Proceedings of the Royal Society of Edinburgh: Section A Mathematics, **156**, 1835–1868, *2026*
DOI:10.1017/prm.2024.125

The admissible KMS bundles on classifiable C*-algebras

Robert Neagu
Department of mathematics, KU Leuven, Celestijnenlaan 200B, 3001 Leuven, Belgium (robert.neagu@kuleuven.be)

(Received 9 February 2024; revised 19 October 2024; accepted 21 October 2024)

Given any unital, finite, classifiable C*-algebra A with real rank zero and any compact simplex bundle with the fibre at zero being homeomorphic to the space of tracial states on A, we show that there exists a flow on A realizing this simplex. Moreover, we show that given any unital UCT Kirchberg algebra A and any proper simplex bundle with empty fibre at zero, there exists a flow on A realizing this simplex.

Keywords: states; classification of C*-algebras; general theory of C*-algebras; C*-algebras; flows; KMS conditions; equilibrium states

2020 *Mathematics Subject Classification:* Primary: 46L30
Secondary: 46L35; 46L05

1. Introduction

The studyof group actions is ubiquitous within the subject of operator algebras, often providing deep structural properties. While classifying actions of discrete groups proved to be an indispensable tool for understanding the structure and symmetries of operator algebras, certain applications to geometry or physics are usually related to time evolutions, or continuous actions of $\mathbb{R}$. In the setting of von Neumann algebras, such actions appeared most prominently in the form of modular automorphism groups through the theory of Tomita and Takesaki [51]. One convenient feature of the Tomita-Takesaki theory is that one can check if a certain flow is a modular flow by verifying the so-called KMS condition. Precisely,

*Funded by the European Union. Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the EU nor the ERC can be held responsible for them. For the purpose of open access, the author has applied a CC-BW license to any author accepted manuscript arising from this submission.

© The Author(s), 2024. Published by Cambridge University Press on behalf of The Royal Society of Edinburgh This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

a faithful normal state φ on a von Neumann algebra satisfies the KMS conditions for a flow if and only if the given flow is the modular flow induced by φ. Therefore, investigating the set of corresponding KMS states sheds light on questions regarding structure and classification of continuous actions of $\mathbb{R}$ on von Neumann algebras.

On the C*-algebraic side, named after mathematical physicists Kubo, Martin, and Schwinger, KMS states are a special class of states on any C*-algebra admitting a continuous action of $\mathbb{R}$. However, the collection of KMS states for a given flow on a C*-algebra can be quite intricate. Indeed, given a flow on a unital separable C*-algebra, the collection of KMS states corresponding to that flow, also called the KMS bundle of the flow, has the structure of a *proper simplex bundle* (see for example [27, §2.1]).

The quest of constructing flows inducing a given KMS simplex has its roots in work of Bratteli, Elliott, Herman, and Kishimoto in [7–9], where flows were constructed on certain classes of simple C*-algebras. More recently, extensive work has been done to examine possible KMS bundles on various special classes of C*-algebras (see for example [1, 3, 6, 15–17, 53]). Moreover, in [27, 54] Elliott and Thomsen built flows on simple AF-algebras with prescribed KMS behaviour. These results were then extended by Elliott, Sato, and Thomsen in [25, 26] to any unital classifiable C*-algebra with a unique trace, as well as any unital UCT Kirchberg algebra with no torsion in the K_1-group.

Using the new abstract classification of morphisms from [13], we will prove a similar result for any unital tracial classifiable C*-algebra with real rank zero. However, as in [26, theorem 3.14], the assumption that the KMS bundle is compact is still needed in the case of tracial C*-algebras. In [25], this assumption was removed in the case when the given classifiable C*-algebra has a unique trace. Moreover, without any restrictions on the given KMS bundle, we will remove the assumption of no torsion in the K_1-group from [26, theorem 5.1] to obtain the definitive result for unital UCT Kirchberg algebras. The main results of this paper are the following.

THEOREM A *Let A be a unital* UCT *Kirchberg algebra and let (S, π) be a proper simplex bundle such that $\pi^{-1}(0) = \emptyset$. Then there exists a 2π-periodic flow θ on A such that its induced* KMS *bundle (S^θ, π^θ) is isomorphic to (S, π).*

THEOREM B *Let A be a unital, stably finite, classifiable* C*-*algebra with real rank zero and let (S, π) be a compact simplex bundle such that $\pi^{-1}(0)$ is affinely homeomorphic to $T(A)$. Then there exists a 2π-periodic flow θ on A such that its induced* KMS *bundle (S^θ, π^θ) is isomorphic to (S, π).*

The proof of theorem A follows the skeleton provided by [26, theorem 5.1]. Essentially, given a unital UCT Kirchberg algebra A and a suitable simplex bundle (S, π), one can build a pair of abelian groups containing both information from the K-theory of A as well as the bundle (S, π). Then, using the classification results in [21], we can obtain a stable $A\mathbb{T}$-algebra of real rank zero B realizing the constructed pair of abelian groups. Then, A can be identified with a full corner of a crossed product of B by $\mathbb{Z}$. It is precisely the fact that B is an $A\mathbb{T}$-algebra rather than an AF-algebra (see [26, theorem 5.1]) that allows us to remove the assumption that A has no torsion in K_1.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

The proof of theorem B uses a similar strategy. Given a unital classifiable C*-algebra A with real rank zero, we realize it as a full corner of a crossed product of a stabilized unital classifiable C*-algebra $B\otimes\mathcal{K}$ by $\mathbb{Z}$. A key ingredient in obtaining an appropriate automorphism of $B\otimes\mathcal{K}$ is the classification of unital *-homomorphisms from [13]. Moreover, compared to [26, theorem 3.14], where the corresponding result was proved for the Jiang-Su algebra $\mathcal{Z}$, extra care has to be taken in checking that A and the corner of the crossed product built above have the same pairing between K-theory and traces. The assumption that A has real rank zero is instrumental in constructing an Elliott invariant of a classifiable C*-algebra B. Precisely, one can build an ordered abelian group G and then realize the trace space of B as the space of states on G. However, to make sure that the state space on G is a metrisable Choquet simplex, the fact that A has real rank zero is heavily used. The assumption that the space S is compact in theorem B seems to be more difficult to remove. In particular, it ensures that the constructed ordered abelian group is simple, and hence the C*-algebra B is simple. Then, one can use a classification of automorphisms of $B\otimes\mathcal{K}$ to obtain a suitable automorphism γ. Note that a definitive classification theorem for automorphisms of general nonsimple $\mathcal{Z}$-stable C*-algebras is currently out of reach, so the compactness assumption appears necessary at this point.

This paper is organized as follows. In §2 we collect some definitions regarding KMS states and classification invariants. Then, §3 is concerned with the proof of theorem A, while §4 will focus on the proof of theorem B.

2. Preliminaries

2.1 KMS states

In this subsection, we record some facts about KMS states. We refer the reader to [55] for a detailed account of the topic.

A *flow* θ on a C*-algebra A is a continuous one-parameter group of automorphisms $(\theta_t)_{t\in\mathbb{R}}$.

DEFINITION 2.1. *Let A be a C*-algebra, θ be a flow on A, and $\beta\in\mathbb{R}$. Then, ω is said to be a β-*KMS *weight for θ on A if ω is a weight on A such that*

$$\omega\circ\theta_t=\omega \text{ for all } t\in\mathbb{R}$$

and

$$\omega(a^*a)=\omega\left(\theta_{-\frac{i\beta}{2}}(a)\theta_{-\frac{i\beta}{2}}(a)^*\right),\ a\in A \text{ analytic.}\ [1]$$

*If, in addition, ω has norm 1, then it is called a β-*KMS *state for θ on A.*

REMARK 2.2. Note that a 0-KMS weight for θ is a θ-invariant trace. Moreover, a 0-KMS state for θ is a θ-invariant tracial state.

[1] An element $a\in A$ is analytic for θ if the map $t\in\mathbb{R}\mapsto\theta_t(a)\in A$ extends to an entire analytic map $\mathbb{C}\to A$.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

If A is a separable unital C*-algebra and θ is a flow on A, we consider the collection of KMS states for θ on A. Following the notation in [26, §2], for each $\beta \in \mathbb{R}$, let S^θ_β denote the set of β-KMS states for θ. If we denote the set of states on A by $S(A)$, let

$$S^\theta = \left\{(\omega, \beta) \in S(A) \times \mathbb{R} : \omega \in S^\theta_\beta\right\},$$

and equip it with the relative topology from $S(A) \times \mathbb{R}$. In particular, since A is separable, the topology on S^θ is metrisable. Let $\pi^\theta : S^\theta \to \mathbb{R}$ be the projection onto the second coordinate. Then, the pair (S^θ, π^θ) will be called the {KMS *bundle* of θ. This was characterized abstractly in [26] as we review below.

Fix a second countable locally compact Hausdorff space S and let π be a continuous map from S to $\mathbb{R}$. The pair (S, π) is said to be a *simplex bundle* if the inverse image $\pi^{-1}(t)$ is a compact metrisable Choquet simplex in the relative topology from S for any $t \in \mathbb{R}$. For such pairs (S, π), we denote by $\mathcal{A}(S, \pi)$ the set of continuous functions f from S to $\mathbb{R}$ such that the restriction $f|_{\pi^{-1}(t)}$ of f to $\pi^{-1}(t)$ is affine for any $t \in \mathbb{R}$.

Definition 2.3. [26, definition 2.1] *A simplex bundle (S, π) is proper, if*

- *$\pi^{-1}(K)$ is compact in S for any compact subset K of $\mathbb{R}$,*
- *For any $x \neq y$ in S, there exists $f \in \mathcal{A}(S, \pi)$ such that $f(x) \neq f(y)$.*

Moreover, if S is compact, then (S, π) is called a compact simplex bundle. Two proper simplex bundles (S, π) and (S', π') are isomorphic if there exists a homeomorphism $\phi : S \to S'$ such that

$$\pi' \circ \phi = \pi \text{ and } \phi : \pi^{-1}(\beta) \to \pi'^{-1}(\beta) \text{ is affine for all } \beta \in \mathbb{R}.$$

Remark 2.4. By [, theorem 9.2.2], the collection of KMS states for a flow on a unital separable C*-algebra is a proper simplex bundle. Hence, this is a necessary assumption for the pair (S, π) to be a suitable invariant. Moreover, a proper simplex bundle (S, π) is second countable, locally compact, Hausdorff, so the topology on S is metrisable.

Lemma 2.5. *Let $\eta : (S', \pi') \to (S, \pi)$ be a bijection between proper simplex bundles such that $\pi \circ \eta = \pi'$. If either η or η^{-1} is continuous, then η is a homeomorphism.*

Proof. Suppose that η is continuous. It remains to show that η^{-1} is continuous. We show that η maps closed sets to closed sets. By [43, corollary], it suffices to check that η is a proper map i.e. the preimage of any compact set is compact. Let $K \subseteq S$ be compact and note that $\pi(K)$ is compact since π is continuous. Since $\pi \circ \eta = \pi'$, it follows that

$$\eta^{-1}(K) \subseteq \eta^{-1}(\pi^{-1}(\pi(K))) = (\pi')^{-1}(\pi(K)).$$

But $(\pi')^{-1}(\pi(K))$ is compact since (S', π') is a proper simplex bundle (see definition 1.3), so $\eta^{-1}(K)$ is compact as a closed subset of a compact set. This finishes the argument. The case when η^{-1} is assumed to be continuous is completely symmetric. □

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

2.2 Classification invariants

Let A be a separable C*-algebra. We denote by $K_0(A)_+$ the set of positive elements in $K_0(A)$. We write $T(A)$ for the set of tracial states on A. Recall that for a unital C*-algebra A, there exists a natural pairing between the K_0-group and tracial states. For $n \in \mathbb{N}$ and $\tau \in T(A)$, we denote the canonical non-normalized extension of τ to a tracial functional on $M_n(A)$ by τ_n. Then, the pairing map $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ is given by

$$\rho_A([p]_0 - [q]_0)(\tau) := \tau_n(p - q)$$

for all $\tau \in T(A)$ and all projections $p, q \in M_n(A)$.

Definition 2.6. *The Elliott invariant of a unital separable* C*-*algebra A, denoted* $\mathrm{Ell}(A)$, *is given by*

$$\mathrm{Ell}(A) = (K_0(A), K_0(A)_+, [1_A]_0, K_1(A), T(A), \rho_A).$$

Remark 2.7. In the case of a stable C*-algebra, one needs to remove the class of the unit and consider the cone of densely defined lower semicontinuous traces instead of the set of tracial states.

Due to work by many mathematicians (e.g. [13, 24, 29, 37, 44, 57]), this invariant was shown to classify a large class of simple C*-algebras. In this paper, we will use the following classification theorem.

Theorem 2.8 (see [13, theorem A]). *Unital simple separable nuclear $\mathcal{Z}$-stable* C*-*algebras satisfying Rosenberg and Schochet's universal coefficient theorem (UCT) are classified by Elliott's invariant consisting of K-theory and traces.*

Remark 2.9. On the class of C*-algebras in the statement of theorem 2.8, the order on the K_0-group is induced by traces.[2] Precisely, non-zero positive elements in K_0 are those elements that are strictly positive when evaluated on traces. Therefore, if A is a C*-algebra as in the statement of theorem 2.8, then the invariant $(K_0(A), [1_A]_0, K_1(A), T(A), \rho_A)$ carries the same information as $\mathrm{Ell}(A)$. We will use this observation in the proof of lemma 4.6.

Since we will need to check that certain corners of crossed products by $\mathbb{Z}$ satisfy the assumptions of the classification theorem, we collect below a number of permanence properties for the hypotheses in the statement of theorem 2.8.

Proposition 2.10. *Let A, B be separable* C*-*algebras such that $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$. If A is nuclear, $\mathcal{Z}$-stable, and satisfies the UCT, then B is nuclear, $\mathcal{Z}$-stable, and satisfies the UCT.*

Proof. We first note that $B \otimes \mathcal{K} \cong A \otimes \mathcal{K}$ is nuclear and $\mathcal{Z}$-stable. Since nuclearity passes to hereditary subalgebras, B is nuclear. Moreover, B is $\mathcal{Z}$-stable by [58,

[2] This is a consequence of [13, proposition 4.13(ii)] which follows from [49] (one can also see [28, theorem 1]).

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

corollary 3.1]. Finally, the UCT is closed under stable isomorphisms ([4, 22.3.5 (a)]), so B satisfies the UCT. □

PROPOSITION 2.11. *Let B be a separable C^*-algebra, γ be an automorphism of B, and $e \in B$ be a full projection in $B \rtimes_\gamma \mathbb{Z}$. Then the following hold.*

(i) The corner $e(B \rtimes_\gamma \mathbb{Z})e$ is separable and unital.
(ii) If $B \rtimes_\gamma \mathbb{Z}$ is simple, then $e(B \rtimes_\gamma \mathbb{Z})e$ is simple.
(iii) If B is nuclear, then $e(B \rtimes_\gamma \mathbb{Z})e$ is nuclear.
(iv) If B satisfies the UCT, then so does $e(B \rtimes_\gamma \mathbb{Z})e$.
(v) If $B \rtimes_\gamma \mathbb{Z}$ is $\mathcal{Z}$-stable, then so is $e(B \rtimes_\gamma \mathbb{Z})e$.

Proof. Note that $e(B \rtimes_\gamma \mathbb{Z})e$ is unital as e is a projection, and it is separable since B is separable and $\mathbb{Z}$ is a countable discrete group. If $B \rtimes_\gamma \mathbb{Z}$ is simple and e is full, then $e(B \rtimes_\gamma \mathbb{Z})e$ is simple. If B is nuclear, then $B \rtimes_\gamma \mathbb{Z}$ is nuclear by [11, theorem 4.2.4]. Since nuclearity passes to hereditary subalgebras, $e(B \rtimes_\gamma \mathbb{Z})e$ is nuclear. Suppose now that B satisfies the UCT. Since the UCT is closed under crossed products by $\mathbb{Z}$ ([4, 22.3.5 (g)]), $B \rtimes_\gamma \mathbb{Z}$ satisfies the UCT. Furthermore, $B \rtimes_\gamma \mathbb{Z}$ and $e(B \rtimes_\gamma \mathbb{Z})e$ are stably isomorphic (see [10]) and the UCT is closed under stable isomorphisms ([4, 22.3.5 (a)]). Hence, $e(B \rtimes_\gamma \mathbb{Z})e$ satisfies the UCT. Since $\mathcal{Z}$-stability passes to hereditary subalgebras ([58, corollary 3.1]) and $B \rtimes_\gamma \mathbb{Z}$ is $\mathcal{Z}$-stable, it follows that $e(B \rtimes_\gamma \mathbb{Z})e$ is $\mathcal{Z}$-stable. □

Furthermore, theorem 2.8 is complemented by a range-of-the-invariant result from [22]. Before stating the result, we need to record a few notions regarding ordered abelian groups.

2.3 Ordered groups and the range of the invariant

We refer the reader to [30] for a detailed account on the content of this subsection.

An ordered abelian group (G, G_+) is called *unperforated* if for any $n \in \mathbb{N}$ and any $g \in G$ such that $ng \in G_+$, then $g \in G_+$. Moreover, an ordered abelian group (G, G_+) is called *weakly unperforated* if for any $n \in \mathbb{N}$ and any $g \in G$ such that $ng \in G_+ \setminus \{0\}$, then $g \in G_+ \setminus \{0\}$. Also, (G, G_+) is said to have the *Riesz interpolation property* if for any $g_1, g_2, h_1, h_2 \in G$ such that $g_i \leq h_j$ for any $i, j = 1, 2$ there exists $z \in G$ such that $g_i \leq z \leq h_j$ for any $i, j = 1, 2$. If all inequalities are replaced by strict inequalities, then (G, G_+) is said to satisfy the *strong Riesz interpolation property*. An ordered abelian group (G, G_+) is said to have the *Riesz decomposition property* if for any $g, h_1, h_2 \in G_+$ such that $g \leq h_1 + h_2$, there exist $g_1, g_2 \in G_+$ such that $g = g_1 + g_2$ and $g_j \leq h_j$ for each $j = 1, 2$.

Recall also that a countable ordered abelian group (G, G_+) is called a *dimension group* if it is isomorphic to the inductive limit of a sequence

$$\mathbb{Z}^{r_1} \xrightarrow{\alpha_1} \mathbb{Z}^{r_2} \xrightarrow{\alpha_2} \mathbb{Z}^{r_3} \xrightarrow{\alpha_3} \dots,$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

for some natural numbers r_n, some positive group homomorphisms α_n, where $\mathbb{Z}^r$ is equipped with its standard ordering given by

$$(\mathbb{Z}^r)_+ = \{(x_1, x_2, \ldots, x_r) : x_j \geq 0\}.$$

Using a classical theorem of Effros, Handelman, and Shen ([19, theorem 2.2]), a countable ordered abelian group is a dimension group if and only if it is *unperforated* and has the *Riesz interpolation property*.

Recall that I is an *order ideal* of (G, G_+) if I is a subgroup of G such that

$$I = (I \cap G_+) - (I \cap G_+)$$

and

$$\text{if } 0 \leq y \leq x \text{ in } G_+ \text{ and } x \in I, \text{ then } y \in I.$$

Moreover, an ordered abelian group (G, G_+) is called *simple* if any non-zero positive element is an order unit.

In §3, we will often use some of the properties of ordered groups defined above in the case when the group G is a priori graded. To avoid any confusion, we remark that the meaning of these properties is unchanged. In particular, $G = G_0 \oplus G_1$ is a *graded ordered group* if it is graded and there exists a distinguished subset G_+ such that (G, G_+) is an ordered group in the usual sense.

We can now state the range-of-the-invariant result in [22]. The theorem below is folklore (see also [13, remark 2.5] stated in terms of the KT_u-invariant) and it is implicitly contained in [22] as all of the inductive limit constructions provided in [22] are simple and have finite nuclear dimension, and hence are $\mathcal{Z}$-stable. We will use this result in the proof of theorem B.

THEOREM 2.12 ([22]). *Let (G, G_+, v) be a simple, countable, ordered abelian group which is weakly unperforated and has distinguished order unit v, H be a countable abelian group, and X be a compact, metrisable Choquet simplex together with a weakly unperforated pairing $\rho : G \times X \to \mathbb{R}$ which determines G_+. Suppose also that the pairing ρ induces a surjection from X onto the positive homomorphisms on (G, G_+) which send v to 1. Then there exists a simple, separable, unital, nuclear, $\mathcal{Z}$-stable* C*-*algebra A satisfying the UCT such that*

$$\mathrm{Ell}(A) \cong (G, G_+, v, H, X, \rho).$$

Proof. Let (G, G_+, v, H, X, ρ) be as in the statement of the theorem. Then there exists a simple separable stable nuclear C*-algebra B such that $\mathrm{Ell}(B) \cong (G, G_+, H, \tilde{X}, \tilde{\rho})$, where $\tilde{X}$ is the cone with base X and $\tilde{\rho}$ is the unique extension of the pairing ρ ([22, theorem 5.2.3.2]). Since the C*-algebra B is constructed as an inductive limit, one can simply tensor with $\mathcal{Z}$ at each stage to assume that B is $\mathcal{Z}$-stable, or notice that B is simple and has finite nuclear dimension as each building block has finite nuclear dimension ([61, proposition 2.3]). Thus, B is $\mathcal{Z}$-stable by [56, corollary 8.7]. Moreover, the UCT is preserved by inductive limits (see [4, 22.3.5 (e)]), so B satisfies the UCT.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

If $p \in B$ is a projection such that $[p]_0 = v$ in $K_0(B)$, then we take $A = pBp$. Since B is simple, p is a full projection, so A is simple separable and unital (see [10]). Moreover, A is nuclear, $\mathcal{Z}$-stable, and it satisfies the UCT by proposition 2.10. Finally, it is immediate to see that $\mathrm{Ell}(A) \cong (G, G_+, v, H, X, \rho)$. □

2.4 Elliott's classification of AT-algebras of real rank zero

This subsection contains a summary of Elliott's classification of (possibly non-simple) AT-algebras of real rank zero from [21].

Let A be a unital C*-algebra, $K_*(A)$ be the graded group $K_0(A) \oplus K_1(A)$, equipped with the order

$$K_*(A)_+ = \left\{ \begin{array}{c} ([p]_0, [u \oplus (1_n - p)]_1) : p \in \mathcal{P}(M_n(A)) \text{ for some } n \in \mathbb{N}, \\ u \in \mathcal{U}(pM_n(A)p) \end{array} \right\},$$

and

$$D_*(A) = \{([p]_0, [u \oplus (1-p)]_1) \in K_*(A)_+ : p \in A\}.$$

If A is nonunital, then define

$$K_*(A)_+ = K_*(A) \cap K_*(\tilde{A})_+,$$

and

$$D_*(A) = K_*(A) \cap D_*(\tilde{A}),$$

where $\tilde{A}$ is the minimal unitization of A. If A is an AT-algebra of real rank zero, consider the pair $(K_*(A), K_*(A)_+)$. Since A also has stable rank one, the pair $(K_*(A), K_*(A)_+)$ is a graded ordered group ([21, theorem 3.2]). Then, the graded group $K_*(A)$ together with the *graded dimension range* $D_*(A)$ determine the pair $(K_*(A), K_*(A)_+)$ and constitutes the invariant used by Elliott in his classification of AT-algebras of real rank zero. If A is simple, the invariant $(K_*(A), K_*(A)_+)$ reduces to the usual K-theory triple $(K_0(A), K_0(A)_+, K_1(A))$ appearing in the Elliott invariant of a nonunital C*-algebra.

We record the following classification of isomorphisms between AT-algebras of real rank zero from [21]. Note that the statement does not appear in [21], but, as observed in [21, remark 7.3], the proof of [21, theorem 7.1] not only produces a classification of AT-algebras of real rank zero, but of isomorphisms between them.

THEOREM 2.13 ([21]). *Let A and B be AT-algebras of real rank zero. Then A is isomorphic to B if and only if there is a graded group isomorphism $\alpha : K_*(A) \to K_*(B)$ such that $\alpha(D_*(A)) = D_*(B)$. Moreover, for each such α, there exists an isomorphism $\phi : A \to B$ such that $K_*(\phi) = \alpha$.*

We also recall the range-of-the-invariant result in [21] for AT-algebras of real rank zero.

THEOREM 2.14 ([21, theorem 8.1]). *Let $G_* = G_0 \oplus G_1$ be a graded ordered group which is the inductive limit of a sequence $G_{*1} \to G_{*2} \to \ldots$ of graded ordered groups*

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

such that each $G_{*i} = G_{0i} \oplus G_{1i}$ is the direct sum of finitely many basic building blocks of the form $\mathbb{Z} \oplus \mathbb{Z}$ with the order determined by strict positivity in the first component. If G_* has the Riesz interpolation property, then there exists a stable $A\mathbb{T}$-algebra of real rank zero A such that $K_*(A) \cong G_*$ and $K_*(A)_+ \cong (G_*)_+$.

We record the following folklore result relating traces and positive homomorphisms on the K_0-group of A$\mathbb{T}$-algebras of real rank zero.

PROPOSITION 2.15. *Let B be a separable, stable $A\mathbb{T}$-algebra of real rank zero. Then the canonical map from the densely defined lower semicontinuous traces on B to positive group homomorphisms $K_0(B) \to \mathbb{R}$ is an affine homeomorphism.*

Proof. This follows from [31, theorem 12.3] (or [5, theorem III.1.3]) since any lower semicontinuous extended quasitrace on an A$\mathbb{T}$-algebra is an extended trace ([12, theorem 6]). □

2.5 Finite Rokhlin dimension

The Rokhlin dimension of a group action on a unital C*-algebra was introduced in [34] as a tool to obtain finite nuclear dimension and hence $\mathcal{Z}$-stability of the corresponding crossed product. In [33], Hirshberg and Phillips extended finite Rokhlin dimension to actions on possibly nonunital C*-algebras. We will use the simplified definition from [35].

DEFINITION 2.16. ([35, definition 4.6]). *An automorphism γ of a C*-algebra A is said to have Rokhlin dimension d if d is the least nonnegative integer with the following property. For any finite set $F \subseteq A$, integer $p \geq 1$ and $\epsilon > 0$, there are positive contractions $f_0^{(l)}, \ldots, f_{p-1}^{(l)} \in A$, $l \in \{0, 1, \ldots, d\}$, such that:*

(i) $\|f_k^{(l)} f_j^{(l)} a\| < \epsilon$ *for every* $a \in F$, $l \in \{0, 1, \ldots, d\}$, $j \neq k \in \{0, 1, \ldots, p-1\}$;
(ii) $\left\| \left(\sum_{l=0}^{d} \sum_{k=0}^{p-1} f_k^{(l)} \right) a - a \right\| < \epsilon$ *for every* $a \in F$;
(iii) $\|[f_j^{(l)}, a]\| < \epsilon$ *for every* $a \in F$, $l \in \{0, 1, \ldots, d\}$, $j \in \{0, 1, \ldots, p-1\}$;
(iv) $\left\| \left(\gamma(f_j^{(l)}) - f_{j+1}^{(l)} \right) a \right\| < \epsilon$ *for every* $a \in F$, $l \in \{0, 1, \ldots, d\}$, $j \in \{0, 1, \ldots, p-1\}$, *where* $f_p^{(l)} := f_0^{(l)}$.

We will use finite Rokhlin dimension as a tool to detect when densely defined lower semicontinuous traces on a crossed product by $\mathbb{Z}$ can be identified with densely defined lower semicontinuous invariant traces on the algebra. The next lemma essentially follows from [41, proposition 2.3], as the proof also works in the nonunital and nonsimple setting. That the proof works in the nonunital setting was previously observed in [35, lemma 4.8]. However, the proof in [41, proposition 2.3], only deals with bounded traces. To generalize the result for densely defined lower semicontinuous traces, we will further assume the existence of an approximate unit of projections.

LEMMA 2.17. (cf.[41, proposition 2.3]). *Let A be a separable C*-algebra with an approximate unit of projections $(q_r)_{r \in \mathbb{N}}$ and γ be an automorphism of A which has*

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

finite Rokhlin dimension. Then the restriction map from densely defined lower semicontinuous traces on $A \rtimes_\gamma \mathbb{Z}$ to γ-invariant densely defined lower semicontinuous traces on A is bijective.

Proof. Using the strategy in [54, lemma 3.4], we will adapt Liao's proof of [41, proposition 2.3]. Let $P : A \rtimes_\gamma \mathbb{Z} \to A$ be the canonical conditional expectation. Given a densely defined lower semicontinuous trace τ on $A \rtimes_\gamma \mathbb{Z}$, we will show that $\tau = \tau|_A \circ P$. Since q_r is a projection and hence contained in the Pedersen ideal of A, we have that $\tau(q_r) < \infty$. Moreover,

$$\tau(x) = \lim_{r\to\infty} \tau(q_r x q_r)$$

and

$$\tau(P(x)) = \lim_{r\to\infty} \tau(q_r P(x) q_r)$$

for any positive element x in $A \rtimes_\gamma \mathbb{Z}$. Therefore, to show that $\tau = \tau|_A \circ P$, it suffices to show that

$$\tau(q_r x q_r) = \tau(q_r P(x) q_r)$$

for any positive element x in $A \rtimes_\gamma \mathbb{Z}$ and any $r \in \mathbb{N}$. Since τ is nonzero, we can assume that $\tau(q_r) > 0$ for any $r \in \mathbb{N}$.

Let u be the unitary in $\mathcal{M}(A \rtimes_\gamma \mathbb{Z})$ implementing the action. Since the map $x \mapsto \tau(q_r x q_r)$ is bounded on $A \rtimes_\gamma \mathbb{Z}$ with norm $\tau(q_r)$, it suffices to show that $\tau(q_r a u^n q_r) = 0$ for any $a \in A$, and any $r, n \in \mathbb{N}$.

Recall that the automorphism γ has finite Rokhlin dimension, say $d \in \mathbb{N}$. Let $r, n \in \mathbb{N}$, $a \in A$ be a contraction, and $\epsilon > 0$. Let $p = 2n$ and $\epsilon' = \frac{\epsilon}{(6(d+1)n+1)\tau(q_r)}$. Using (ii) of definition 1.16, it follows that there exist $\{f_k^{(\ell)} : 0 \le k \le p-1,\ 0 \le \ell \le d\}$ in A such that

$$\left\| \left(\sum_{\ell=0}^{d} \sum_{k=0}^{p-1} f_k^{(\ell)} \right) a - a \right\| < \epsilon'. \tag{1.1}$$

Combining (i), (iii), and (iv) of definition 1.16, we can further assume that

$$\| f_k^{(\ell)\frac{1}{2}} a \gamma^n (f_k^{(\ell)\frac{1}{2}}) \| < \epsilon' \quad (\ell = 0, 1, ..., d;\ k = 0, ..., p-1). \tag{1.2}$$

Moreover, by (iii) of definition 1.16, we can ensure that

$$\| f_k^{(l)\frac{1}{2}} q_r - q_r f_k^{(l)\frac{1}{2}} \| < \epsilon' \tag{1.3}$$

for any $0 \le k \le p-1$ and any $0 \le l \le d$.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

Then

$$|\tau(q_r a u^n q_r)| \overset{(1.1)}{\leq} \left|\sum_{\ell=0}^{d}\sum_{k=0}^{p-1}\tau(q_r f_k^{(\ell)} a u^n q_r)\right| + \tau(q_r)\epsilon'. \tag{1.4}$$

Fix k, l and let $x := f_k^{(l)\frac{1}{2}} a u^n$. Using that τ is a trace and that x is a contraction, we get that

$$\begin{aligned} |\tau(q_r x(q_r f_k^{(l)\frac{1}{2}} - f_k^{(l)\frac{1}{2}} q_r))| &= |\tau(q_r x(q_r f_k^{(l)\frac{1}{2}} - f_k^{(l)\frac{1}{2}} q_r)q_r)| \\ &\leq \tau(q_r)\|x(q_r f_k^{(l)\frac{1}{2}} - f_k^{(l)\frac{1}{2}} q_r)\| \\ &\overset{(1.3)}{\leq} \tau(q_r)\epsilon'. \end{aligned} \tag{1.5}$$

Similarly, one has that

$$\begin{aligned} |\tau(f_k^{(\ell)\frac{1}{2}} q_r f_k^{(\ell)\frac{1}{2}} a u^n q_r) - \tau(q_r f_k^{(\ell)} a u^n q_r)| &= |\tau(q_r(f_k^{(\ell)\frac{1}{2}} q_r - q_r f_k^{(\ell)\frac{1}{2}})x q_r)| \\ &\leq \tau(q_r)\epsilon'. \end{aligned}$$

Therefore, using the last two estimations above, we get that

$$\begin{aligned} &|\tau(q_r f_k^{(\ell)\frac{1}{2}} a u^n f_k^{(\ell)\frac{1}{2}} q_r) - \tau(q_r f_k^{(\ell)} a u^n q_r)| \\ &\leq |\tau(q_r f_k^{(\ell)\frac{1}{2}} a u^n q_r f_k^{(\ell)\frac{1}{2}}) - \tau(q_r f_k^{(\ell)} a u^n q_r)| + \tau(q_r)\epsilon' \\ &= |\tau(f_k^{(\ell)\frac{1}{2}} q_r f_k^{(\ell)\frac{1}{2}} a u^n q_r) - \tau(q_r f_k^{(\ell)} a u^n q_r)| + \tau(q_r)\epsilon' \\ &\leq 2\tau(q_r)\epsilon'. \end{aligned}$$

Thus, from (1.4) we now get that

$$\begin{aligned} |\tau(q_r a u^n q_r)| &\leq \sum_{\ell=0}^{d}\sum_{k=0}^{p-1}\left|\tau(q_r f_k^{(\ell)\frac{1}{2}} a u^n f_k^{(\ell)\frac{1}{2}} q_r)\right| + (2(d+1)p+1)\tau(q_r)\epsilon' \\ &= \sum_{\ell=0}^{d}\sum_{k=0}^{p-1}\left|\tau(q_r f_k^{(\ell)\frac{1}{2}} a\gamma^n(f_k^{(\ell)\frac{1}{2}}) u^n q_r)\right| + (2(d+1)p+1)\tau(q_r)\epsilon' \\ &\overset{(1.2)}{\leq} ((d+1)p + 2(d+1)p + 1)\tau(q_r)\epsilon' \\ &= (6(d+1)n+1)\tau(q_r)\epsilon' = \epsilon. \end{aligned}$$

Hence, $\tau(q_r a u^n q_r) = 0$ for any $r, n \in \mathbb{N}$, which yields that $\tau(q_r x q_r) = \tau(q_r P(x) q_r)$ for any $r \in \mathbb{N}$ and any $x \in A \rtimes_\gamma \mathbb{Z}$. Therefore, the canonical restriction induces an injection from the space of densely defined lower semicontinuous traces on $A \rtimes_\gamma \mathbb{Z}$ into the space of γ-invariant densely defined lower semicontinuous traces on A. Since this restriction is also surjective, the conclusion follows. □

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

3. Bundles on Kirchberg algebras

In this section, we will prove theorem A. The construction of the required flow follows the strategy in [26, §5], together with some modifications that allow us to remove the assumption of no torsion in the K_1-group. We briefly describe the construction for the convenience of the reader, but we will often refer to [26, §5].

3.1 Realizing Kirchberg algebras as corners of crossed products by $\mathbb{Z}$

Throughout this section, let A be a unital UCT Kirchberg algebra and (S, π) be a proper simplex bundle such that $\pi^{-1}(0) = \emptyset$. The main step in proving theorem A is realizing A as a full corner of a crossed product $B \rtimes_\gamma \mathbb{Z}$, with B being a stable, possibly nonsimple, A$\mathbb{T}$-algebra with real rank zero. Comparing with the construction in [26, theorem 5.1], it is precisely constructing B to be an A$\mathbb{T}$-algebra rather than an AF-algebra that allows for torsion in $K_1(A)$. We collect the differences to the proof of [26, theorem 5.1] in remark 3.10.

We first need to construct suitable K-theory groups, together with group isomorphisms between them. We will then obtain B and γ by using Elliott's classification of morphisms between stable, possibly nonsimple A$\mathbb{T}$-algebras with real rank zero from [21]. We proceed to build the K-theory of the desired A$\mathbb{T}$-algebra B. Since A will be identified with a full corner of a crossed product of B with $\mathbb{Z}$, the construction of $K_0(B)$ needs to recover $K_0(A)$ via the Pimsner–Voiculescu exact sequence. To recover the simplex bundle (S, π), we will ensure that $K_0(B)$ contains a dense subset of $\mathcal{A}(S, \pi)$. We will now sketch the relevant parts of the construction in [26, theorem 5.1]. If $S = \emptyset$, then we can take θ to be the trivial flow on A, so assume that $S \neq \emptyset$ and $\pi^{-1}(0) = \emptyset$.

LEMMA 3.1. (cf.[26, theorem 5.1]). *Let (S, π) be a nonempty proper simplex bundle such that $\pi^{-1}(0) = \emptyset$. Then there exists a countable torsion free ordered group (G, G_+) together with an automorphism α of (G, G_+) such that $G = \mathbb{Q}G_0$, where G_0 is a countable subgroup of $\mathcal{A}(S, \pi)$, $\alpha(g)(x) = e^{-\pi(x)}g(x)$ for any $g \in G$ and $x \in S$, and the following conditions hold:*

(i) For any $n \in \mathbb{N}$, $\epsilon > 0$, and $f \in \mathcal{A}(S, \pi)$ such that $\operatorname{supp}(f) \subseteq \pi^{-1}([-n, n])$, there exists $g \in G_0$ with $\operatorname{supp}(g) \subseteq \pi^{-1}([-n, n])$ and

$$\sup_{x \in S} |f(x) - g(x)| < \epsilon;$$

(ii) (G, G_+) is a dimension group satisfying the strong Riesz interpolation property;
(iii) The maps α and $\mathrm{id} - \alpha$ are automorphisms of G, and $\alpha(G_+) = G_+$;
(iiv) If I is an order ideal in G such that $\alpha(I) = I$, then $I = \{0\}$ or $I = G$;
(v) If $\psi \colon G \to \mathbb{R}$ is a positive group homomorphism such that $\psi(1) = 1$, where 1 denotes the constant function 1 on S and $\beta \in \mathbb{R}$ satisfies $\psi \circ \alpha = e^{-\beta}\psi$, then there exists a unique $\omega \in \pi^{-1}(\beta)$ such that $\psi(g) = g(\omega)$ for all $g \in G$.

Proof. The construction of the pair (G, G_+) is the one provided in [26, theorem 5.1]. Since the topology on S is second countable, we can choose a countable subgroup

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

G_0 of $\mathcal{A}(S,\pi)$ satisfying (i) in the statement of the lemma. Moreover, we can ensure that

- The function
$$x \mapsto e^{n\pi(x)}(1-e^{-\pi(x)})^m f(x)$$
is in G_0 for any $f \in G_0$ and any $n, m \in \mathbb{Z}$;
- The functions
$$x \mapsto (\chi_{(-\infty,0]} \circ \pi)(x) e^{n\pi(x)}(1-e^{-\pi(x)})^m$$
and
$$x \mapsto (\chi_{[0,\infty)} \circ \pi)(x) e^{n\pi(x)}(1-e^{-\pi(x)})^m$$
are in G_0 for any $n, m \in \mathbb{Z}$. The functions $\chi_{(-\infty,0]} \circ \pi$ and $\chi_{[0,\infty)} \circ \pi$ are continuous on S because $\pi^{-1}(0) = \emptyset$.

Let $G = \mathbb{Q}G_0$ and $G_+ = \{f \in G : f(x) > 0, x \in S\} \cup \{0\}$.

That (G, G_+) is a dimension group satisfying the strong Riesz interpolation property follows from [26, lemma 5.3]. Since the function
$$x \mapsto e^{n\pi(x)}(1-e^{-\pi(x)})^m f(x)$$
is in G_0 for any $f \in G_0$ and any $n, m \in \mathbb{Z}$, we get that α is an automorphism of G and $\alpha(G_+) = G_+$. Furthermore, note that $\pi^{-1}(0) = \emptyset$, so $\mathrm{id} - \alpha$ is also an automorphism of G. This proves (iii). Condition (iv) follows from the proof of [26, lemma 5.5], while the existence of an element $\omega \in \pi^{-1}(\beta)$ as in (v) is shown in [26, lemma 5.6] (note that both the group G and the map ψ appearing in the proof of [26, lemma 5.6] coincide with the 'G' and 'ψ' in the statement of the lemma).

Suppose that there exists $\omega' \neq \omega$ in $\pi^{-1}(\beta)$ such that $g(\omega) = g(\omega')$ for all $g \in G$. Then, by (i) above, $h(\omega) = h(\omega')$ for any $h \in \mathcal{A}(S,\pi)$ with compact support. As $\omega, \omega' \in \pi^{-1}(\beta)$, there exists $f \in \mathrm{Aff}(\pi^{-1}(\beta))$ such that $f(\omega) \neq f(\omega')$. Therefore, there exist $n \in \mathbb{N}$ and $h \in \mathcal{A}(S,\pi)$ such that the support of h is contained in $\pi^{-1}([-n,n])$ and $h(\omega) \neq h(\omega')$, which is a contradiction. Hence, ω is unique. □

The pair (G, G_+) only contains information about the simplex S. To complete the construction, the key ingredient is the following proposition which follows as a direct application of [47, proposition 3.5].

PROPOSITION 3.2. *There exist countable, abelian, torsion free groups $H_0 \neq \{0\}$ and H_1, automorphisms κ_0 and κ_1 of H_0 and H_1 respectively and homomorphisms $q_i : H_i \to K_i(A)$ for $i = 0, 1$ such that*
$$0 \longrightarrow H_i \xrightarrow{\mathrm{id}-\kappa_i} H_i \xrightarrow{q_i} K_i(A) \longrightarrow 0$$
is exact for $i = 0, 1$.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

Proof. We apply [47, proposition 3.5] to the pairs $(K_0(A), 0)$ and $(K_1(A), 0)$. Moreover, if $H_0 = \{0\}$, which happens only when $K_0(A) = 0$, we take $H_0 = \mathbb{Q}$ and $\kappa_0(x) = 2x$. □

LEMMA 3.3. *Set* $\Gamma = H_0 \oplus G$ *and if* $p : \Gamma \to G$ *is the canonical projection, let*

$$\Gamma_+ = \{x \in \Gamma : p(x) \in G_+ \setminus \{0\}\} \cup \{0\}.$$

Then the following conditions hold:

(i) The pair (Γ, Γ_+) *is a dimension group;*
(ii) For any $g \in \Gamma_+$*, there exist* $h_1, h_2 \in \Gamma_+$ *such that* $g = 2h_1 + 3h_2$ *i.e.* Γ_+ *is weakly divisible;*
(iii) If I *is a nonzero order ideal of* Γ*, then* $I = H_0 \oplus p(I)$*;*
(iv) The only order ideals I *in* Γ *such that* $(\kappa_0 \oplus \alpha)(I) = I$ *are* $I = \{0\}$ *and* $I = \Gamma$*;*
(v) The ordered group (Γ, Γ_+) *has no nonzero liminary subquotients i.e. no subquotient has all its prime quotients isomorphic to* $\mathbb{Z}$*.*

Proof. We first check (i). As (G, G_+) is a dimension group satisfying the strong Riesz interpolation property (lemma 3.1(ii)) and H_0 is nonzero and torsion free, (Γ, Γ_+) is a dimension group by [19, lemma 3.2].

To check (ii), it suffices to find $h \in \Gamma_+$ such that $2h \leq g \leq 3h$. We then let $h_1 = 3h - g$ and $h_2 = g - 2h$. Let $g \in \Gamma_+$ be nonzero. Since $G = \mathbb{Q}G$, there exists $h' \in G_+$ nonzero such that $\frac{7}{3}h' \leq p(g) \leq \frac{8}{3}h'$. Then, $p(g - 2(0, h')) = p(g) - 2h' > 0$ and $p(3(0, h') - g) = 3h' - p(g) > 0$, so we can take $h = (0, h')$ to conclude that $2h \leq g \leq 3h$ in Γ.

We now check (iii). Let I be a nonzero order ideal of Γ. Since $I = (I \cap \Gamma_+) - (I \cap \Gamma_+)$, there exist $h' \in H_0$ and $g \in G_+ \setminus \{0\}$ such that $(h', g) \in I$. Then, for any $h \in H_0$, one has that

$$0 < (h + h', g) < 2(h', g)$$

in Γ. Since I is an order ideal, it follows that $(h + h', g) \in I$. Combining this with the fact that $(h', g) \in I$, yields that $(h, 0) \in I$. Hence $H_0 \oplus 0 \subseteq I$. If $g \in p(I)$, there exists $h' \in H_0$ such that $(h', g) \in I$. If $h \in H_0$, then $(h + h', 0) \in I$, so $(h, g) \in I$. Hence, we get that $I = H_0 \oplus p(I)$.

The proof of (iv) is contained in [26, lemma 5.5], but we will include the details for the convenience of the reader. Let I be a nonzero order ideal of Γ such that $(\kappa_0 \oplus \alpha)(I) = I$. By condition (iii), it follows that $I = H_0 \oplus p(I)$. Moreover, $p(I)$ is an order ideal in G such that $\alpha(p(I)) = p(I)$. Condition (iv) of lemma 3.1 yields that $G = p(I)$. Thus, $I = \Gamma$.

We prove (v) by contradiction. Let $J \subseteq I$ be order ideals of Γ such that I/J is nonzero and liminary in the category of ordered groups i.e. any prime quotient of I/J is isomorphic to $\mathbb{Z}$. Then, by (iii) above, $I = H_0 \oplus p(I)$. Suppose that there exists a surjection $q : (H_0 \oplus p(I))/J \to \mathbb{Z}$ and let (h, i) be positive in $H_0 \oplus p(I)$ such that $q(h, i) = 1$. As $p(I) \subseteq G$ is an order ideal, it follows that $\frac{1}{2}i \in p(I)$. If

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

$(h, \frac{1}{2}i) \in J$, then $(0, \frac{1}{2}i) \in J$ by (iii). Therefore, $(h, i) \in J$, which is a contradiction, so $(h, \frac{1}{2}i) \notin J$. Hence, one gets that

$$0 \leq q(h, \frac{1}{2}i) \leq 1.$$

If $q(h, \frac{1}{2}i) = 0$, then $q(4h, 2i) = 0$. But $(h, i) \leq (4h, 2i)$, so $1 = q(h, i) \leq 0$, which is a contradiction. If $q(h, \frac{1}{2}i) = 1$, then $q(0, \frac{1}{2}i) = 0$, which gives that $q(0, 2i) = 0$. But $(h, i) \leq (0, 2i)$, so $1 = q(h, i) \leq 0$, which is a contradiction. Thus, the subquotient I/J is not liminary. □

LEMMA 3.4. *Choose $w \in H_0$ such that $q_0(w) = [1_A]_0$. Let $v = (w, 1) \in \Gamma_+$, where 1 stands for the constant function 1 on S. Let $\varphi : \Gamma \to \mathbb{R}$ be a positive homomorphism and $\beta \in \mathbb{R}$ such that $\varphi(v) = 1$ and $\varphi \circ (\kappa_0 \oplus \alpha) = e^{-\beta}\varphi$. Then, there is a unique $\omega \in \pi^{-1}(\beta)$ such that $\varphi(h, g) = g(\omega)$ for all $(h, g) \in \Gamma$. In particular, there are no positive homomorphisms on Γ sending v to 1 that are invariant under the automorphism $\kappa_0 \oplus \alpha$.*

Proof. This follows as in [26, lemma 5.6]. Let $h \in H_0$ and $n \geq 1$. Then $n(h, 0) + v \in \Gamma_+$ and $-n(h, 0) + v \in \Gamma_+$, which yields that $n\varphi(h, 0) + 1 \geq 0$ and $-n\varphi(h, 0) + 1 \geq 0$ for all $n \geq 1$. Hence, $\varphi(h, 0) = 0$, which yields a positive homomorphism $\psi : G \to \mathbb{R}$ such that $\psi \circ p = \varphi$. Since $\varphi(v) = 1$ and $p(v) = 1$, it follows that $\psi(1) = 1$. Moreover, $\varphi \circ (\kappa_0 \oplus \alpha) = e^{-\beta}\varphi$ and $p \circ (\kappa_0 \oplus \alpha)(g) = \alpha(g)$ for any $g \in G$, so $\psi \circ \alpha = e^{-\beta}\psi$. Thus, (v) of lemma 3.1 shows that there is a unique $\omega \in \pi^{-1}(\beta)$ such that $\psi(g) = g(\omega)$ for all $g \in G$. Hence, $\varphi(h, g) = g(\omega)$ for all $(h, g) \in \Gamma$. The last statement follows by taking $\beta = 0$, as $\pi^{-1}(0) = \emptyset$. □

We consider the triple $(\Gamma, \Gamma_+, H_1 \oplus G)$ and we claim that it realizes the K-theory of some stable A$\mathbb{T}$-algebra with real rank zero. This will follow from Elliott's classification of A$\mathbb{T}$-algebras with real rank zero from [21]. We first need to impose a suitable order on the graded group $\Gamma \oplus (H_1 \oplus G)$. We will do so using [23, theorem 4.28].

LEMMA 3.5. *Consider the graded group $\Lambda_* = \Gamma \oplus (H_1 \oplus G)$ and define*

$$(\Lambda_*)_+ = \{((h_0, g_0), (h_1, g_1)) \in \Lambda_* : (h_0, g_0) \in \Gamma_+ \setminus \{0\}, g_1 \in I(g_0)\} \cup \{0\},$$

where $I(g_0)$ is the order ideal of G generated by g_0. Then the following conditions hold:

- *(i)* $(\Lambda_*, (\Lambda_*)_+)$ *is a countable graded ordered group;*
- *(ii)* $(\Lambda_*, (\Lambda_*)_+)$ *is unperforated;*
- *(iii)* $(\Lambda_*, (\Lambda_*)_+)$ *has the Riesz decomposition property and the Riesz interpolation property;*
- *(iv)* $(\Lambda_*, (\Lambda_*)_+)$ *is the inductive limit of a sequence $\Lambda_{*1} \to \Lambda_{*2} \to \ldots$ of graded ordered groups such that each $\Lambda_{*i} = \Lambda_{0i} \oplus \Lambda_{1i}$ is the direct sum of finitely many basic building blocks of the form $\mathbb{Z} \oplus \mathbb{Z}$ with the order determined by strict positivity in the first component.*

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

Proof. Conditions (i) and (iii) follow as a direct application of [23, theorem 4.28]. It is worth noting that in [23], the term ideal stands for what is called order ideal in this paper (see [23, remark 4.27]).

We first need to define a map from order ideals of Γ to subgroups of $H_1 \oplus G$. If I is a nonzero order ideal of Γ, then $I = H_0 \oplus p(I)$ by (iii) of lemma 3.3. We then consider the map from order ideals of Γ to subgroups of $H_1 \oplus G$ given by $H_0 \oplus p(I) \mapsto H_1 \oplus p(I)$ which maps 0 to 0. This map also preserves inclusions, upward directed unions, and maps Γ into $H_1 \oplus G$. Condition (i) now follows from [23, theorem 4.28].

We now prove (ii). Let $((h_0, g_0), (h_1, g_1)) \in \Lambda_*$ and $n \in \mathbb{N}$ such that $n((h_0, g_0), (h_1, g_1)) \in (\Lambda_*)_+$. Recall from proposition 3.2 that H_0 and H_1 are torsion free. Then Λ_* is torsion free, so we can assume that $(h_0, g_0) \in \Gamma \setminus \{0\}$. By (i) of lemma 3.3, Γ is a dimension group and hence unperforated, so $(h_0, g_0) \in \Gamma_+$. Moreover, since $n((h_0, g_0), (h_1, g_1)) \in (\Lambda_*)_+$, we have that ng_1 is in the order ideal of G generated by ng_0, say I_0. As g_0 is positive, it follows that $g_0 \in I_0$, so I_0 is generated by g_0. Since $ng_1 \in I_0$, there exists $k \in \mathbb{N}$ such that

$$-kng_0 \leq ng_1 \leq kng_0$$

in I_0. Recall from lemma 3.1 that $G = \mathbb{Q}G$. In particular, G is divisible, so it follows that $-kg_0 \leq g_1 \leq kg_0$, which implies that $0 \leq g_1 + kg_0 \leq 2kg_0$. This yields that $g_1 + kg_0 \in I_0$ and so $g_1 \in I_0$. Thus $((h_0, g_0), (h_1, g_1)) \in (\Lambda_*)_+$, which shows (ii).

By (i) of lemma 3.3, (Γ, Γ_+) is a dimension group, so it is weakly unperforated and satisfies the Riesz interpolation property, which coincides with the Riesz decomposition property (see [30, proposition 2.1]). The correspondence $H_0 \oplus p(I) \mapsto H_1 \oplus p(I)$ preserves finite sums and intersections. Moreover, no subquotient of Γ is liminary by (v) of lemma 3.3, so Λ_* satisfies the Riesz decomposition property by [23, theorem 4.28], thus proving (iii).

Condition (iv) follows from [20, theorem 5.2] and the fact that $H_1 \oplus G$ is torsion free. □

To apply classification results, we will need to construct an A$\mathbb{T}$-algebra which is $\mathcal{Z}$-stable. Building on Winter's techniques from [60], in [46], Robert and Tikuisis show that under some extra assumptions, finite nuclear dimension implies $\mathcal{Z}$-stability even in the nonsimple setting. We will use this result to prove that the A$\mathbb{T}$-algebra we obtain is $\mathcal{Z}$-stable.

Lemma 3.6. *There exists a separable stable A$\mathbb{T}$-algebra B with real rank zero such that the following conditions hold:*

(i) $(K_0(B), K_0(B)_+, K_1(B)) \cong (\Gamma, \Gamma_+, H_1 \oplus G)$;
(ii) $D_*(B) \cong K_*(B)_+ \cong (\Lambda_*)_+$;
(iii) B *is* $\mathcal{Z}$*-stable;*

Proof. Consider the graded group $\Lambda_* = \Gamma \oplus (H_1 \oplus G)$ as in lemma 3.5. Combining (iii) and (iv) of lemma 3.5 with [21, theorem 8.1], it follows that there exists a separable stable A$\mathbb{T}$-algebra B with real rank zero such that (i) holds and $K_*(B)_+ \cong$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

$(\Lambda_*)_+$. Since B is stable, we can identify $K_*(B)_+$ with the graded dimension range $D_*(B)$ (see §2.4), so (ii) is also satisfied.

We will prove that B is $\mathcal{Z}$-stable. Since B is an AT-algebra, it follows that B has finite decomposition rank (see [38, example 4.2]). We are going to use [46, corollary 7.11]. Since B has finite decomposition rank and real rank zero, no quotient of B has a simple purely infinite ideal and the space of primitive ideals of B has a basis of compact open sets (see the remark (a) after [46, corollary 7.11]). Moreover, the Murray-von Neumann semigroup of B is weakly divisible by (ii) of lemma 3.3, which yields that B has no nonzero elementary ideal quotients (see [52, theorem 9.1] together with the implication (1) $\implies$ (3) of [52, theorem B]). Therefore, B is $\mathcal{Z}$-stable by [46, corollary 7.11]. □

In the next lemma, we will realize the automorphisms $\kappa_0 \oplus \alpha$ and $\kappa_1 \oplus \alpha$ as the K-theory of an automorphism γ of B. Moreover, we choose γ such that the crossed product $B \rtimes_\gamma \mathbb{Z}$ is simple using a variation of a result by Kishimoto from [39]. Recall that an automorphism σ on a C*-algebra E is called *properly outer* if for every nonzero σ-invariant closed two-sided ideal I of E and for every unitary u in $\mathcal{M}(I)$ one has $\|\sigma|_I - \mathrm{Ad}(u)\| = 2$.

LEMMA 3.7. *Let B be the AT-algebra obtained in lemma 3.6. Then there exists an automorphism γ of B such that the following conditions hold:*

(i) $K_0(\gamma) = \kappa_0 \oplus \alpha$ *and* $K_1(\gamma) = \kappa_1 \oplus \alpha$*;*
(ii) $B \rtimes_\gamma \mathbb{Z}$ *is simple;*
(iii) $B \rtimes_\gamma \mathbb{Z}$ *is $\mathcal{Z}$-stable;*
(iv) The canonical restriction induces a bijection between the cone of densely defined lower semicontinuous traces on $B \rtimes_\gamma \mathbb{Z}$ and the cone of γ-invariant densely defined lower semicontinuous traces on B.

Proof. Consider the automorphism of $\Gamma \oplus (H_1 \oplus G)$ given by $(\kappa_0 \oplus \alpha) \oplus (\kappa_1 \oplus \alpha)$. Since $\alpha(G_+) = G_+$ by (iii) of lemma 3.1, $(\kappa_0 \oplus \alpha) \oplus (\kappa_1 \oplus \alpha)$ preserves $(\Lambda_*)_+$. Therefore, we can apply Elliott's classification of automorphisms of AT-algebras with real rank zero. Thus, by theorem 2.13, there exists $\gamma' \in \mathrm{Aut}(B)$ such that $K_0(\gamma') = \kappa_0 \oplus \alpha$ and $K_1(\gamma') = \kappa_1 \oplus \alpha$.

We claim that we can replace γ' by another automorphism γ which induces the same maps in K-theory and the crossed product has finite nuclear dimension. Building on the work in [50], we can take an automorphism γ of B with finite Rokhlin dimension such that $K_i(\gamma) = K_i(\gamma')$ for $i = 0, 1$ ([35, lemma 4.7]). Since B has finite nuclear dimension and γ has finite Rokhlin dimension, it follows that the crossed product $B \rtimes_\gamma \mathbb{Z}$ has finite nuclear dimension ([50, theorem 6.2] or [33, theorem 3.1]).

We now show that $B \rtimes_\gamma \mathbb{Z}$ is simple. Since α^k is non-trivial for all $k \neq 0$, then $K_0(\gamma)^k$ is non-trivial, which implies that no non-trivial power of γ is inner. Moreover, since B is stable, the fact that $K_0(\gamma)^k$ is non-trivial, implies that there exists a projection $p_k \in B$ such that $\gamma^k(p_k)$ is not equivalent to p_k. Then, if u is a unitary in $\mathcal{M}(B)$, $\gamma^k(p_k)$ is not equivalent to up_ku^*, so

$$\|\gamma^k(p_k) - up_ku^*\| = 1.$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

By (iv) of lemma 3.3, the only order ideals I in Γ such that $(\kappa_0 \oplus \alpha)(I) = I$ are $I = \{0\}$ and $I = \Gamma$. Thus, γ^k is properly outer by the implication (iii) $\Longrightarrow$ (ii) in [42, theorem 6.6]. Moreover, by (iv) of lemma 3.3, $K_0(B)$ has no non-trivial $K_0(\gamma)$-invariant ideals. Since B is an A$\mathbb{T}$-algebra of real rank zero, this implies that B has no non-trivial γ-invariant ideals ([48, proposition 1.5.3]), and hence the crossed product $B \rtimes_\gamma \mathbb{Z}$ is simple by [42, theorem 7.2]. Furthermore, $B \rtimes_\gamma \mathbb{Z}$ has finite nuclear dimension, so it is $\mathcal{Z}$-stable by [56, corollary 8.7].

Note that B has real rank zero, so it has an approximate unit of projections. Moreover, γ has finite Rokhlin dimension. Hence, (iv) follows from lemma 2.17. □

We now have all the necessary ingredients to show that A can be realized as a corner of a crossed product by the integers.

LEMMA 3.8. *Let B the stable A$\mathbb{T}$-algebra with real rank zero from lemma 3.6, and let γ be the automorphism of B from lemma 3.7. Then, there exists a projection $e \in B$ such that $e\,(B \rtimes_\gamma \mathbb{Z})\,e \cong A$.*

Proof. Recall that $v = (w, 1) \in \Gamma_+$, where $w \in H_0$ is such that $q_0(w) = [1_A]_0$. Let e be a projection in B such that $[e]_0 = v$ in $K_0(B)$. We will prove that $A \cong e\,(B \rtimes_\gamma \mathbb{Z})\,e$ using the Kirchberg–Phillips classification theorem ([37, 44]). First, we claim that $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is a simple, separable, unital, nuclear, purely infinite C*-algebra satisfying the UCT.

Note that $B \rtimes_\gamma \mathbb{Z}$ is simple by (ii) of lemma 3.7, so e is a full projection. Moreover, B is nuclear, satisfies the UCT, and $B \rtimes_\gamma \mathbb{Z}$ is $\mathcal{Z}$-stable by (iii) of lemma 3.7. Therefore, by proposition 2.11, $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is a simple, separable, unital, nuclear, $\mathcal{Z}$-stable C*-algebra which satisfies the UCT. By [49, corollary 5.1], to show that $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is purely infinite, it now suffices to prove that $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ has no quasitraces. Since $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ is unital and nuclear, any quasitrace is a trace by [32]. Therefore, it suffices to show that $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ has no tracial states. If τ is a tracial state on $e\,(B \rtimes_\gamma \mathbb{Z})\,e$, then it extends to a lower semicontinuous (possibly unbounded) trace τ' on $B \rtimes_\gamma \mathbb{Z}$ by [61, corollary 5.2]. By (iv) of lemma 3.7, the restriction of τ' to B induces a positive group homomorphism $\tau'_* : K_0(B) \to \mathbb{R}$ such that $\tau'_*(v) = 1$ and $\tau'_* \circ K_0(\gamma) = \tau'_*$. Recall from (i) of lemma 3.6 and (i) of lemma 3.7 that $K_0(B) = \Gamma$ and $K_0(\gamma) = \kappa_0 \oplus \alpha$. Thus, we obtain a contradiction with lemma 3.4. Hence, $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ has no tracial states. Moreover, it is nuclear and $\mathcal{Z}$-stable, so it is purely infinite by [49, corollary 5.1].

Therefore, by the Kirchberg–Phillips' classification theorem, it suffices to check that $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ and A have the same Elliott invariant. First note that $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ and $B \rtimes_\gamma \mathbb{Z}$ have the same K-groups. Applying the Pimsner–Voiculescu exact sequence ([45, theorem 2.4]) to B and the automorphism γ, we get the six-term exact sequence

$$\begin{array}{ccccc} H_0 \oplus G & \xrightarrow{\mathrm{id}-K_0(\gamma)} & H_0 \oplus G & \longrightarrow & K_0(B \rtimes_\gamma \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(B \rtimes_\gamma \mathbb{Z}) & \longleftarrow & H_1 \oplus G & \xleftarrow[\mathrm{id}-K_1(\gamma)]{} & H_1 \oplus G. \end{array}$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

By (i) of lemma 3.7, $\mathrm{id}_{H_1\oplus G} - K_1(\gamma) = (\mathrm{id}_{H_1} - \kappa_1) \oplus (\mathrm{id}_G - \alpha)$, which is injective by proposition 3.2 and since $\mathrm{id}_G - \alpha$ is an automorphism of G ((iii) of lemma 3.1). Then, the downward map $K_0(B \rtimes_\gamma \mathbb{Z}) \to H_1 \oplus G$ is zero. This further implies that the map $H_0 \oplus G \to K_0(B \rtimes_\gamma \mathbb{Z})$ is surjective, which yields that

$$K_0(B \rtimes_\gamma \mathbb{Z}) \cong (H_0 \oplus G)/(\mathrm{id}_{H_0\oplus G} - K_0(\gamma))(H_0 \oplus G).$$

By lemma (i) of 2.7, $\mathrm{id}_{H_0\oplus G} - K_0(\gamma) = (\mathrm{id}_{H_0} - \kappa_0) \oplus (\mathrm{id}_G - \alpha)$. By (iii) of lemma 3.1, $\mathrm{id}_G - \alpha$ is an automorphism of G, so

$$K_0(B \rtimes_\gamma \mathbb{Z}) \cong H_0/(\mathrm{id}_{H_0} - \kappa_0)(H_0) \cong K_0(A),$$

where the last isomorphism is given by proposition 3.2. Note that $[e]_0 = (w, 1)$ in $K_0(B)$, where w is mapped to $[1_A]_0$ by the isomorphism $H_0/(\mathrm{id}_{H_0} - \kappa_0)(H_0) \cong K_0(A)$. Therefore, the resulting isomorphism $K_0(e\,(B \rtimes_\gamma \mathbb{Z})\,e) \to K_0(A)$ sends $[e]_0$ to $[1_A]_0$.

Further examining the Pimsner–Voiculescu exact sequence and using that $\mathrm{id}_{H_0\oplus G} - K_0(\gamma)$ is injective, it follows that the map $K_1(B \rtimes_\gamma \mathbb{Z}) \to H_0 \oplus G$ is zero. Therefore, we obtain the exact sequence

$$0 \longrightarrow H_1 \oplus G \xrightarrow{\mathrm{id}-K_1(\gamma)} H_1 \oplus G \longrightarrow K_1(B \rtimes_\gamma \mathbb{Z}) \longrightarrow 0,$$

which yields that $K_1(B \rtimes_\gamma \mathbb{Z}) \cong (H_1 \oplus G)/(\mathrm{id}_{H_1\oplus G} - (\kappa_1 \oplus \alpha))(H_1 \oplus G) \cong H_1/(\mathrm{id}_{H_1\oplus G} - \kappa_1)(H_1)$, where the last identification follows as $\mathrm{id}_G - \alpha$ is an automorphism on G ((iii) of lemma 3.1). Hence $K_1(B \rtimes_\gamma \mathbb{Z}) \cong K_1(A)$ by proposition 3.2. Thus, $e\,(B \rtimes_\gamma \mathbb{Z})\,e \cong A$ by the Kirchberg–Phillips' classification theorem (see for example [48, theorem 8.4.1(iv)]). □

3.2 KMS bundles on Kirchberg algebras

In this subsection, we will finish the proof of theorem A. Recall from lemmas 3.6 and 2.7 that B is a stable $A\mathbb{T}$-algebra with real rank zero and γ is an automorphism on B with specified behaviour on K-theory such that $B \rtimes_\gamma \mathbb{Z}$ is $\mathcal{Z}$-stable.

Consider the dual action $\hat{\gamma}$ on $B \rtimes_\gamma \mathbb{Z}$ as a 2π-periodic flow. Recall that $\hat{\gamma}_t(f)(x) = e^{-ixt} f(x)$ for any $t \in \mathbb{R}$, $f \in C_c(\mathbb{Z}, B)$ and $x \in \mathbb{Z}$, so that $\hat{\gamma}$ acts trivially on B. Since $e \in B$, $\hat{\gamma}$ restricts to an action on $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ which we denote by θ. First, we need a result which will relate KMS states on $e\,(B \rtimes_\gamma \mathbb{Z})\,e$ to positive homomorphisms on $K_0(B)$. The last statement of the following proposition is not explicitly contained in [27, lemma 4.1]. However, it is implied by the remark leading to [27, corollary 4.2].

Proposition 3.9. ([27, lemma 4.1]). *Let D be a C*-algebra. Let ρ be an automorphism of D and let $q \in D$ a projection in D which is full in $D \rtimes_\rho \mathbb{Z}$. Let $\hat{\rho}$ denote the restriction of the dual action on $D \rtimes_\rho \mathbb{Z}$ to $q(D \rtimes_\rho \mathbb{Z})q$ and consider it as a 2π-periodic flow. If $P : D \rtimes_\rho \mathbb{Z} \to D$ is the canonical conditional expectation, then for each $\beta \in \mathbb{R}$, the map $\tau \mapsto \tau \circ P|_{q(D\rtimes_\rho\mathbb{Z})q}$ is an affine homeomorphism from the*

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

densely defined lower semicontinuous traces on D satisfying

$$\tau \circ \rho = e^{-\beta}\tau \quad \textit{and} \quad \tau(q) = 1,$$

onto the set of β-KMS states for the dual action $\hat{\rho}$ on $q(D \rtimes_\rho \mathbb{Z})q$. Moreover, the inverse is given by the map $\omega \mapsto \hat{\omega}|_D$, where $\hat{\omega}$ is a β-KMS weight for the dual action on $D \rtimes_\rho \mathbb{Z}$, which extends ω.

Proof. We will only comment on the proof of the last statement. Let ω be a β-KMS state for the dual action $\hat{\rho}$ on $q(D \rtimes_\rho \mathbb{Z})q$. Then, by [40, remark 3.3], there exists a unique β-KMS weight $\hat{\omega}$ for the dual action on $D \rtimes_\rho \mathbb{Z}$ which extends ω. The result now follows by [54, lemma 3.1]. □

Proof of theorem A. The proof follows the strategy in [26, lemma 5.8]. Recall that if $S = \emptyset$, then we take θ to be the trivial flow on A. Therefore, we assume that S is nonempty and we claim that the KMS bundle (S^θ, π^θ) of θ is isomorphic to (S, π). This will finish the proof.

Let $(\omega, \beta) \in S^\theta$. By [40, remark 3.3], there exists a unique β-KMS weight $\hat{\omega}$ for the dual action $\hat{\gamma}$ on $B \rtimes_\gamma \mathbb{Z}$ which extends ω. Furthermore, $B \rtimes_\gamma \mathbb{Z}$ is simple by (ii) of lemma 3.7, so e is full in $B \rtimes_\gamma \mathbb{Z}$. Then, by proposition 3.9, $\hat{\omega}|_B$ is a densely defined lower semicontinuous trace on B such that $\hat{\omega}|_B \circ \gamma = e^{-\beta}\hat{\omega}|_B$ and $\hat{\omega}|_B(e) = 1$. Therefore, $(\hat{\omega}|_B)_*$ is a positive homomorphism on $K_0(B)$ such that

$$(\hat{\omega}|_B)_* \circ K_0(\gamma) = e^{-\beta}(\hat{\omega}|_B)_* \text{ and } (\hat{\omega}|_B)_*([e]_0) = 1.$$

Then, by lemma 3.4, there exists a unique $\omega' \in \pi^{-1}(\beta)$ such that

$$(\hat{\omega}|_B)_*(h, g) = g(\omega'), \tag{2.1}$$

for all $(h, g) \in \Gamma \cong K_0(B)$. Define $\varphi : S^\theta \to S$ by $\varphi(\omega, \beta) = \omega'$ and note that $\pi \circ \varphi = \pi^\theta$ as $\omega' \in \pi^{-1}(\beta)$. Moreover, the restriction $\varphi : (\pi^\theta)^{-1}(\beta) \to \pi^{-1}(\beta)$ is affine by construction for any $\beta \in \mathbb{R}$.

We claim that φ is surjective. Let $\mu \in S$ and define $\mathrm{ev}_\mu : K_0(B) \to \mathbb{R}$ by $\mathrm{ev}_\mu(h, g) = g(\mu)$. Recall that $K_0(\gamma) = \kappa_0 \oplus \alpha$ by (i) of lemma 3.7 and $\alpha(g)(x) = e^{-\pi(x)}g(x)$ for any $g \in G$. Then, ev_μ is a positive group homomorphism such that $\mathrm{ev}_\mu \circ K_0(\gamma) = e^{-\pi(\mu)}\mathrm{ev}_\mu$ and $\mathrm{ev}_\mu(v) = 1$ as $v = (w, 1)$. Then, by proposition 2.15, there exists a unique densely defined lower semicontinuous trace τ_μ on B such that $(\tau_\mu)_* = \mathrm{ev}_\mu$. By proposition 3.9, there exists a $\pi(\mu)$-KMS state ω for θ such that $(\hat{\omega}|_B)_* = \mathrm{ev}_\mu$. Then $\varphi(\omega, \pi(\mu)) = \mu$, so φ is surjective.

We further check that φ is injective. Let $(\omega_1, \beta_1), (\omega_2, \beta_2) \in S^\theta$ such that $\varphi(\omega_1, \beta_1) = \varphi(\omega_2, \beta_2)$. Then,

$$\beta_1 = \pi(\varphi(\omega_1, \beta_1)) = \pi(\varphi(\omega_2, \beta_2)) = \beta_2.$$

Moreover, by the definition of the map φ,

$$(\hat{\omega}_1|_B)_* = (\hat{\omega}_2|_B)_*.$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

By proposition 3.9, $\hat{\omega}_1|_B$ and $\hat{\omega}_2|_B$ are densely defined lower semicontinuous traces on B such that

$$(\hat{\omega}_1|_B)(e) = (\hat{\omega}_2|_B)(e) = 1.$$

By proposition 2.15, it follows that $\hat{\omega}_1|_B = \hat{\omega}_2|_B$. Then, proposition 3.9 yields that $\omega_1 = \omega_2$, which shows that φ is injective.

If we show that $\varphi^{-1} : S \to S^\theta$ is continuous, then φ is a homeomorphism by lemma 2.5. Recall from remark 2.4 that both S^θ and S are metrisable and let ω'_n be a sequence in S which converges to ω'. If $\varphi^{-1}(\omega'_n) = (\omega_n, \beta_n)$ and $\varphi^{-1}(\omega') = (\omega, \beta)$, then (2.1) yields that $(\hat{\omega}_n|_B)_*(h, g)$ converges to $(\hat{\omega}|_B)_*(h, g)$ for any $(h, g) \in \Gamma \cong K_0(B)$. Then, $\hat{\omega}_n|_B$ converges to $\hat{\omega}|_B$ by proposition 2.15, so ω_n converges to ω by proposition 3.9. Moreover, $\beta_n = \pi(\omega'_n)$, which converges to $\pi(\omega') = \beta$ by continuity of π. Thus, (ω_n, β_n) converges to (ω, β), so φ^{-1} is continuous. Together with lemma 3.8, this yields the conclusion of theorem A. □

REMARK 3.10. Note that in [26, §5], to realize the K-theory of a stable AF-algebra, the authors apply proposition 3.2 to the pair $(K_0(A), K_1(A))$. Applying proposition 3.2 to the pairs $(K_0(A), 0)$ and $(K_1(A), 0)$ instead of the pair $(K_0(A), K_1(A))$ is precisely what allows us to avoid assuming that $K_1(A)$ is torsion free. This idea has its roots in [47, theorem 3.6], where Rørdam showed that any pair of countable discrete abelian groups can be realized as the K-theory of an endomorphism crossed product of an $A\mathbb{T}$-algebra. Then, since we are working with an A$\mathbb{T}$-algebra, rather than an AF-algebra, different arguments are needed in lemmas 3.6 and 2.7 to show that B is $\mathcal{Z}$-stable and the crossed product is simple and $\mathcal{Z}$-stable. Moreover, the invariant for classifying possibly nonsimple A$\mathbb{T}$-algebras of real rank zero is more intricate than in the case of AF-algebras (see §2.4), so more detailed K-theory groups have to be constructed.

4. Bundles on tracial classifiable C*-algebras

In this section, we will prove theorem B. The construction of the required flow is in the spirit of [26, §3]. Adapting the construction in [26, §3] and using classification of unital *-homomorphisms between classifiable C*-algebras from [13], we will extend [26, theorem 3.14] to any simple, separable, unital, nuclear, stably finite, $\mathcal{Z}$-stable C*-algebra, with real rank zero, and which satisfies the UCT.

4.1 Realizing tracial C*-algebras as corners of crossed products by $\mathbb{Z}$

We will follow a similar strategy as in §3, where we proved the corresponding result for Kirchberg algebras. Throughout this section, A will be a simple, separable, unital, nuclear, stably finite, $\mathcal{Z}$-stable C*-algebra, with real rank zero, and which satisfies the UCT. We also let (S, π) be a compact simplex bundle such that $\pi^{-1}(0) \cong T(A)$. We will freely identify these simplices. The main step in proving theorem B is realizing A as a unital corner of a crossed product $(B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}$, with B being a simple, separable, unital, nuclear, stably finite, $\mathcal{Z}$-stable C*-algebra satisfying the UCT. We first need to build the K-theory of the desired C*-algebra

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

B. Let $\mathcal{A}_0(S,\pi)$ denote the set of elements $f \in \mathcal{A}(S,\pi)$ for which $f(T(A)) = 0$, and let

$$\mathcal{A}_0(S,\pi)_+ := \{f \in \mathcal{A}_0(S,\pi)\colon f(x) > 0 \text{ for any } x \in \pi^{-1}(t)\ \forall\ t \neq 0\}.$$

LEMMA 4.1. *Let (S,π) be a compact simplex bundle such that $\pi^{-1}(0) \cong T(A)$. Then there exists a countable subgroup G_0 of $\mathcal{A}_0(S,\pi)$ such that the following conditions hold:*

(i) For any $f \in \mathcal{A}_0(S,\pi)$ and any $\epsilon > 0$, there is an element $g \in G_0$ such that

$$\sup_{x\in S} |f(x) - g(x)| < \epsilon;$$

(ii) The map $\eta : g \mapsto e^{-\pi}g$ is an automorphism on G_0 and $\mathrm{id} - \eta$ is also an automorphism on G_0;
(iii) For any $g \in G_0$, there exist $g_1, g_2 \in G_0 \cap \mathcal{A}_0(S,\pi)_+$ such that $g = g_1 - g_2$.

Proof. Since S is compact and the topology on S is second countable, we can choose a countable subgroup G_0 of $\mathcal{A}_0(S,\pi)$ such that for all $\epsilon > 0$ and all $f \in \mathcal{A}_0(S,\pi)$, there is an element $g \in G_0$ such that

$$\sup_{x\in S} |f(x) - g(x)| < \epsilon.$$

Moreover, we can ensure that if $f \in G_0$, then

$$x \mapsto e^{n\pi(x)}(1 - e^{-\pi(x)})^m f(x)$$

is in G_0 for all $m, n \in \mathbb{Z}$. Then, a similar construction to the one in [27, §4.2 pp 109-110] allows to choose G_0 such that both η and $\mathrm{id} - \eta$ are surjective, hence giving (ii). To check (iii), let $g \in G_0$ and $g_1, g_2 \in \mathcal{A}_0(S,\pi)_+$ such that $g = g_1 - g_2$. Using compactness of S, we can then ensure that $g_1, g_2 \in G_0$. □

LEMMA 4.2. *There exist a countable ordered abelian group (G, G_+, v) with distinguished order unit v, a homomorphism $\hat{L} : G \to \mathcal{A}(S,\pi)$, and an automorphism α of (G, G_+) such that the following conditions hold:*

(i) (G, G_+, v) is simple and weakly unperforated;
(ii) $\hat{L}(G)$ is uniformly dense in $\mathcal{A}(S,\pi)$;
(iii) The space of states on G, denoted by S(G), is a metrisable Choquet simplex;
(iv) The homomorphism $\mathrm{id} - \alpha : G \to G$ is injective;
(v) There is an isomorphism $\Sigma : G/(\mathrm{id} - \alpha)(G) \to K_0(A)$;
(vi) If $\beta \in \mathbb{R}$ and φ is a state on (G, G_+, v) with the property that

$$\varphi \circ \alpha = e^{-\beta}\varphi,$$

then there is a unique $s \in \pi^{-1}(\beta)$ such that $\varphi = \mathrm{ev}_s \circ \hat{L}$.[3] In particular, the space of α-invariant states on G is affinely homeomorphic to $T(A)$.

[3]The map ev_s denotes evaluation at $s \in \pi^{-1}(\beta)$.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

Proof. Set

$$G = \left(\bigoplus_{\mathbb{Z}} K_0(A) \right) \oplus G_0.$$

We will define an order on G adapting the construction in [27, §4.2]. The map

$$r : \mathcal{A}(S, \pi) \to \mathrm{Aff}(T(A))$$

given by restriction is surjective ([27, lemma 4.4(1)]), so we can choose a positive linear map

$$L : \mathrm{Aff}(T(A)) \to \mathcal{A}(S, \pi) \text{ such that } r \circ L = \mathrm{id} \text{ and } L(1) = 1. \tag{3.1}$$

If $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ is the canonical pairing map of A, let us consider the homomorphism $\hat{L} : G \to \mathcal{A}(S, \pi)$ given by

$$\hat{L}(\xi, g)(x) = g(x) + \sum_{n \in \mathbb{Z}} L(\rho_A(\xi_n))(x) e^{n\pi(x)} \tag{3.2}$$

for all $\xi = (\xi_n)_{n \in \mathbb{Z}}$, $g \in G_0$, and $x \in S$. Note that the map $\hat{L}$ is well-defined since ξ has only finitely many nonzero entries.

Define

$$G_+ = \{(\xi, g) \in G : \hat{L}(\xi, g)(x) > 0,\ x \in S\} \cup \{0\} \tag{3.3}$$

and set $v = (1^{(0)}, 0)$, where $1^{(0)}$ is the sequence with $[1_A]_0$ in the zero entry and zero elsewhere.

To show (i), we will first show that the triple (G, G_+, v) defines an ordered abelian group. The construction of G_+ in (3.3) yields that $G_+ \cap (-G_+) = \{0\}$. We will check that $G = G_+ - G_+$. Let $(\xi, g) \in G$. As A is stably finite, $(K_0(A), K_0(A)_+)$ is an ordered abelian group ([4, proposition 6.3.3]), so there exist sequences $\xi^{(1)}, \xi^{(2)}$ such that $\xi_n^{(i)} \in K_0(A)_+$, for any $i = 1, 2$ and $n \in \mathbb{N}$ and $\xi = \xi^{(1)} - \xi^{(2)}$. Moreover, by (iii) of lemma 4.1, there exist $g_1, g_2 \in G_0 \cap \mathcal{A}_0(S, \pi)_+$ such that $g = g_1 - g_2$. Thus, $(\xi, g) = (\xi^{(1)}, g_1) - (\xi^{(2)}, g_2)$, which shows that $G = G_+ - G_+$.

To show that (G, G_+, v) is simple, we use the assumption that S is compact. Let $(\xi, g) \in G_+ \setminus \{0\}$. By definition, this yields that $\hat{L}(\xi, g)(x) > 0$ for any $x \in S$. Since S is compact, there exists $n \in \mathbb{N}$ such that $\hat{L}(\xi, g)(x) \geq \frac{1}{n}$ for any $x \in S$. This implies that (ξ, g) is an order unit, so (G, G_+, v) is a simple ordered abelian group.

To see that (G, G_+) is weakly unperforated, let $n \in \mathbb{N}$ and $(\xi, g) \in G$ such that $n(\xi, g) \in G_+ \setminus \{0\}$. Therefore, by linearity of L, it follows that $\hat{L}(\xi, g)(x) > 0$, $x \in S$. Hence, $(\xi, g) \in G_+ \setminus \{0\}$, so (G, G_+) is weakly unperforated.

We claim that (ii) follows by combining (i) of lemma 4.1 and the fact that A has real rank zero. Let $f \in \mathcal{A}(S, \pi)$ and $\epsilon > 0$. Note $f|_{T(A)} \in \mathrm{Aff}(T(A))$. Since A is unital, simple, exact, finite, $\mathcal{Z}$-stable, and has real rank zero, it follows that $\rho_A(K_0(A))$ is uniformly dense in $\mathrm{Aff}(T(A))$ ([49, theorem 7.2]). Therefore, there

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

exists $y \in K_0(A)$ such that

$$f(\tau) - \epsilon/2 < \rho_A(y)(\tau) < f(\tau) + \epsilon/2$$

for any $\tau \in T(A)$. As $T(A)$ is closed in S and S is compact, [9, lemma 2.3] shows that there exists $h \in \mathcal{A}(S, \pi)$ extending $\rho_A(y)$ such that

$$f(x) - \epsilon/2 < h(x) < f(x) + \epsilon/2 \tag{3.4}$$

for any $x \in S$. Then, $h - L(\rho_A(y)) \in \mathcal{A}_0(S, \pi)$, so by (i) of lemma 4.1, there exists $g \in G_0$ such that

$$|h(x) - L(\rho_A(y))(x) - g(x)| < \epsilon/2 \tag{3.5}$$

for any $x \in S$. Now consider the element $(y^{(0)}, g) \in G$, where $y^{(0)}$ is the sequence which is constant 0 apart from the zero entry which is equal to y. Then,

$$\begin{aligned} \sup_{x \in S} |\hat{L}(y^{(0)}, g)(x) - f(x)| &= \sup_{x \in S} |g(x) + L(\rho_A(y))(x) - f(x)| \\ &\overset{(3.5)}{\leq} \sup_{x \in S} |f(x) - h(x)| + \epsilon/2 \\ &\overset{(3.4)}{\leq} \epsilon. \end{aligned}$$

Thus, $\hat{L}(G)$ is uniformly dense in $\mathcal{A}(S, \pi)$.

We will now show (iii). This follows the strategy in (ii). Using [2, corollary II.3.11] and Kadison's duality ([36]), it suffices to check that G satisfies the strong Riesz interpolation property. Suppose that $(\xi_i, g_i) < (\eta_j, h_j)$ in G for any $i, j = 1, 2$. Therefore, $\hat{L}(\xi_i, g_i)(x) < \hat{L}(\eta_j, h_j)(x)$ for any $x \in S$. Moreover, $\hat{L}(\xi_i, g_i)$ and $\hat{L}(\eta_j, h_j)$ restrict to affine functions on $T(A)$.

Since A is unital, simple, exact, finite, $\mathcal{Z}$-stable, and has real rank zero, it follows that $\rho_A(K_0(A))$ is uniformly dense in $\mathrm{Aff}(T(A))$ ([49, theorem 7.2]). Thus, $\rho_A(K_0(A))$ satisfies the strong Riesz interpolation property with respect to the strict ordering in $\mathrm{Aff}(T(A))$ ([19, lemma 3.1]), so there exists $y \in K_0(A)$ such that

$$\hat{L}(\xi_i, g_i)(\tau) < \rho_A(y)(\tau) < \hat{L}(\eta_j, h_j)(\tau) \tag{3.6}$$

for any $\tau \in T(A)$ and any $i, j = 1, 2$. As $T(A)$ is closed in S and S is compact, [9, lemma 2.3] shows that there exists $f \in \mathcal{A}(S, \pi)$ extending $\rho_A(y)$ such that

$$\hat{L}(\xi_i, g_i)(x) < f(x) < \hat{L}(\eta_j, h_j)(x) \tag{3.7}$$

for any $x \in S$ and any $i, j = 1, 2$.

Since S is compact, we can let $\delta > 0$ be smaller than $f(x) - \hat{L}(\xi_i, g_i)(x)$ and $\hat{L}(\eta_j, h_j)(x) - f(x)$ for any $x \in S$ and any $i, j = 1, 2$. By (i) of lemma 4.1, there

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

exists $g \in G_0$ such that

$$|f(x) - L(\rho_A(y))(x) - g(x)| < \delta \tag{3.8}$$

for any $x \in S$. Now consider the element $(y^{(0)}, g) \in G$, where $y^{(0)}$ is the sequence which is constant 0 apart from the zero entry which is equal to y. We claim that

$$\hat{L}(\xi_i, g_i)(x) < \hat{L}(y^{(0)}, \quad g(x) < \hat{L}(\eta_j, h_j)(x)$$

for any $x \in S$. First note that for any $x \in S$ and $i = 1, 2$ we have that

$$\begin{aligned} \hat{L}(y^{(0)}, g)(x) - \hat{L}(\xi_i, g_i)(x) &= g(x) + L(\rho_A(y))(x) - \hat{L}(\xi_i, g_i)(x) \\ &\overset{(3.8)}{>} f(x) - \delta - \hat{L}(\xi_i, g_i)(x) \\ &> 0, \end{aligned}$$

by the choice of δ. A similar calculation shows that $\hat{L}(\eta_j, h_j)(x) > \hat{L}(y^{(0)}, g)(x)$ for any $x \in S$ and $j = 1, 2$. Hence, by the definition of the order on G, we obtain that

$$(\xi_i, g_i) < (y^{(0)}, g) < (\eta_j, h_j)$$

for any $i, j = 1, 2$, which shows that G satisfies the strong Riesz interpolation property. Thus, $S(G)$ is a Choquet simplex. Moreover, $S(G)$ is metrisable since G is countable.

We will now define the automorphism α of G. Let α be the automorphism of (G, G_+, v) given by

$$\alpha(\xi, f) = (\sigma(\xi), e^{-\pi} f)$$

where $\sigma(\xi) = (\xi_{n+1})_{n \in \mathbb{Z}}$ is the left shift on $\bigoplus_{\mathbb{Z}} K_0(A)$ and $e^{-\pi}$ denotes the function $x \mapsto e^{-\pi(x)}$. To check (iv), let $(\xi, g) \in G$ such that $(\mathrm{id} - \alpha)(\xi, g) = 0$. Then $\sigma(\xi) = \xi$ and $e^{-\pi} g = g$. Since σ is the shift on $\bigoplus_{\mathbb{Z}} K_0(A)$, it follows that $\xi = 0$. Moreover, g is supported away from $\pi^{-1}(0)$ by lemma 4.1, so $g = 0$. Thus, $\mathrm{id} - \alpha$ is injective.

We will now check (v). Consider the map $\Sigma_0 : G \to K_0(A)$ given by

$$\Sigma_0((\xi_n), f) = \sum_{n \in \mathbb{Z}} \xi_n,$$

for any $(\xi_n) \in \bigoplus_{\mathbb{Z}} K_0(A)$ and $f \in G_0$. We claim that the kernel of Σ_0 is $(\mathrm{id} - \alpha)(G)$. If $(\xi, g) \in G$, then $\Sigma_0((\mathrm{id} - \alpha)(\xi, g)) = 0$. Conversely, let $(\xi, g) \in G$ such that $\Sigma_0(\xi, g) = 0$ and choose $N \in \mathbb{N}$ such that $\xi_n = 0$ for all $|n| \geq N$. We follow the proof in [54, lemma 4.6]. Set $z_n = 0$ for $n \geq N$ and

$$z_n = \xi_n + z_{n+1}, \ n < N.$$

Then

$$z_{-N} = \xi_{-N} + \sum_{i=1}^{2N} \xi_{-N+i} = 0$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

since $\Sigma_0(\xi, g) = 0$. This yields that $z_n = 0$ for any $n \leq -N$. Moreover, if we set $z = (z_n)_{n\in\mathbb{Z}}$, we get that $(\mathrm{id} - \sigma)(z) = \xi$. By condition (ii) in lemma 4.1, it also follows that there exists $f \in G_0$ such that $g = (1 - e^{-\pi})f$, so $(\mathrm{id} - \alpha)(z, f) = (\xi, g)$. Thus, the kernel of $\mathbf{\Sigma}_0$ is $(\mathrm{id} - \alpha)(G)$, so $\mathbf{\Sigma}_0$ induces an isomorphism

$$\Sigma : G/(\mathrm{id} - \alpha)(G) \to K_0(A).$$

The proof of (vi) follows as in [26, lemma 3.8]. Let $\beta \in \mathbb{R}$ and φ be a state on (G, G_+, v) such that $\varphi \circ \alpha = e^{-\beta}\varphi$.

The map $\hat{L}$ can be extended uniquely to a $\mathbb{Q}$-linear map on $\mathbb{Q} \otimes_{\mathbb{Z}} G$. For ease of notation, we will identify $\hat{L}$ with its extension $\hat{L} : \mathbb{Q} \otimes_{\mathbb{Z}} G \to \mathcal{A}(S, \pi)$. Then, we can also extend φ uniquely to a $\mathbb{Q}$-linear state $\hat{\varphi}$ of $\mathbb{Q} \otimes_{\mathbb{Z}} G$. Suppose that $\hat{L}(\xi, g) = 0$ for some $(\xi, g) \in \mathbb{Q} \otimes_{\mathbb{Z}} G$. Then, $\frac{1}{n}v + (\xi, g), \frac{1}{n}v - (\xi, g) \in (\mathbb{Q} \otimes_{\mathbb{Z}} G)_+$, so

$$-\frac{1}{n} \leq \hat{\varphi}(\xi, g) \leq \frac{1}{n}$$

for any $n \in \mathbb{N}$. Thus, $\hat{\varphi}(\xi, g) = 0$, which implies that $\hat{\varphi}$ factors through $\hat{L}$. Indeed, there is a $\mathbb{Q}$-linear map

$$\psi : \hat{L}(\mathbb{Q} \otimes_{\mathbb{Z}} G) \to \mathbb{R}, \text{ such that } \psi \circ \hat{L} = \hat{\varphi}.$$

We claim that ψ is continuous. Let $f \in \hat{L}(\mathbb{Q} \otimes_{\mathbb{Z}} G)$ and suppose by compactness of S that $|f(x)| < \frac{n}{m}$ for any $x \in S$, for some $n, m \in \mathbb{N}$. Then, there exists $w \in \mathbb{Q} \otimes_{\mathbb{Z}} G$ such that $\hat{L}(w) = f$. Since

$$-n = -n\hat{L}(v)(x) < m\hat{L}(w)(x) < n\hat{L}(v)(x) = n$$

for any $x \in S$, we get that $-nv < mw < nv$ in $\mathbb{Q} \otimes_{\mathbb{Z}} G$. Since $\psi \circ \hat{L} = \hat{\varphi}$ and $\hat{\varphi}(v) = 1$, it follows that $|\psi(f)| = |\hat{\varphi}(w)| \leq \frac{n}{m}$. If there exists $g \in \hat{L}(\mathbb{Q} \otimes_{\mathbb{Z}} G)$ such that $|\psi(g)| > \sup_{x\in S} |g(x)|$, then there exist $n, m \in \mathbb{N}$ such that

$$|\psi(g)| > \frac{n}{m} > \sup_{x\in S} |g(x)|.$$

But by the argument above, if $|g(x)| < \frac{n}{m}$ for any $x \in S$, then $|\psi(g)| \leq \frac{n}{m}$, so ψ must be norm-contractive.

It follows from (ii) above that $\hat{L}(\mathbb{Q} \otimes_{\mathbb{Z}} G)$ is uniformly dense in $\mathcal{A}(S, \pi)$. By continuity of ψ, we get that ψ extends uniquely to a linear norm-contractive map $\psi : \mathcal{A}(S, \pi) \to \mathbb{R}$. The fact that there exists a unique $s \in \pi^{-1}(\beta)$ such that $\psi = \mathrm{ev}_s$ follows as in [26, lemma 3.8]. Hence, $\varphi = \mathrm{ev}_s \circ \hat{L}$ as required. □

Remark 4.3. Note that the assumption that A has real rank zero was instrumental in showing that $S(G)$ is a Choquet simplex.

As it was done in proposition 3.2, we now build a suitable K_1-group.

Proposition 4.4. *There exist a countable, abelian, torsion free group H, an automorphism κ of H and a homomorphism $q : H \to K_1(A)$ such that*

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

$$0 \longrightarrow H \xrightarrow{\mathrm{id}-\kappa} H \xrightarrow{q} K_1(A) \longrightarrow 0$$

is exact.

Proof. We apply [47, proposition 3.5] to the pair $(K_1(A), 0)$. □

We can now construct a classifiable C*-algebra B. To show that we can identify A with a corner of a crossed product of $B \otimes \mathcal{K}$, we will produce an automorphism of $B \otimes \mathcal{K}$ such that the crossed product is $\mathcal{Z}$-stable. The strategy is similar to the one in §3. We first show that we can choose an automorphism of $B \otimes \mathcal{K}$ which has finite Rokhlin dimension. Combining this with the fact that $B \otimes \mathcal{K}$ has finite nuclear dimension will yield that the obtained crossed product has finite nuclear dimension.

LEMMA 4.5. *Let $\rho_G : G \to \mathrm{Aff}(S(G))$ be the canonical map given by $\rho_G(g)(\phi) = \phi(g)$ for any $g \in G$ and $\phi \in S(G)$. There exist a simple, separable, unital, nuclear, $\mathcal{Z}$-stable* C*-*algebra B, satisfying the* UCT *and an automorphism γ of $B \otimes \mathcal{K}$ such that the following conditions hold:*

(i) $\mathrm{Ell}(B) \cong (G, G_+, v, H, S(G), \rho_G)$;
(ii) $K_0(\gamma) = \alpha$ *and* $K_1(\gamma) = \kappa$;
(iii) $(B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}$ *is simple and $\mathcal{Z}$-stable;*
(iv) the restriction map is a bijection from the densely defined lower semicontinuous traces on $(B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}$ onto the γ-invariant densely defined lower semicontinuous traces on $B \otimes \mathcal{K}$.

Proof. To construct a C*-algebra B as in the statement of the lemma, we apply Elliott's range-of-the-invariant result (see theorem 2.12). Lemma 4.2 shows that (G, G_+, v) is a simple, weakly unperforated, countable ordered abelian group and $S(G)$ is a metrisable Choquet simplex. Thus, by theorem 2.12, there exists a simple, separable, unital, nuclear, $\mathcal{Z}$-stable C*-algebra B, satisfying the UCT such that $\mathrm{Ell}(B) \cong (G, G_+, v, H, S(G), \rho_G)$.

We construct an automorphism γ' which satisfies (ii). Set $p_1 = 1_B \otimes e_{11}$, where e_{11} is a minimal projection in $\mathcal{K}$, and say that $\alpha([p_1]_0) = [p_2]_0$ for some projection $p_2 \in B \otimes \mathcal{K}$. Then, essentially since any matrix amplification of $B \otimes \mathcal{K}$ is isomorphic to $B \otimes \mathcal{K}$, there exists a *-isomorphism $\theta : B \otimes \mathcal{K} \to B \otimes \mathcal{K} \otimes \mathcal{K}$ such that $K_i(\theta) = K_i(\phi)$ for $i = 0, 1$ (see for example the argument in [59, corollary 6.2.11]), where $\phi : B \otimes \mathcal{K} \to B \otimes \mathcal{K} \otimes \mathcal{K}$ is given by $\phi(x) = x \otimes e_{11}$. By [10], we get isometries $v_i \in \mathcal{M}(B \otimes \mathcal{K} \otimes \mathcal{K})$ such that $v_i v_i^* = p_i \otimes 1_{\mathcal{M}(\mathcal{K})}$ for $i = 1, 2$. Thus, the maps

$$\eta_i := \mathrm{Ad}(v_i) \circ \phi : B \otimes \mathcal{K} \to p_i(B \otimes \mathcal{K})p_i \otimes \mathcal{K}$$

induce isomorphisms at the level of K_0 and K_1 for any $i = 1, 2$.

We then set

$$\alpha' := K_0(\eta_2) \circ \alpha \circ K_0(\eta_1)^{-1}$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

to be a unital ordered group isomorphism from $K_0(p_1(B\otimes\mathcal{K})p_1)$ to $K_0(p_2(B\otimes\mathcal{K})p_2)$. Moreover, set

$$\kappa' := K_1(\eta_2)\circ\kappa\circ K_1(\eta_1)^{-1}$$

to be a group isomorphism from $K_1(p_1(B\otimes\mathcal{K})p_1)$ to $K_1(p_2(B\otimes\mathcal{K})p_2)$.

By construction of B, the canonical map from tracial states on B to states on $K_0(B)$ is an affine homeomorphism. Therefore, the tracial states on $p_i(B\otimes\mathcal{K})p_i$ are determined by states on $K_0(p_i(B\otimes\mathcal{K})p_i)$ for any $i=1,2$. Hence, it is immediate to see that there exists a unique linear isomorphism $\sigma : \mathrm{Aff}(T(p_1(B\otimes\mathcal{K})p_1))\to \mathrm{Aff}(T(p_2(B\otimes\mathcal{K})p_2))$, which is compatible with α' via the canonical pairing maps. Moreover, since B is nuclear, $\mathcal{Z}$-stable, and satisfies the UCT, and all these properties are preserved by stable isomorphisms (see proposition 2.10), so are $p_i(B\otimes\mathcal{K})p_i$ for $i=1,2$. Then, p_1,p_2 are full projections, so $p_i(B\otimes\mathcal{K})p_i$ is simple, separable, unital for $i=1,2$. Thus, by [13, corollary 9.5], there exists a unital *-isomorphism $\psi : p_1(B\otimes\mathcal{K})p_1\to p_2(B\otimes\mathcal{K})p_2$ such that $(K_0(\psi),K_1(\psi),\mathrm{Aff}(T(\psi)))=(\alpha',\kappa',\sigma)$.

Set γ' to be the following sequence of maps

$$\gamma' : B\otimes\mathcal{K}\xrightarrow{\mathrm{Ad}(v_1)\circ\theta} p_1(B\otimes\mathcal{K})p_1\otimes\mathcal{K}\xrightarrow{\psi\otimes\mathrm{id}_{\mathcal{K}}} p_2(B\otimes\mathcal{K})p_2\otimes\mathcal{K}\xrightarrow{\theta^{-1}\circ\mathrm{Ad}(v_2^*)} B\otimes\mathcal{K}.$$

By construction, we have that $K_0(\gamma')=\alpha$ and $K_1(\gamma')=\kappa$.

To check condition (iii), we will first show that we can choose γ such that the crossed product is simple and has finite nuclear dimension. Building on work in [50], we can take an automorphism γ of $B\otimes\mathcal{K}$ with finite Rokhlin dimension such that $K_i(\gamma)=K_i(\gamma')$ for $i=0,1$ ([35, lemma 4.7] or [34, theorem 3.4]). Since $B\otimes\mathcal{K}$ is simple, nuclear, and $\mathcal{Z}$-stable, it has finite nuclear dimension ([14, theorem A]). Moreover, γ has finite Rokhlin dimension, so the crossed product $(B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z}$ has finite nuclear dimension ([50, theorem 5.2] or [33, theorem 3.1]). Since $K_0(\gamma)^n\neq\mathrm{id}$ for all $n\neq 0$, no non-trivial power of γ is inner. Moreover, $B\otimes\mathcal{K}$ is simple, so $(B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z}$ is simple by [39, theorem 3.1]. Thus, we get (iii) by [56, corollary 8.7].

Since B is unital and $\mathcal{K}$ has real rank zero, $B\otimes\mathcal{K}$ has an approximate unit of projections. Moreover, γ has finite Rokhlin dimension, so (iv) follows from lemma 2.17. □

As B is unital and simple, the restriction map $\tau\mapsto\tau|_{B\otimes e_{11}}$ is a linear order-preserving isomorphism from the cone of densely defined lower semicontinuous traces on $B\otimes\mathcal{K}$ to the space of tracial states on B ([18, proposition 4.7]). But, by construction, the space of tracial states on B is affinely homeomorphic to the space of states on $K_0(B)$, so we identify states on $K_0(B)$ with densely defined lower semicontinuous traces on $B\otimes\mathcal{K}$. We will now show that A can be identified with a corner of the crossed product $(B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z}$.

Lemma 4.6. *Let B be the C*-algebra and γ be the automorphism of $B\otimes\mathcal{K}$ both given by lemma 4.5 and $p=1\otimes e_{11}\in B\otimes\mathcal{K}$. Then $p\,((B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z})\,p\cong A$.*

Proof. We will use the classification theorem in [13, theorem 9.9]. First, we check that $p\,((B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z})\,p$ is a simple, separable, unital, nuclear, $\mathcal{Z}$-stable C*-algebra satisfying the UCT. Note that $(B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z}$ is simple by (iii) of lemma 4.5, so p is a

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

full projection. Moreover, $B \otimes \mathcal{K}$ is nuclear, satisfies the UCT, and $(B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}$ is $\mathcal{Z}$-stable by (iii) of lemma 4.5. Therefore, by proposition 2.11, $p\left((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}\right) p$ is a simple, separable, unital, nuclear, $\mathcal{Z}$-stable C*-algebra which satisfies the UCT.

It remains to check that the Elliott invariant of $p\left((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}\right) p$ is isomorphic to the Elliott invariant of A. As discussed in remark 2.9, we do not need to check that the positive cones in the K_0-groups coincide. Note first that the space of tracial states on $p\left((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}\right) p$ is linearly isomorphic to the space of densely defined lower semicontinuous traces on $(B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}$ by [18, proposition 4.7]. By (iv) of lemma 4.5, the latter is in a bijective correspondence to the space of γ-invariant densely defined lower semicontinuous traces on $B \otimes \mathcal{K}$. As observed previously, since B is simple, unital and $S(K_0(B)) \cong T(B)$, it follows that we can identify the space of γ-invariant densely defined lower semicontinuous traces on $B \otimes \mathcal{K}$ with the space of $K_0(\gamma)$-invariant states on $K_0(B)$. But the latter is homeomorphic to $\pi^{-1}(0) \cong T(A)$ by (vi) of lemma 4.2. Hence, $T\left(p\left((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}\right) p\right) \cong T(A)$.

To compute the K-groups of $p\left((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}\right) p$, we apply the Pimsner–Voiculescu exact sequence in [45, theorem 2.4] to the C*-algebra $B \otimes \mathcal{K}$ and the automorphism γ. Identifying $K_i(B \otimes \mathcal{K})$ with $K_i(B)$ for $i = 0, 1$, we get that

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{\mathrm{id}-K_0(\gamma)} & K_0(B) & \longrightarrow & K_0((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}) & \longleftarrow & K_1(B) & \xleftarrow[\mathrm{id}-K_1(\gamma)]{} & K_1(B). \end{array}$$

Recall from (i) of lemma 4.5 that $K_1(B) = H$ and from (ii) of lemma 4.5 that $K_1(\gamma) = \kappa$. Then, proposition 4.4 gives that $\mathrm{id} - K_1(\gamma)$ is injective. Therefore, the map $K_0((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}) \to K_1(B)$ is zero, which yields that the map $K_0(B) \to K_0((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z})$ is surjective. Thus,

$$K_0((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}) \cong K_0(B)/(\mathrm{id} - K_0(\gamma))(K_0(B)). \tag{3.9}$$

Since $K_0(B) = G$ by (i) of lemma 4.5 and $K_0(\gamma) = \alpha$ by (ii) of lemma 4.5, we get that $K_0((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}) \cong K_0(A)$ by (v) of lemma 4.2. This gives that,

$$K_0\left(p\left((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}\right) p\right) \cong K_0(A).$$

Since $\Sigma([p]_0) = [1_A]_0$, it follows that the K_0-isomorphism is compatible with the position of the unit.

Combining (ii) of lemma 4.5 and (iv) of lemma 4.2 yields that the map $\mathrm{id} - K_0(\gamma)$ is injective. Therefore, we obtain the short exact sequence

$$0 \longrightarrow H \xrightarrow{\mathrm{id}-K_1(\gamma)} H \longrightarrow K_1((B \otimes \mathcal{K}) \rtimes_\gamma \mathbb{Z}) \longrightarrow 0.$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

It follows that $K_1((B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z})\cong K_1(A)$ by proposition 4.4. Hence,

$$K_1\left(p\left((B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z}\right)p\right)\cong K_1(A).$$

To apply [13, theorem 9.9], it remains to check that $p\left((B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z}\right)p$ and A have the same pairing between K-theory and traces. Let τ be a tracial state on A and τ_* be the induced state on $K_0(A)$. Recall that the homeomorphism from $T(A)$ onto $T(p\left((B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z}\right)p)$ is given by following sequence of mappings. One first uses (vi) of lemma 4.2 to send τ to $\mathrm{ev}_\tau\circ\hat{L}$, which is a $K_0(\gamma)$-invariant state on $K_0(B)$. Then, (i) of lemma 4.5 yields that $\mathrm{ev}_\tau\circ\hat{L}$ corresponds to a unique γ-invariant tracial state on $B\otimes\mathcal{K}$. By (iv) of lemma 4.5, this extends uniquely to a densely defined lower semicontinuous trace on $(B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z}$. As mentioned previously, the latter corresponds to a tracial state on $p\left((B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z}\right)p$ ([18, proposition 4.7]).

Therefore, it suffices to check that

$$\tau_*\circ\Sigma_0=\mathrm{ev}_\tau\circ\hat{L},$$

where $\Sigma_0: K_0(B)\to K_0(A)$ is the homomorphism in (v) of lemma 4.2 which induces the isomorphism $\boldsymbol{\Sigma}$. If $(\xi,g)\in K_0(B)$, then

$$\tau_*\circ\Sigma_0(\xi,g)=\tau_*\left(\sum_{n\in\mathbb{Z}}\xi_n\right)=\sum_{n\in\mathbb{Z}}\rho_A(\xi_n)(\tau).$$

On the other hand,

$$(\mathrm{ev}_\tau\circ\hat{L})(\xi,g)=g(\tau)+\sum_{n\in\mathbb{Z}}L(\rho_A(\xi_n))(\tau)e^{n\pi(\tau)}.$$

Since $\pi(\tau)=0$ and $g\in G_0$ is supported away from $T(A)$ by lemma 4.1, it follows that

$$(\mathrm{ev}_\tau\circ\hat{L})(\xi,g)=\sum_{n\in\mathbb{Z}}L(\rho_A(\xi_n))(\tau).$$

Furthermore, recall from (3.1) that for any $f\in\mathrm{Aff}(T(A))$, $L(f)|_{T(A)}=f$. Thus,

$$(\mathrm{ev}_\tau\circ\hat{L})(\xi,g)=\sum_{n\in\mathbb{Z}}\rho_A(\xi_n)(\tau)=\tau_*\circ\Sigma_0(\xi,g).$$

Hence, as A and $p\left((B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z}\right)p$ have the same Elliott invariant, they are isomorphic by [13, theorem 9.9]. □

4.2 KMS bundles on unital tracial classifiable C*-algebras

In this subsection, we will finish the proof of theorem B. Consider the dual action $\hat{\gamma}$ on $C=(B\otimes\mathcal{K})\rtimes_\gamma\mathbb{Z}$ as a 2π-periodic flow. Recall that $\hat{\gamma}_t(f)(x)=e^{-ixt}f(x)$ for any $t\in\mathbb{R}, f\in C_c(\mathbb{Z},B\otimes\mathcal{K})$ and $x\in\mathbb{Z}$. Since $p\in B\otimes\mathcal{K}$, $\hat{\gamma}$ restricts to an action on $pCp\cong A$ which we denote by θ. We claim that the KMS-bundle (S^θ,π^θ)

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

is isomorphic to (S, π). We will crucially use [54, lemma 3.1], so we state it for the convenience of the reader.

LEMMA 4.7. ([54, lemma 3.1]). *Let D be a C*-algebra and $\sigma \in \mathrm{Aut}(D)$ an automorphism of D. Let $\hat{\sigma}$ be the dual action on $D \rtimes_\sigma \mathbb{Z}$ considered as a 2π-periodic flow. For $\beta \in \mathbb{R}$, the restriction map $\omega \mapsto \omega|_D$ is a bijection from the β-KMS weights for $\hat{\sigma}$ onto the densely defined lower semicontinuous traces τ on D with the property that $\tau \circ \sigma = e^{-\beta}\tau$. The inverse is the map $\tau \mapsto \tau \circ P$, where $P : D \rtimes_\sigma \mathbb{Z} \to D$ is the canonical conditional expectation.*

Proof of theorem B. This follows the strategy in [26, lemma 3.13]. Let $(\omega, \beta) \in S^\theta$. By [40, remark 3.3], ω extends uniquely to a β-KMS weight $\hat{\omega}$ for $\hat{\gamma}$ on C. By lemma 4.7, the restriction $\hat{\omega}|_{B\otimes\mathcal{K}}$ is a densely defined lower semicontinuous trace on $B \otimes \mathcal{K}$ such that

$$\hat{\omega}|_{B\otimes\mathcal{K}} \circ \gamma = e^{-\beta}\hat{\omega}|_{B\otimes\mathcal{K}}.$$

Since $\hat{\omega}(1 \otimes e_{11}) = 1$, (vi) of lemma 4.2 gives a unique $s \in \pi^{-1}(\beta)$ such that

$$(\hat{\omega}|_{B\otimes\mathcal{K}})_* = \mathrm{ev}_s \circ \hat{L}. \tag{3.10}$$

Then, we can define a map $\xi : S^\theta \to S$ by $\xi(\omega, \beta) = s$. By construction, $\pi \circ \xi = \pi^\theta$ and the restriction $\xi : (\pi^\theta)^{-1}(\beta) \to \pi^{-1}(\beta)$ is affine for any $\beta \in \mathbb{R}$.

We will first check that the map ξ is injective. Let $(\omega_i, \beta_i) \in S^\theta$ such that $\xi(\omega_1, \beta_1) = \xi(\omega_2, \beta_2) = s$ for $i = 1, 2$. Since $\pi \circ \xi = \pi^\theta$, it follows that $\beta_1 = \beta_2$. By construction of the map ξ, we have that

$$(\hat{\omega}_1|_{B\otimes\mathcal{K}})_* = (\hat{\omega}_2|_{B\otimes\mathcal{K}})_*.$$

As the space of densely defined lower semicontinuous traces on $B\otimes\mathcal{K}$ is in a bijective correspondence to the space of states on $K_0(B)$, it follows that $\hat{\omega}_1|_{B\otimes\mathcal{K}} = \hat{\omega}_2|_{B\otimes\mathcal{K}}$, so $\hat{\omega}_1 = \hat{\omega}_2$ by lemma 4.7. Since $\hat{\omega}_i$ is an extension of ω_i for $i = 1, 2$, we get that $\omega_1 = \omega_2$, so the map ξ is indeed injective.

We will now check that ξ is also surjective. Let $s \in S \cap \pi^{-1}(\beta)$ for some $\beta \in \mathbb{R}$. Then $\mathrm{ev}_s \circ \hat{L}$ is a state on $K_0(B)$ such that

$$\mathrm{ev}_s \circ \hat{L} \circ K_0(\gamma) = e^{-\beta}\mathrm{ev}_s \circ \hat{L}.$$

By construction, states on $K_0(B)$ are uniquely induced by tracial states on B (lemma 4.5). These are in one-to-one correspondence with densely defined lower semicontinuous traces on $B \otimes \mathcal{K}$ ([18, proposition 4.7]). Therefore, there exists a unique densely defined lower semicontinuous trace τ on $B\otimes\mathcal{K}$ such that $\tau_* = \mathrm{ev}_s \circ \hat{L}$ and $\tau_* \circ K_0(\gamma) = e^{-\beta}\tau_*$. Since the canonical map from densely defined lower semicontinuous traces on $B \otimes \mathcal{K}$ to states on $K_0(B)$ is a bijection (lemma 4.5 and [18, proposition 4.7]), we get that $\tau \circ \gamma = e^{-\beta}\tau$. If $P : C \to B \otimes \mathcal{K}$ is the canonical conditional expectation, then the restriction $\tau \circ P|_{pCp}$ is a β-KMS state for θ by lemma 4.7. By construction, $\xi(\tau \circ P|_{pCp}, \beta) = s$, so ξ is surjective.

If we show that $\xi^{-1} : S \to S^\theta$ is continuous, then ξ is a homeomorphism by lemma 2.5. Recall from remark 2.4 that both S^θ and S are metrisable and let s_n

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

be a sequence in S which converges to s. Then $\hat{L}(g)(s_n)$ converges to $\hat{L}(g)(s)$ for any $g \in G \cong K_0(B)$. If $\xi^{-1}(s_n) = (\omega_n, \beta_n)$ and $\xi^{-1}(s) = (\omega, \beta)$, then (3.10) yields that

$$\lim_{n\to\infty} (\hat{\omega}_n|_{B\otimes\mathcal{K}})_*(g) = (\hat{\omega}|_{B\otimes\mathcal{K}})_*(g)$$

for any $g \in K_0(B)$. Since the canonical map from the densely defined lower semicontinuous traces on $B\otimes\mathcal{K}$ to states on $K_0(B)$ is a bijection (follows again from lemma 4.5 and [18, proposition 4.7]), $\hat{\omega}_n|_{B\otimes\mathcal{K}}$ converges pointwise to $\hat{\omega}|_{B\otimes\mathcal{K}}$. Thus, $\hat{\omega}_n$ converges pointwise to $\hat{\omega}$ by lemma 4.7, so ω_n converges pointwise to ω. Moreover, $\beta_n = \pi(s_n)$ converges to $\pi(s) = \beta$ by continuity of π, so (ω_n, β_n) converges to (ω, β). This shows that ξ^{-1} is continuous. Hence, the KMS-bundle (S^θ, π^θ) is isomorphic to (S, π). Combining this with lemma 4.6 finishes the proof of theorem B. □

Acknowledgements

The author was partly supported by the EPSRC grant EP/R513295/1 and by the European Research Council under the European Union's Horizon Europe research and innovation programme (ERC grant AMEN-101124789). The author would like to thank Stuart White and Jamie Gabe for their supervision on this project and Stuart White for numerous helpful comments on earlier drafts of this paper. The author would also like to thank Hung-Chang Liao for allowing him to include a proof of lemma 2.17. This article will form part of the author's DPhil thesis.

References

[1] Z. Afsar, N. S. Larsen and S. Neshveyev. KMS states on Nica-Toeplitz C*-algebras. *Commun. Math. Phys*. **378** (2020), 1875–1929.

[2] E. M. Alfsen. *Compact Convex Sets and Boundary integrals*, Ergebnisse der Mathematik und Ihrer Grenzgebiete [Results in Mathematics and Related Areas], Band 57. (Springer-Verlag, New York-Heidelberg, 1971).

[3] A. an Huef, M. Laca, I. Raeburn and A. Sims. KMS states on the C*-algebras of reducible graphs. *Ergodic Theory Dynam. Systems*. **35** (2015), 2535–2558.

[4] B. Blackadar. *K-Theory for Operator algebras*, Mathematical Sciences Research Institute Publications, second edition, Vol.5 (Cambridge University Press, Cambridge, 1998).

[5] B. Blackadar and D. Handelman. Dimension functions and traces on C*-algebras. *J. Funct. Anal*. **45** (1982), 297–340.

[6] B. M. Braga and R. Exel. KMS states on uniform Roe algebras. ArXiv:2304.05873, 2023.

[7] O. Bratteli, G. A. Elliott and R. H. Herman. On the possible temperatures of a dynamical system. *Commun. Math. Phys*. **74** (1980), 281–295.

[8] O. Bratteli, G. A. Elliott and A. Kishimoto. The temperature state space of a C*-dynamical system. I. *Yokohama Math. J*. **28** (1980), 125–167.

[9] O. Bratteli, G. A. Elliott and A. Kishimoto. The temperature state space of a C*-dynamical system. II. *Ann. Math. (2)*. **123** (1986), 205–263.

[10] L. G. Brown. Stable isomorphism of hereditary subalgebras of C*-algebras. *Pacific J. Math*. **71** (1977), 335–348.

[11] N. P. Brown and N. Ozawa. C*-*Algebras and finite-Dimensional approximations*, Graduate Studies in Mathematics. Vol.88 (American Mathematical Society, Providence, RI, 2008).

[12] N. P. Brown and W. Winter. Quasitraces are traces: a short proof of the finite-nuclear-dimension case. *C. R. Math. Acad. Sci. Soc. R. Can*. **33** (2011), 44–49.

[13] J. R. Carrión, J. Gabe, C. Schafhauser, A. Tikuisis and S. White. Classifying *-homomorphisms I: Unital simple nuclear C*-algebras. *ArXiv:2307.06480*, 2023.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

[14] J. Castillejos and S. Evington. Nuclear dimension of simple stably projectionless C*-algebras. *Anal. PDE*. **13** (2020), 2205–2240.

[15] J. Christensen. The structure of KMS weights on étale groupoid C*-algebras. *J. Noncommut. Geom*. **17** (2023), 663–691.

[16] J. Christensen and K. Thomsen. KMS states on the crossed product C*-algebra of a homeomorphism. *Ergodic Theory Dynam. Systems*. **42** (2022), 1373–1414.

[17] J. Christensen and S. Vaes. KMS spectra for group actions on compact spaces. *Commun. Math. Phys*. **390** (2022), 1341–1367.

[18] J. Cuntz and G. K. Pedersen. Equivalence and traces on C*-algebras. *J. Funct. Anal*. **33** (1979), 135–164.

[19] E. G. Effros, D. E. Handelman and C. L. Shen. Dimension groups and their affine representations. *Am. J. Math*. **102** (1980), 385–407.

[20] G. A. Elliott. Dimension groups with torsion. *Int. J. Math*. **1** (1990), 361–380.

[21] G. A. Elliott. On the classification of C*-algebras of real rank zero. *J. Reine Angew. Math*. **443** (1993), 179–219.

[22] G. A. Elliott. An invariant for simple C*-algebras. *Canadian Mathematical Society. 1945–1995, Vol. 3*, pp. 61–90 (Canadian Math. Soc, Ottawa, ON, 1996).

[23] G. A. Elliott and G. Gong. On the classification of C*-algebras of real rank zero. II. *Ann. Math. (2)*. **144** (1996), 497–610.

[24] G. A. Elliott, G. Gong, H. Lin and Z. Niu. The classification of simple separable unital $\mathcal{Z}$ -stable locally ASH algebras. *J. Funct. Anal*. **272** (2017), 5307–5359.

[25] G. A. Elliott and Y. Sato. Rationally AF algebras and KMS states of $\mathcal{Z}$ -absorbing C*-algebras. *ArXiv:2207.11653*, 2022.

[26] G. A. Elliott, Y. Sato and K. Thomsen. On the bundle of KMS state spaces for flows on a $\mathcal{Z}$-absorbing C*-algebra. *Commun. Math. Phys*. **393** (2022), 1105–1123.

[27] G. A. Elliott and K. Thomsen. The bundle of KMS state spaces for flows on a unital AF C*-algebra. *C. R. Math. Acad. Sci. Soc. R. Can*. **43** (2021), 103–121.

[28] G. Gong, X. Jiang and H. Su. Obstructions to $\mathcal{Z}$ -stability for unital simple C*-algebras. *Canad. Math. Bull*. **43** (2000), 418–426.

[29] G. Gong, H. Lin and Z. Niu. A classification of finite simple amenable $\mathcal{Z}$ -stable C*-algebras, I: C*-algebras with generalized tracial rank one. *C. R. Math. Acad. Sci. Soc. R. Can*. **42** (2020), 63–450.

[30] K. R. Goodearl. *Partially Ordered Abelian Groups With interpolation* Mathematical Surveys and Monographs. Vol.20 (American Mathematical Society, Providence, RI, 1986).

[31] K. R. Goodearl. K_0 of multiplier algebras of C*-algebras with real rank zero. K-*Theory*. **10** (1996), 419–489.

[32] U. Haagerup. Quasitraces on exact C*-algebras are traces. *C. R. Math. Acad. Sci. Soc. R. Can*. **36** (2014),67–92.

[33] I. Hirshberg and N. C. Phillips. Rokhlin dimension: obstructions and permanence properties. *Doc. Math*. **20** (2015), 199–236.

[34] I. Hirshberg, W. Winter and J. Zacharias. Rokhlin dimension and C*-dynamics. *Commun. Math. Phys*. **335** (2015), 637–670.

[35] B. Jacelon. Chaotic tracial dynamics. *Forum Math. Sigma*. **11** (2023), Paper No. e39, 21.

[36] R. V. Kadison. A representation theory for commutative topological algebra. *Mem. Amer. Math. Soc*. **7** (1951), 39.

[37] E. Kirchberg. The classification of purely infinite C*-algebras using Kasparov's theory. manuscript available at https://ivv5hpp.uni-muenster.de/u/echters/ekneu1.pdf., 1994.

[38] E. Kirchberg and W. Winter. Covering dimension and quasidiagonality. *Int. J. Math*. **15** (2004), 63–85.

[39] A. Kishimoto. Outer automorphisms and reduced crossed products of simple C*-algebras. *Commun. Math. Phys*. **81** (1981), 429–435.

[40] M. Laca and S. Neshveyev. KMS states of quasi-free dynamics on Pimsner algebras. *J. Funct. Anal*. **211** (2004), 457–482.

[41] H. -C. Liao. A Rokhlin type theorem for simple C*-algebras of finite nuclear dimension. *J. Funct. Anal*. **270** (2016), 3675–3708.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use , available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125

[42] D. Olesen and G. K. Pedersen. Applications of the Connes spectrum to C*-dynamical systems. III. *J. Funct. Anal.* **45** (1982), 357–390.
[43] R. S. Palais. When proper maps are closed. *Proc. Am. Math. Soc.* **24** (1970), 835–836.
[44] N. C. Phillips. A classification theorem for nuclear purely infinite simple C*-algebras. *Doc. Math.* **5** (2000), 49–114.
[45] M. Pimsner and D. Voiculescu. Exact sequences for K-groups and Ext-groups of certain cross-product C*-algebras. *J. Operator Theory.* **4** (1980), 93–118.
[46] L. Robert and A. Tikuisis. Nuclear dimension and $\mathcal{Z}$ -stability of non-simple C*-algebras. *Trans. Am. Math. Soc.* **369** (2017), 4631–4670.
[47] M. Rørdam. Classification of certain infinite simple C*-algebras. *J. Funct. Anal.* **131** (1995), 415–458.
[48] M. Rørdam. Classification of nuclear, simple C*-algebras. *Classification of Nuclear C*-algebras. Entropy in Operator algebras, Encyclopaedia Math. Sci.*, Vol. 126, pp. 1–145 (Springer, Berlin, 2002).
[49] M. Rørdam. The stable and the real rank of $\mathcal{Z}$-absorbing C*-algebras. *Int. J. Math.* **15** (2004), 1065–1084.
[50] G. Szabó, J. Wu and J. Zacharias. Rokhlin dimension for actions of residually finite groups. *Ergodic Theory Dyn. Syst.*. **39** (2019), 2248–2304.
[51] M. Takesaki. *Tomita's Theory of Modular Hilbert Algebras and its Applications, Lecture Notes in Mathematics*. Vol.128 (Springer-Verlag, Berlin-New York, 1970).
[52] H. Thiel and E. Vilalta. Nowhere scattered C*-algebras. *J. Noncommut. Geom.* **18** (2024), 231–263.
[53] K. Thomsen. KMS weights on graph C*-algebras. *Adv. Math.* **309** (2017), 334–391.
[54] K. Thomsen. The possible temperatures for flows on a simple AF algebra. *Commun. Math. Phys.* **386** (2021), 1489–1518.
[55] K. Thomsen. *An introduction to KMS weights $I + II + III$* , Vol. 2362 (Springer Cham, 2023), https://link.springer.com/book/9783031756290.
[56] A. Tikuisis. Nuclear dimension, $\mathcal{Z}$ -stability, and algebraic simplicity for stably projectionless C*-algebras. *Math. Ann.* **358** (2014), 729–778.
[57] A. Tikuisis, S. White and W. Winter. Quasidiagonality of nuclear C*-algebras. *Ann. Math. (2).* **185** (2017), 229–284.
[58] A. S. Toms and W. Winter. Strongly self-absorbing C*-algebras. *Trans. Am. Math. Soc.* **359** (2007), 3999–4029.
[59] N. E. Wegge-Olsen. *K-Theory and C*-algebras*. (Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993) A friendly approach..
[60] W. Winter. Nuclear dimension and $\mathcal{Z}$ -stability of pure C*-algebras. *Invent. Math.* **187** (2012), 259–342.
[61] W. Winter and J. Zacharias. The nuclear dimension of C*-algebras. *Adv. Math.* **224** (2010), 461–498.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 10:46:24, subject to the Cambridge Core terms of use , available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2024.125