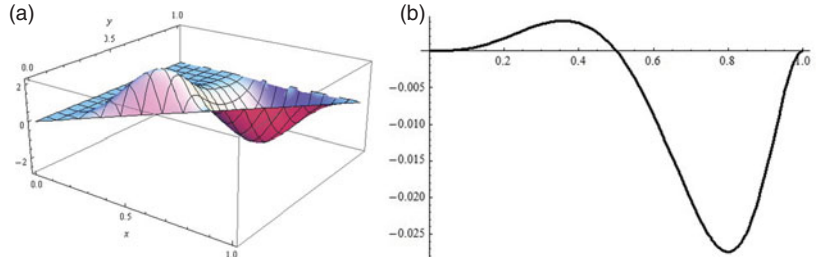


FIGURE **1.** Differences between density functions and survival functions. (a) $f(x,y,0.7)-f(y,x,0.7)$ for $0\le x\le y\le 1$. (b) $\mathsf{P}(T(k;\boldsymbol{\omega})>t)-\mathsf{P}(T\big(k;\tau_{12}(\boldsymbol{\omega})\big)>t)$.

COROLLARY 3.2: *Suppose component lifetime vector* $\boldsymbol{X}$ *is SAI. Then, for any* $1\le i<j\le n$,$T(k;\boldsymbol{\omega},\boldsymbol{X})\ge_{\mathrm{st}} T(k;\tau_{ij}(\boldsymbol{\omega}),\boldsymbol{X})$ *for any* $k>0$ *whenever* $\omega_i\le\omega_j$.

As per the example below, if component lifetime vector $\boldsymbol{X}$ is not SAI, then $\omega_i\le\omega_j$ no longer guarantee the usual stochastic order between $T(k;\boldsymbol{\omega},\boldsymbol{X})$ and $T\big(k;\tau_{ij}(\boldsymbol{\omega}),\boldsymbol{X}\big)$ for $1\le i<j\le n$.

Example 3.3 Clayton copula: A Beta distribution $\mathcal{B}(a,b)$ has density function $x^{a-1}(1-x)^{b-1}/B(a,b)$ for $x\in(0,1)$, where $B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$ for parameters $a>0$ and $b>0$. Assume that component lifetimes $X_1\sim\mathcal{B}(2,2)$, $X_2\sim\mathcal{B}(3,3)$ and $X_3\sim\mathcal{B}(3,2)$ have Clayton copula

$$C(u_1,u_2,u_3)=\big(u_1^{-2}+u_2^{-2}+u_3^{-2}-2\big)^{-1/2},\quad 0<u_1,u_2,u_3<1.$$

As is shown in Figure 1(a), the joint density function $f(x,y,z)$ is not arrangement increasing in $(x,y)\in[0,1]^2$ for given $z=0.7$, let alone being arrangement increasing for $(x,y,z)\in[0,1]^3$. Then, in accordance with Remark 3.7 of Cai and Wei [4] we conclude that lifetime vector (X_1,X_2,X_3) is not SAI.

For the threshold $k=2$ and the component weight vector $\boldsymbol{\omega}=(1,2,3)$, Figure 1(b) plots the difference between $\mathrm{P}(T(k;\boldsymbol{\omega},\boldsymbol{X})>t)$ and $\mathrm{P}(T(k;\tau_{12}(\boldsymbol{\omega}),\boldsymbol{X})>t)$. It is seen that such a difference is neither negative nor positive on the whole interval $\in[0,1]$, and this implies that there is no usual stochastic order between $T(k;\boldsymbol{\omega},\boldsymbol{X})$ and $T\big(k;\tau_{12}(\boldsymbol{\omega}),\boldsymbol{X}\big)$ at all.

Based on Theorem 3.1 and Corollary 3.2, *the system with components having SAI lifetimes gets larger total weight and hence stochastically longer lifetime whenever the weight vector is increasing*. So, from now on we only consider systems with increasing weight vector. Denote

$$\mathcal{I}_n=\{(a_1,\ldots,a_n)\in\mathbb{R}^n: a_n\ge\cdots\ge a_1>0\}$$

the set of all increasing weight vectors and

$$\mathcal{D}_n=\{(a_1,\ldots,a_n)\in\mathbb{R}^n: a_1\ge\cdots\ge a_n>0\}$$

the set of all decreasing weight vectors. The next theorem discusses the expected total system weight of a sub-majorized increasing weight vector.

THEOREM 3.4: *For two weight vectors* $\boldsymbol{a},\boldsymbol{b}\in\mathcal{I}_n$, *if* $X_1\le_{\mathrm{st}}\cdots\le_{\mathrm{st}}X_n$, *then* $\boldsymbol{b}\preceq_{\mathrm{w}}\boldsymbol{a}$ *implies* $\mathsf{E}[W(t;\boldsymbol{a}),\boldsymbol{X}]\ge\mathsf{E}[W(t;\boldsymbol{b},\boldsymbol{X})]$ *for any* $t\ge 0$.

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PROOF: Due to $\boldsymbol{b} \preceq_{\mathrm{w}} \boldsymbol{a}$, it holds that $\sum_{i=j}^{n}(a_i - b_i) \geq 0$ for $j = 1, \ldots, n$. Let $\bar{F}_i$ be the survival function of X_i, $i = 1, \ldots, n$. Then the chain of stochastic orders $X_1 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} X_n$ implies

$$\bar{F}_j(t) - \bar{F}_{j-1}(t) \geq 0, \quad \text{for any } t \geq 0, j = 2, \ldots, n.$$

Note that

$$\mathsf{E}\big[W(t; \boldsymbol{a}, \boldsymbol{X})\big] = \sum_{i=1}^{n} a_i \mathsf{E}\big[I(X_i > t)\big] = \sum_{i=1}^{n} a_i \mathrm{P}(X_i > t) = \sum_{i=1}^{n} a_i \bar{F}_i(t).$$

Denote $\bar{F}_0(t) = 0$ for any t. Then, it holds that, for any $t \geq 0$,

$$\begin{aligned}
&\mathsf{E}\big[W(t; \boldsymbol{a}, \boldsymbol{X})\big] - \mathsf{E}\big[W(t; \boldsymbol{b}, \boldsymbol{X})\big] \\
&= \sum_{i=1}^{n} (a_i - b_i) \bar{F}_i(t) \\
&= \sum_{j=1}^{n} \sum_{i=j}^{n} (a_i - b_i) \big[\bar{F}_j(t) - \bar{F}_{j-1}(t)\big] \\
&= \sum_{j=1}^{n} \big[\bar{F}_j(t) - \bar{F}_{j-1}(t)\big] \sum_{i=j}^{n} (a_i - b_i) \\
&\geq 0,
\end{aligned}$$

which yields $\mathsf{E}[W(t; \boldsymbol{a}, \boldsymbol{X})] \geq \mathsf{E}[W(t; \boldsymbol{b}, \boldsymbol{X})]$. ■

As an illustration of Theorem 3.4 above, the next example reveals that the expectation of the total system weight may not be numerically ordered if the weakly sub-majorization on the weight vector is violated.

Example 3.5: Let X_i be of Beta distribution $Beta(a_i, 2)$ with $a_i = i$ for $i = 1, 2, 3$, it is plain to verify $X_1 \leq_{\mathrm{st}} X_2 \leq_{\mathrm{st}} X_3$. For three weight vectors $\boldsymbol{a} = (2, 3, 4)$, $\boldsymbol{b} = (2, 3, 5)$ and $\boldsymbol{c} = (0.5, 3, 5)$, it is clear that $\boldsymbol{a} \preceq_{\mathrm{w}} \boldsymbol{b}$ and there is no weakly submajorization between $\boldsymbol{a}$ and $\boldsymbol{c}$.

It is shown that (i) $\mathsf{E}[W(t; \boldsymbol{a}, \boldsymbol{X})]$ is indeed above $\mathsf{E}[W(t; \boldsymbol{b}, \boldsymbol{X})]$ in Figure 2(a), and (ii) $\mathsf{E}[W(t; \boldsymbol{a}, \boldsymbol{X})]$ and $\mathsf{E}[W(t; \boldsymbol{c}, \boldsymbol{X})]$ cross with each other on the interval $[0, 1]$ in Figure 2(b).

In accordance with Theorem 3.4, irrespective of the dependence among components lifetimes, *the system with a sub-majorized weight vector acquires a larger expected total system weight whenever component lifetimes are arrayed in stochastic order*. However, as is seen in the following example, the statistical dependence among component lifetimes plays a role in determining the distribution of the total system weight and hence the existence of usual stochastic order.

Example 3.6: Suppose X_i has Beta distribution $Beta(a_i, 2)$ with $a_i = i$ for $i = 1, 2, 3$, and the Clayton copula with parameter 0.2. One can easily verify $X_1 \leq_{\mathrm{st}} X_2 \leq_{\mathrm{st}} X_3$.

For two weight vectors $\boldsymbol{a} = (2, 3, 4)$ and $\boldsymbol{b} = (1, 3, 5)$, it is clear that $\boldsymbol{a} \preceq_{\mathrm{m}} \boldsymbol{b}$ is true. Figure 3(a) plots the curves of survival functions of $W(t; \boldsymbol{a}, \boldsymbol{X})$ and $W(t; \boldsymbol{b}, \boldsymbol{X})$ with $t = 0.5$. As can be seen, neither one is always above the other on the whole interval $[0, 9]$, meaning

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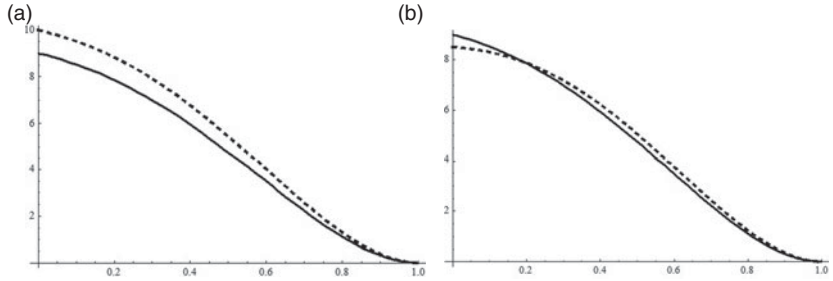


FIGURE **2.** Curves of $\mathsf{E}[W(t;\boldsymbol{a},\boldsymbol{X})]$, $\mathsf{E}[W(t;\boldsymbol{b},\boldsymbol{X})]$ and $\mathsf{E}[W(t;\boldsymbol{c},\boldsymbol{X})]$. (a) $\mathsf{E}[W(t;\boldsymbol{a},\boldsymbol{X})]$ (solid) and $\mathsf{E}[W(t;\boldsymbol{b},\boldsymbol{X})]$ (dashed). (b) $\mathsf{E}[W(t;\boldsymbol{a},\boldsymbol{X})]$ (solid) and $\mathsf{E}[W(t;\boldsymbol{c},\boldsymbol{X})]$ (dashed).

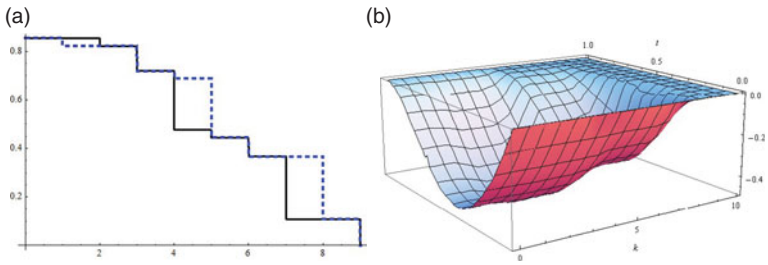


FIGURE **3.** Survival functions and the integrated difference. (a) $\mathrm{P}(W(0.5;\boldsymbol{a},\boldsymbol{X})>k)$ (solid) $\mathrm{P}(W(0.5;\boldsymbol{b},\boldsymbol{X})>k)$ (dashed). (b) $\int_k^{\infty}[\mathrm{P}\big(W(t;\boldsymbol{a},\boldsymbol{X})>x\big)-\mathrm{P}(W(t;\boldsymbol{b},\boldsymbol{X})>x)]\,\mathrm{d}x$.

that the two total system weights cannot be ordered in the usual stochastic order. And based on Figure 3(b)[2], the difference

$$\int_k^{+\infty}[\mathrm{P}(W(t;\boldsymbol{a},\boldsymbol{X})>x)-\mathrm{P}(W(t;\boldsymbol{b},\boldsymbol{X})>x)]\,\mathrm{d}x,$$

is non-negative for any $t\in[0,1]$ and $k\in[0,10]$, and this implies $W(t;\boldsymbol{a},\boldsymbol{X})\le_{\mathrm{icx}}W(t;\boldsymbol{b},\boldsymbol{X})$.

As per Example 3.6, for some dependent component lifetimes the majorization order on component weight vector may result in the increasing convex order on the total system weight. In Theorem 3.7 below, it is found that *the system with a sub-majorized component weight vector acquires a larger total weight in the sense of increasing convex order whenever component lifetime vector is RWSAI.*

THEOREM 3.7: Assume that component lifetime vector $\boldsymbol{X}$ is RWSAI. Then, $\boldsymbol{b}\preceq_{\mathrm{w}}\boldsymbol{a}\in\mathcal{I}_n$ implies $W(t;\boldsymbol{b},\boldsymbol{X})\le_{\mathrm{icx}}W(t;\boldsymbol{a},\boldsymbol{X})$ for any $t\ge 0$.

PROOF: By Lemma 2.2, $\boldsymbol{b}\preceq_{\mathrm{w}}\boldsymbol{a}\in\mathcal{I}_n$ implies that there exists some $\boldsymbol{c}\in\mathcal{I}_n$ such that $\boldsymbol{b}\le\boldsymbol{c}$ and $\boldsymbol{c}\preceq_{\mathrm{m}}\boldsymbol{a}$. Note that

$$\ell(x,y)=\begin{cases}xy, & x\ge 0,\ y\ge 0,\\ 0, & \text{otherwise}\end{cases}$$

[2] Since X_i has the support $[0,1]$ for $i=1,2,3$, and $W(t;\boldsymbol{a},\boldsymbol{X})$ and $W(t;\boldsymbol{b},\boldsymbol{X})$ both have support $[0,9]$, it is sufficient to plot the surface for $t\in[0,1]$ and $k\in[0,10]$.

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is supermodular, increasing in x for any $y \geq 0$ and convex with respect to y for any x. On the other hand, due to Lemma 2.6, it holds that $(I(X_1 > t), \ldots, I(X_n > t))$ is RWSAI. Consequently, from Lemma 2.8 it follows immediately that

$$W(t; \boldsymbol{a}, \boldsymbol{X}) \geq_{\mathrm{icx}} W(t; \boldsymbol{c}, \boldsymbol{X}). \tag{6}$$

Obviously, $\boldsymbol{b} \leq \boldsymbol{c}$ implies $W(t; \boldsymbol{c}, \boldsymbol{X}) \geq_{\mathrm{st}} W(t; \boldsymbol{b}, \boldsymbol{X})$, and hence

$$W(t; \boldsymbol{c}, \boldsymbol{X}) \geq_{\mathrm{icx}} W(t; \boldsymbol{b}, \boldsymbol{X}) \tag{7}$$

follows from the fact that the usual stochastic order implies the increasing convex order. Now, in combination with (6) and (7) we reach $W(t; \boldsymbol{b}, \boldsymbol{X}) \leq_{\mathrm{icx}} W(t; \boldsymbol{a}, \boldsymbol{X})$. ■

To close this section, let us present a dual theorem concerning LWSAI component lifetimes.

THEOREM 3.8: *Suppose component lifetime vector $\boldsymbol{X}$ is LWSAI. Then, $\boldsymbol{b} \preceq^{\mathrm{w}} \boldsymbol{a} \in \mathcal{D}_n$ implies $W(t; \boldsymbol{b}, \boldsymbol{X}) \geq_{\mathrm{icv}} W(t; \boldsymbol{a}, \boldsymbol{X})$ for any $t \geq 0$.*

PROOF: According to Lemma 2.2, $\boldsymbol{b} \preceq^{\mathrm{w}} \boldsymbol{a} \in \mathcal{D}_n$ implies that there exists some $\boldsymbol{c} \in \mathcal{D}_n$ such that $\boldsymbol{b} \geq \boldsymbol{c}$ and $\boldsymbol{c} \preceq_{\mathrm{m}} \boldsymbol{a}$.

Let us first prove $W(t; \boldsymbol{c}) \geq_{\mathrm{icv}} W(t; \boldsymbol{a})$. As per Lemma 2.B.1 of Marshall, Olkin, and Arnold [23] and transitivity of the increasing concave order, it suffices to show that

$$\sum_{i=1}^{n} c_i I(X_i > t) \geq_{\mathrm{icv}} a_l I(X_l > t) + a_r I(X_r > t) + \sum_{k \notin \{l,r\}} c_k I(X_k > t), \tag{8}$$

where $(c_l, c_r) \preceq_{\mathrm{m}} (a_l, a_r)$ and $a_l \geq a_r$ for some $l < r$. Without loss of generality, here we assume $l = 1$ and $r = 2$.

For any increasing and concave u, $(c_1, c_2) \preceq_{\mathrm{m}} (a_1, a_2)$ and $a_1 \geq a_2$, denote

$$g_2(x_1, x_2) = u(c_1 x_1 + c_2 x_2), \quad g_1(x_1, x_2) = u\big(a_1 x_1 + a_2 x_2\big).$$

Due to Lemma 2.3, it holds that $g_2(x_1, x_2) - g_1(x_1, x_2)$ is decreasing in $x_1 \leq x_2$ and

$$g_2(x_1, x_2) + g_2(x_2, x_1) \geq g_1(x_1, x_2) + g_1(x_2, x_1), \quad \text{for any } x_2 \geq x_1.$$

From Lemma 2.6 it follows that $(I(X_1 > t), I(X_2 > t))$ is LWSAI. Thus, by Lemma 2.9 we have, for any increasing and concave u and $(c_1, c_2) \preceq_{\mathrm{m}} (a_1, a_2)$,

$$\begin{aligned} &\mathsf{E}\big[u\big(c_1 I(X_1 > t) + c_2 I(X_2 > t)\big)\big] \\ &\quad = \mathsf{E}\big[g_2(X_1, X_2)\big] \\ &\quad \geq \mathsf{E}\big[g_1(X_1, X_2)\big] \\ &\quad = \mathsf{E}\big[u\big(a_1 I(X_1 > t) + a_2 I(X_2 > t)\big)\big], \end{aligned}$$

which is equivalent to, for $(c_1, c_2) \preceq_{\mathrm{m}} (a_1, a_2)$ and $a_1 \geq a_2$,

$$c_1 I(X_1 > t) + c_2 I(X_2 > t) \geq_{\mathrm{icv}} a_1 I(X_1 > t) + a_2 I(X_2 > t). \tag{9}$$

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Denote $I(\boldsymbol{X} > t)_{\{1,2\}^c} = (I(X_3 > t), \ldots, I(X_n > t))$. According to Lemma 2.7, the LWSAI property of $(I(X_1 > t), \ldots, I(X_n > t))$ implies that $\left[\left(I(X_1 > t), I(X_2 > t)\right) \mid I(\boldsymbol{X} > t)_{\{1,2\}^c}\right]$ is LWSAI. For any increasing and concave u, it follows from (9) that

$$\begin{aligned}
&\mathsf{E}\left[u\left(\sum_{i=1}^{n} c_i I(X_i > t)\right)\right] \\
&\quad = \mathsf{E}\left[\mathsf{E}\left[u\left(c_1 I(X_1 > t) + c_2 I(X_2 > t) + \sum_{k=3}^{n} c_k I(X_k > t)\right) \middle| I(\boldsymbol{X} > t)_{\{1,2\}^c}\right]\right] \\
&\quad \geq \mathsf{E}\left[\mathsf{E}\left[u\left(a_1 I(X_1 > t) + a_2 I(X_2 > t) + \sum_{k=3}^{n} c_k I(X_k > t)\right) \middle| I(\boldsymbol{X} > t)_{\{1,2\}^c}\right]\right] \\
&\quad = \mathsf{E}\left[u\left(a_1 I(X_1 > t) + a_2 I(X_2 > t) + \sum_{k=3}^{n} c_k I(X_k > t)\right)\right],
\end{aligned}$$

for $(c_1, c_2) \preceq_{\mathrm{m}} (a_1, a_2)$ and $a_1 \geq a_2$. This invokes

$$\sum_{i=1}^{n} c_i I(X_i > t) \geq_{\mathrm{icv}} a_1 I(X_1 > t) + a_2 I(X_2 > t) + \sum_{k=3}^{n} c_k I(X_k > t),$$

and hence we reach (8). By reiterating the above procedure we have,

$$W(t; \boldsymbol{c}, \boldsymbol{X}) \geq_{\mathrm{icv}} W(t; \boldsymbol{a}, \boldsymbol{X}). \tag{10}$$

Moreover, $\boldsymbol{b} \geq \boldsymbol{c}$ implies $W(t; \boldsymbol{c}, \boldsymbol{X}) \leq_{\mathrm{st}} W(t; \boldsymbol{b}, \boldsymbol{X})$ and hence

$$W(t; \boldsymbol{c}, \boldsymbol{X}) \leq_{\mathrm{icv}} W(t; \boldsymbol{b}, \boldsymbol{X}). \tag{11}$$

By combining (10) and (11) we reach $W(t; \boldsymbol{b}, \boldsymbol{X}) \geq_{\mathrm{icv}} W(t; \boldsymbol{a}, \boldsymbol{X})$. ■

4. APPLICATIONS

The concept of system signature, introduced by Samaniego [29] for coherent systems with independent and identically distributed (i.i.d.) components, is becoming more and more useful as a research tool for studying the reliability of coherent systems. Afterward, the concept of signature of a systems with exchangeable component lifetimes has been studied in Navarro et al. [26]. In this section, we apply the theoretical results obtained in Section 3 to the signature of coherent system. For more details on the notion of signature and its applications in reliability engineering, one may refer to Kochar, Mukerjee and Samaniego [16], Boland [2], Boland and Samaniego [3], Navarro and Eryilmaz [24], Navarro and Rychlik [25], Navarro, Samaniego, and Balakrishnan [27], Da, Zheng and Hu [7] and related references therein.

Assume $\boldsymbol{X} = (X_1, \ldots, X_n)$ is an exchangeable random vector with absolutely continuous joint distribution. Let $X_{i:n}$ be the ith order statistics associated with $X_1, \ldots, X_n$. The

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signature of the system with lifetime $T_{\mathbf{p}}(\boldsymbol{X})$ is the probability vector $\boldsymbol{p} = (p_1, \ldots, p_n)$, where

$$p_i = \mathrm{P}(T_{\mathbf{p}} = X_{i:n}), \quad i = 1, \ldots, n.$$

According to Lemma 1 of Navarro et al. [26], the system reliability function can be represented as follows:

$$\mathrm{P}(T_{\boldsymbol{p}}(\boldsymbol{X}) > t) = \sum_{i=1}^{n} p_i \mathrm{P}(X_{i:n} > t), \quad \text{for any } t \geq 0. \tag{12}$$

In what follows, we focus on the system having exchangeable component lifetimes. As an application of Theorem 3.4, we immediately get the following stochastic order of two systems with signatures having stochastic ordering.

PROPOSITION 4.1: *For two systems with exchangeable component lifetimes* $\boldsymbol{X}$ *and their signatures* $\boldsymbol{p} = (p_1, \ldots, p_n)$ *and* $\boldsymbol{q} = (q_1, \ldots, q_n)$*, respectively,* $\boldsymbol{p} \preceq_{\mathrm{m}} \boldsymbol{q}$ *implies* $T_{\boldsymbol{p}}(\boldsymbol{X}) \leq_{\mathrm{st}} T_{\boldsymbol{q}}(\boldsymbol{X})$.

PROOF: Note that $X_{1:n} \leq \cdots \leq X_{n:n}$ and $\boldsymbol{p} \preceq_{\mathrm{m}} \boldsymbol{q}$ implies $\boldsymbol{p} \preceq_{\mathrm{w}} \boldsymbol{q}$. By Theorem 3.4 we have

$$\begin{aligned} \sum_{i=1}^{n} p_i \mathrm{P}(X_{i:n} > t) &= \mathsf{E}\left[\sum_{i=1}^{n} p_i I(X_{i:n} > t)\right] \\ &\leq \mathsf{E}\left[\sum_{i=1}^{n} q_i I(X_{i:n} > t)\right] = \sum_{i=1}^{n} q_i \mathrm{P}(X_{i:n} > t). \end{aligned}$$

In light of (12), we conclude that $\mathrm{P}(T_{\boldsymbol{p}}(\boldsymbol{X}) > t) \leq \mathrm{P}(T_{\boldsymbol{q}}(\boldsymbol{X}) > t)$ for any $t \geq 0$, and this is the desired result. ■

It should be remarked here that Theorem 3 of Kochar, Mukerjee and Samaniego [16] also established the above conclusion in Proposition 4.1 for systems with components having independent and identically distributed lifetimes. So, Proposition 4.1 serves as an extension to systems with components having exchangeable lifetimes.

Also, as an application of Theorems 3.7 and 3.8 we get the next proposition.

PROPOSITION 4.2: *For two systems with exchangeable component lifetimes* $\boldsymbol{X}$ *and signatures* $\boldsymbol{p}$ *and* $\boldsymbol{q}$*, respectively,* $\boldsymbol{p} \preceq_{\mathrm{m}} \boldsymbol{q} \in \mathcal{I}_n$ *implies* $T_{\boldsymbol{p}}(\boldsymbol{X}) \leq_{\mathrm{st}} T_{\boldsymbol{q}}(\boldsymbol{X})$.

PROOF: Due to $X_{1:n} \leq \cdots \leq X_{n:n}$, it is easy to check that $(X_{1:n}, \ldots, X_{n:n})$ is SAI. Moreover, $\boldsymbol{p} \preceq_{\mathrm{m}} \boldsymbol{q} \in \mathcal{I}_n$ implies $\boldsymbol{p} \preceq_{\mathrm{w}} \boldsymbol{q} \in \mathcal{I}_n$. Thus, as per Theorem 3.7, $\boldsymbol{p} \preceq_{\mathrm{m}} \boldsymbol{q} \in \mathcal{I}_n$ implies

$$\sum_{i=1}^{n} p_i I(X_{i:n} > t) \leq_{\mathrm{icx}} \sum_{i=1}^{n} q_i I(X_{i:n} > t), \quad \text{for any } t \geq 0.$$

Since $u(x) = x$ is increasing convex, we immediately get, for any $t \geq 0$,

$$\begin{aligned} \mathrm{P}\big(T_{\boldsymbol{p}}(\boldsymbol{X}) > t\big) &= \mathsf{E}\left[\sum_{i=1}^{n} p_i I(X_{i:n} > t)\right] \\ &\leq \mathsf{E}\left[\sum_{i=1}^{n} q_i I(X_{i:n} > t)\right] = \mathrm{P}\big(T_{\boldsymbol{q}}(\boldsymbol{X}) > t\big). \end{aligned}$$

That is the desired result. ■

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By Theorem 3.8 it is not difficult to come up with the last proposition, we omit the proof due to the similarity.

PROPOSITION 4.3: *For two systems with exchangeable component lifetimes* $\boldsymbol{X}$ *and their signatures* $\boldsymbol{p}$ *and* $\boldsymbol{q}$*, respectively,* $\boldsymbol{p} \preceq_{\mathrm{m}} \boldsymbol{q} \in \mathcal{D}_n$ *implies* $T_{\boldsymbol{p}}(\boldsymbol{X}) \geq_{\mathrm{st}} T_{\boldsymbol{q}}(\boldsymbol{X})$.

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