EXTREME ESTIMATES FOR INTERPOLATED OPERATORS BY THE REAL METHOD

FERNANDO COBOS AND ANTÓN MARTÍNEZ

ABSTRACT

The paper establishes estimates for ideal measures of operators interpolated by the real method in terms of the measures of their restrictions to a sequence of spaces modelled on the intersection. It also shows that the estimates are optimal.

Introduction

Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be Banach couples, and let $T \in \mathscr{L}(\bar{A}, \bar{B})$, that is, T is a linear operator from $A_0 + A_1$ into $B_0 + B_1$ whose restriction on each A_k defines a bounded operator from A_k into B_k $(k = 0, 1)$. We put

$$\|T\|_{\bar{A},\bar{B}} = \max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\}.$$

A classical problem in interpolation theory is to determine if the interpolated operator by the real method $T: (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$ inherits a certain property that $T: A_0 \longrightarrow B_0$ has. Sometimes the answer is positive as in the case of compactness (see [**9**], and also [**6**]); some other times it is negative as in the case of having non-trivial cotype (see [**11**]).

Fewer properties pass from the restriction $T: A_0 \cap A_1 \longrightarrow B_0 + B_1$ to the interpolated operator, but there is one quite important one: Beauzamy showed in [**4**] that if the inclusion $I: A_0 \cap A_1 \longrightarrow A_0 + A_1$ is weakly compact, then $(A_0, A_1)_{\theta,q}$ is reflexive (that is, $I: (A_0, A_1)_{\theta,q} \longrightarrow (A_0, A_1)_{\theta,q}$ is also weakly compact) provided that $1 < q < \infty$. Beauzamy's result has attracted considerable attention, and a number of authors have extended it in several directions (see, for example, [**13**, **15**, **16**]).

The recent papers by Aksoy and Maligranda [**1**] and by the present authors [**8**] are also related to this result. They derive estimates for the measure of weak non-compactness $w(T)$ of the interpolated operator in terms of the measure of the restriction of T to the intersection and of the norm $\|T\|_{\bar{A},\bar{B}}$. However, these results are only valid in a degenerated case, namely when the last couple reduces to a single Banach space.

In this paper we continue that research by dealing with the general case. Our techniques use some ideas introduced in [**8**], geared to the construction of the real interpolation method. The approach we follow works for the (more general) case of the measure $\gamma_{\mathscr{I}}$ associated in a natural way to a surjective closed operator ideal $\mathscr{I}$. We show that the measure $\gamma_{\mathscr{I}}(T)$ of the interpolated operator can be estimated in terms of the norm $\|T\|_{\bar{A},\bar{B}}$ and of the measures of the restrictions of T to a sequence of spaces modelled on the intersection $A_0 \cap A_1$. We also show the optimality of the estimate.

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When $\mathscr{I} = \mathscr{W}$, the ideal of weakly compact operators, $\gamma_{\mathscr{I}}$ coincides with the measure of weak non-compactness w, and we obtain a quantitative version of Beauzamy's result.

Finally, we establish a similar estimate for the measure $\beta_{\mathscr{I}}$ which is the relevant measure when the ideal $\mathscr{I}$ is injective and closed.

1. *Operator ideals and measures*

In what follows, E, F denote Banach spaces and $\mathscr{L}(E,F)$ designates the collection of all bounded linear operators from E into F, endowed with the usual operator norm. We put U_E for the closed unit ball of E, and E^* for the dual space of E.

A class $\mathscr{I}$ of bounded linear operators is called an operator ideal if each component $\mathscr{I}(E,F) = \mathscr{I} \cap \mathscr{L}(E,F)$ is a linear subspace of $\mathscr{L}(E,F)$, contains the finite rank operators and satisfies that $STR \in \mathscr{I}(G,V)$ whenever $R \in \mathscr{L}(G,E)$, $T \in \mathscr{I}(E,F)$ and $S \in \mathscr{L}(F,V)$.

Among the different properties that an ideal may have, the main ones for our purposes are surjectivity, injectivity and being closed. Recall that the ideal $\mathscr{I}$ is said to be closed if each component $\mathscr{I}(E,F)$ is closed in $\mathscr{L}(E,F)$. The ideal $\mathscr{I}$ is called surjective if for every $T \in \mathscr{L}(E,F)$ it follows from $TQ_E \in \mathscr{I}(\ell_1(U_E),F)$ that $T \in \mathscr{I}(E,F)$. Here $\ell_1(U_E)$ stands for the Banach space of all absolutely summable families of scalars $(\lambda_x)_{x\in U_E}$ with U_E as index set and where $Q_E : \ell_1(U_E) \longrightarrow E$ is the surjection $Q_E(\lambda_x) = \sum_{x\in U_E} \lambda_x x$. The ideal $\mathscr{I}$ is said to be injective if for every operator $T \in \mathscr{L}(E,F)$ it follows from $J_F T \in \mathscr{I}(E, \ell_\infty(U_{F^*}))$ that $T \in \mathscr{I}(E,F)$. This time $\ell_\infty(U_{F^*})$ is the Banach space of all bounded families of scalars indexed by the elements of U_{F^*}, and $J_F : F \longrightarrow \ell_\infty(U_{F^*})$ is the isometric embedding given by $J_F y = (\langle f, y\rangle)_{f\in U_{F^*}}$.

Related to an ideal $\mathscr{I}$ there are two natural functionals that we may consider. For $T \in \mathscr{L}(E,F)$, we write

$$\begin{aligned}\gamma_{\mathscr{I}}(T) &= \gamma_{\mathscr{I}}(T_{E,F}) \\ &= \inf\{\sigma > 0 : T(U_E) \subseteq \sigma U_F + R(U_Z),\ R \in \mathscr{I}(Z,F),\ Z \text{ any Banach space}\}\end{aligned}$$

and

$$\begin{aligned}\beta_{\mathscr{I}}(T) = \beta_{\mathscr{I}}(T_{E,F}) = \mathit{inf}\{\sigma > 0 : &\text{there is a Banach space } Z \text{ and } R \in \mathscr{I}(E,Z) \\ &\text{such that } \|Tx\|_F \leqslant \sigma\|x\|_E + \|Rx\|_Z,\ x \in E\}.\end{aligned}$$

Obviously, if $T \in \mathscr{I}$ then $\gamma_{\mathscr{I}}(T) = \beta_{\mathscr{I}}(T) = 0$. On the other hand, $\gamma_{\mathscr{I}}(T) = 0$ implies that T belongs to the surjective hull of the closure of $\mathscr{I}$ (see [**2**, Theorem 3.11]). Therefore, if the ideal $\mathscr{I}$ is closed and surjective, we may use the functional $\gamma_{\mathscr{I}}$ to measure the deviation of T from $\mathscr{I}$.

Similarly (see [**14**, Theorem 20.7.3]) if $\mathscr{I}$ is closed and injective then

$$\beta_{\mathscr{I}}(T) = 0 \text{ if and only if } T \in \mathscr{I}.$$

These measures satisfy

$$\max\{\gamma_{\mathscr{I}}(T), \beta_{\mathscr{I}}(T)\} \leqslant \|T\|$$

and they are submultiplicative:

$$\gamma_{\mathscr{I}}(RT) \leqslant \gamma_{\mathscr{I}}(R)\gamma_{\mathscr{I}}(T), \quad \beta_{\mathscr{I}}(RT) \leqslant \beta_{\mathscr{I}}(R)\beta_{\mathscr{I}}(T).$$

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Moreover, if $T \in \mathscr{L}(E,F)$ then operators J_F and Q_E have the following minimal properties:

$$\gamma_{\mathscr{I}}(J_F T) = \min\{\gamma_{\mathscr{I}}(ST) : S \in \mathscr{L}(F,V) \text{ is an isometric embedding}\}, \tag{1.1}$$

$$\beta_{\mathscr{I}}(TQ_E) = \min\{\beta_{\mathscr{I}}(TR) : R \in \mathscr{L}(G,E) \text{ is a metric surjection}\} \tag{1.2}$$

(see [**2**, p. 21; **7**, §2]).

The (outer) measure $\gamma_{\mathscr{I}}$ was introduced by Astala in [**2**], while the (inner) measure $\beta_{\mathscr{I}}$ was defined by Tylli in [**19**]. When $\mathscr{I} = \mathscr{W}$, the ideal of weakly compact operators, then $\gamma_{\mathscr{W}}(T)$ is equal to the measure of weak non-compactness of T (see [**10**] and [**1**]) and $\beta_{\mathscr{W}}(T) = \gamma_{\mathscr{W}}(T^*)$ (see [**12**, Proposition 1.2]).

Besides weakly compact operators, other examples of injective surjective closed operator ideals are the Rosenthal operators, the Banach–Saks operators and the dual Radon–Nikodym operators (see [**17**] and [**13**]). All these ideals satisfy also the so-called 'Σ_q-condition' (see [**13**]). This property will be important in our later considerations. In order to recall its definition we first introduce some more notation.

Given any sequence of Banach spaces (E_m) and any sequence of non-negative numbers (λ_m), we denote by $\ell_q(\lambda_m E_m)$ the vector valued ℓ_q-space defined by

$$\ell_q(\lambda_m E_m) = \left\{ x = (x_m) : x_m \in E_m \text{ and } \|x\|_{\ell_q(\lambda_m E_m)} = \left(\sum_{m=-\infty}^{\infty} (\lambda_m \|x_m\|_{E_m})^q \right)^{1/q} < \infty \right\}.$$

When $\lambda_m = 1$ for all $m \in \mathbb{Z}$, we write simply $\ell_q(E_m)$.

Let $Q_k : \ell_q(\lambda_m E_m) \longrightarrow \lambda_k E_k$ be the projection $Q_k(x_m) = x_k$, and let $P_n : \lambda_n E_n \longrightarrow \ell_q(\lambda_m E_m)$ be the embedding $P_n y = (\delta_m^n y)$ where δ_m^n is the Kronecker delta.

For $1 < q < \infty$, we say that $\mathscr{I}$ satisfies the Σ_q-condition if for any sequences of Banach spaces (E_m), (F_m) and for any $T \in \mathscr{L}(\ell_q(E_m), \ell_q(F_m))$, it follows from $Q_k T P_n \in \mathscr{I}(E_n, F_k)$ for any n,k that $T \in \mathscr{I}(\ell_q(E_m), \ell_q(F_m))$.

2. *Real interpolation and measures*

Let $\bar{B} = (B_0, B_1)$ be a Banach couple. For $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$ the real interpolation space realized as a K-space $\bar{B}_{\theta,q;K} = (B_0, B_1)_{\theta,q;K}$ consists of all $b \in B_0 + B_1$, which have a finite norm

$$\|b\|_{\theta,q;K} = \left(\sum_{m=-\infty}^{\infty} (2^{-\theta m} K(2^m, b))^q \right)^{1/q}$$

(the sum should be replaced by the supremum if $q = \infty$). Here $K(t,b)$ is the K-functional of Peetre, defined by

$$K(t,b) = K(t,b;\bar{B}) = \inf\{\|b_0\|_{B_0} + t\|b_1\|_{B_1} : b = b_0 + b_1,\ b_k \in B_k\}.$$

So, for each $t > 0$, $K(t,\cdot)$ is a norm on $B_0 + B_1$.

For $m \in \mathbb{Z}$, set $F_m = (B_0 + B_1, K(2^m, \cdot))$. Then the map $j : \bar{B}_{\theta,q;K} \longrightarrow l_q(2^{-\theta m} F_m)$ given by $jb = (\ldots, b, b, b, \ldots)$ turns out to be an isometric embedding.

Note also the following easy consequence of the definition of $\|\cdot\|_{\theta,q;K}$. Given any $\rho > 0$, it holds that

$$\left(\sum_{m=-\infty}^{\infty} ((2^m \rho)^{-\theta} K(2^m \rho, b))^q \right)^{1/q} \leqslant 2\|b\|_{\theta,q;K}. \tag{2.1}$$

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The real interpolation space can be equivalently defined by using a family of norms on the intersection, as we recall next.

Take another Banach couple $\bar{A} = (A_0, A_1)$ and for $a \in A_0 \cap A_1$ let

$$J(t,a) = J(t,a;\bar{A}) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}.$$

The real interpolation space in the form of a J-space $\bar{A}_{\theta,q;J} = (A_0, A_1)_{\theta,q;J}$ is formed by all those $a \in A_0 + A_1$ which can be represented by $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$) where $(u_m) \subset A_0 \cap A_1$ and $(\sum_{m=-\infty}^{\infty} (2^{-\theta m} J(2^m, u_m))^q)^{1/q} < \infty$. This time the norm (which is equivalent to $\|\cdot\|_{\theta,q;K}$) is given by

$$\|a\|_{\theta,q;J} = \inf\left\{\left(\sum_{m=-\infty}^{\infty} (2^{-\theta m} J(2^m, u_m))^q\right)^{1/q} : a = \sum_{m=-\infty}^{\infty} u_m\right\}.$$

In the form of a J-space, the relevant sequence space is $\ell_q(2^{-\theta m} G_m)$, where

$$G_m = (A_0 \cap A_1, J(2^m, \cdot)).$$

The map $\pi : \ell_q(2^{-\theta m} G_m) \longrightarrow \bar{A}_{\theta,q;J}$ defined by $\pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$ is a metric surjection.

We refer to [**5**] or [**18**] for more details on the real interpolation space.

Take now $T \in \mathscr{L}(\bar{A}, \bar{B})$, and let T_m be the restriction of T from $2^{-\theta m} G_m$ into $2^{-\theta m} F_m$. It is clear that

$$\sup\{\gamma_{\mathscr{I}}(T_m) : m \in \mathbb{Z}\} \leqslant \sup\{\|T_m\|_{2^{-\theta m} G_m, 2^{-\theta m} F_m} : m \in \mathbb{Z}\} \leqslant \|T\|_{\bar{A},\bar{B}}. \tag{2.2}$$

The following result shows that the operators T_m can be approximated by operators of $\mathscr{I}$ with a certain regularity.

LEMMA 2.1. *Let $\bar{A}$, $\bar{B}$, T and T_m be as above. Let $1 < q < \infty$, $0 < \theta < 1$ and assume that $\mathscr{I}$ is a surjective closed operator ideal which satisfies the Σ_q-condition. Then given any $\sigma > \sup\{\gamma_{\mathscr{I}}(T_m) : m \in \mathbb{Z}\}$, there are a sequence of Banach spaces $(Z_m)_{m\in\mathbb{Z}}$ and operators $R_m \in \mathscr{I}(Z_m, 2^{-\theta m} F_m)$ such that $T_m(U_{2^{-\theta m} G_m}) \subseteq \sigma U_{2^{-\theta m} F_m} + R_m(U_{Z_m})$, and $\sup\{\|R_m\| : m \in \mathbb{Z}\} < \infty$. Moreover, the operator $R : \ell_q(Z_m) \longrightarrow \ell_q(2^{-\theta m} F_m)$ defined by $R(z_m) = (R_m z_m)$ belongs to the ideal $\mathscr{I}$.*

Proof. Let $\sigma > \sup\{\gamma_{\mathscr{I}}(T_m) : m \in \mathbb{Z}\}$. According to the definition of the measure $\gamma_{\mathscr{I}}$, for each $m \in \mathbb{Z}$, there is a Banach space Y_m and an operator $S_m \in \mathscr{I}(Y_m, 2^{-\theta m} F_m)$ so that

$$T_m(U_{2^{-\theta m} G_m}) \subseteq \sigma U_{2^{-\theta m} F_m} + S_m(U_{Y_m}).$$

If $\sup\{\|S_m\| : m \in \mathbb{Z}\}$ is finite, then we have the first two conditions of the statement. If $\sup\{\|S_m\| : m \in \mathbb{Z}\} = \infty$, then we proceed as follows.

For each $a \in U_{2^{-\theta m} G_m}$ there exists $y_a \in U_{Y_m}$ such that

$$\|T_m a - S_m y_a\|_{2^{-\theta m} F_m} \leqslant \sigma.$$

Set $V_m = \{y_a \in U_{Y_m} : a \in U_{2^{-\theta m} G_m}\}$, let X_m be the subspace generated by V_m, and write W_m for the balanced, convex hull of V_m. The Minkowski functional

$$p_{W_m}(x) = \inf\{\lambda > 0 : x \in \lambda W_m\}$$

is then a norm on X_m and the inclusion $i : (X_m, p_{W_m}) \longrightarrow Y_m$ has norm $\leqslant 1$. Write Z_m for the completion of (X_m, p_{W_m}) and let $R_m = S_m \circ \bar{i}$ where $\bar{i} : Z_m \longrightarrow Y_m$ is the extension of i. Still $R_m \in \mathscr{I}$ with

$$T_m(U_{2^{-\theta m} G_m}) \subseteq \sigma U_{2^{-\theta m} F_m} + R_m(U_{Z_m})$$

and moreover

$$\sup\{\|R_m\| : m \in \mathbb{Z}\} < \infty.$$

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Indeed, if $z \in X_m$ with $p_{W_m}(z) < 1$, z can be represented as a finite sum $z = \sum_j \lambda_{a_j} y_{a_j}$ with $\sum_j |\lambda_{a_j}| \leqslant 1$ and $y_{a_j} \in V_m$. Hence

$$\begin{aligned}\|R_m z\|_{2^{-\theta m}F_m} &\leqslant \|T_m(\sum_j \lambda_{a_j} a_j) - R_m(\sum_j \lambda_{a_j} y_{a_j})\|_{2^{-\theta m}F_m} + \|T_m(\sum_j \lambda_{a_j} a_j)\|_{2^{-\theta m}F_m} \\ &\leqslant \sum_j |\lambda_{a_j}| \, \|T_m a_j - S_m y_{a_j}\|_{2^{-\theta m}F_m} + \|T_m\| \sum_j |\lambda_{a_j}| \leqslant \sigma + \|T\|_{\bar{A},\bar{B}}.\end{aligned}$$

Since $\{z \in X_m : p_{W_m}(z) < 1\}$ is dense in U_{Z_m}, we get

$$\sup\{\|R_m\| : m \in \mathbb{Z}\} \leqslant \sigma + \|T\|_{\bar{A},\bar{B}}.$$

Let $R : \ell_q(Z_m) \longrightarrow \ell_q(2^{-\theta m}F_m)$ be the operator defined by $R(z_m) = (R_m z_m)$. It follows from $\sup\{\|R_m\| : m \in \mathbb{Z}\} < \infty$ that R is bounded. To complete the proof it remains to check that R belongs to $\mathscr{I}$. But this is a consequence of the Σ_q-condition because

$$Q_k R P_n = \begin{cases} 0 & \text{if } n \neq k \\ R_n & \text{if } n = k \end{cases} \in \mathscr{I}(Z_n, 2^{-\theta k}F_k). \qquad \square$$

We are now able to establish the crucial result of this paper.

THEOREM 2.2. *Let $\bar{A}$, $\bar{B}$, G_m, F_m, T and T_m be as before. Let $1 < q < \infty$, $0 < \theta < 1$ and assume that $\mathscr{I}$ is a surjective closed operator ideal which satisfies the Σ_q-condition. Then*

$$\gamma_{\mathscr{I}}([J_{\bar{B}_{\theta,q;K}} T]_{\bar{A}_{\theta,q;K},\ell_\infty(U_{\bar{B}^*_{\theta,q;K}})}) \leqslant 16 \sup\{\gamma_{\mathscr{I}}(T_m)^\Theta : m \in \mathbb{Z}\} \, \|T\|^{1-\Theta}_{\bar{A},\bar{B}}$$

where $\Theta = \min\{\theta, 1-\theta\}$.

Proof. By the minimal property (1.1), it suffices to prove the inequality for jT where

$$j \in \mathscr{L}(\bar{B}_{\theta,q;K}, l_q(2^{-\theta m}F_m)) \quad \text{is the map} \quad j(b) = (\ldots, b, b, b, \ldots).$$

Let $\eta \leqslant 1$ and take any t, s, so that

$$t^\theta \geqslant 1/\eta, \quad s^{1-\theta} \geqslant 1/\eta$$

(and thus $t, s \geqslant 1$). Given any $a \in U_{\bar{A}_{\theta,q;K}}$ and $\delta > 0$, using (2.1) with $\rho = 1/t$ we can find $d_m = d_m(t) > 0$ so that

$$(2^m t^{-1})^{-\theta} K(2^m t^{-1}, a) < d_m \quad \text{and} \quad \sum_{m=-\infty}^{\infty} d_m^q \leqslant (2+\delta)^q.$$

Again by (2.1), but this time with $\rho = s$, we can choose $h_m = h_m(s) > 0$ such that

$$(2^m s)^{-\theta} K(2^m s, a) < h_m \quad \text{and} \quad \sum_{m=-\infty}^{\infty} h_m^q \leqslant (2+\delta)^q.$$

Now we can find decompositions

$$a = a_{0,m} + a_{1,m} = a'_{0,m} + a'_{1,m} \quad \text{with} \quad a_{k,m}, a'_{k,m} \in A_k$$

and

$$\begin{aligned}\|a_{0,m}\|_{A_0} + 2^m t^{-1} \|a_{1,m}\|_{A_1} &\leqslant d_m \, 2^{\theta m} t^{-\theta} \leqslant (d_m + h_m) \, 2^{\theta m} \eta, \\ \|a'_{0,m}\|_{A_0} + 2^m s \|a'_{1,m}\|_{A_1} &\leqslant h_m \, 2^{\theta m} s^\theta \leqslant (d_m + h_m) \, 2^{\theta m} s\eta.\end{aligned}$$

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Therefore

$$\|a_{0,m}\|_{A_0} \leqslant (d_m+h_m)\,2^{\theta m}\eta, \quad \|a_{1,m}\|_{A_1} \leqslant (d_m+h_m)\,2^{(\theta-1)m}t\eta,$$
$$\|a'_{0,m}\|_{A_0} \leqslant (d_m+h_m)\,2^{\theta m}s\eta, \quad \|a'_{1,m}\|_{A_1} \leqslant (d_m+h_m)\,2^{(\theta-1)m}\eta.$$

Put

$$b_m = a_{1,m}-a'_{1,m} = a'_{0,m}-a_{0,m} \in A_0 \cap A_1.$$

We have

$$\begin{aligned} 2^{-\theta m}J(2^m, b_m) &\leqslant \max\{2^{-\theta m}(\|a'_{0,m}\|_{A_0}+\|a_{0,m}\|_{A_0}), 2^{(1-\theta)m}(\|a_{1,m}\|_{A_1}+\|a'_{1,m}\|_{A_1})\} \\ &\leqslant \max\{(d_m+h_m)\,\eta(1+s), (d_m+h_m)\,\eta(1+t)\} \\ &\leqslant \max\{2(d_m+h_m)\,\eta s, 2(d_m+h_m)\,\eta t\} = 2(d_m+h_m)\,\eta\max\{s,t\}. \end{aligned}$$

Let $c_m = a-b_m$ and $\psi_{m,n} = 2(d_m+h_m)\,\eta\max\{s,t\}$. Then

$$2^{-\theta m}K(2^m, c_m) \leqslant 2^{-\theta m}\|a_{0,m}\|_{A_0}+2^{(1-\theta)m}\|a'_{1,m}\|_{A_1} \leqslant 2(d_m+h_m)\,\eta,$$

whence we can decompose a as $a = b_m+c_m$ with

$$2^{-\theta m}J(2^m, b_m) \leqslant \psi_{m,n} \tag{2.3}$$

and

$$2^{-\theta m}K(2^m, c_m) \leqslant 2(d_m+h_m)\,\eta. \tag{2.4}$$

Take any $\sigma > \sup\{\gamma_{\mathscr{I}}(T_m): m\in\mathbb{Z}\}$, and let (Z_m) and $R_m \in \mathscr{I}(Z_m, 2^{-\theta m}F_m)$ be the Banach spaces and the operators given by Lemma 2.1. The operator S defined by

$$S(z_m) = 4(2+\delta)\,\eta\max\{s,t\}\,R(z_m) = (4(2+\delta)\,\eta\max\{s,t\}\,R_m\,z_m)$$

belongs to $\mathscr{I}(\ell_q(Z_m), \ell_q(2^{-\theta m}F_m))$.

For each $m\in\mathbb{Z}$, there exists $x_m \in U_{Z_m}$ so that

$$2^{-\theta m}K\left(2^m, T_m\left(\frac{b_m}{\psi_{m,n}}\right)-R_m\,x_m\right) \leqslant \sigma,$$

that is,

$$2^{-\theta m}K(2^m, T_m\,b_m-\psi_{m,n}\,R_m\,x_m) \leqslant \sigma\psi_{m,n}. \tag{2.5}$$

Then $x = ((d_m+h_m)\,x_m/2(2+\delta))$ belongs to the closed unit ball of $\ell_q(Z_m)$, and it follows from (2.5) and (2.4) that

$$\begin{aligned} \|jTa-Sx\|_{\ell_q(2^{-\theta m}F_m)} &= \left(\sum_{m=-\infty}^{\infty}(2^{-\theta m}K(2^m, Ta-\psi_{m,n}\,R_m\,x_m))^q\right)^{1/q} \\ &\leqslant \left(\sum_{m=-\infty}^{\infty}(2^{-\theta m}K(2^m, T_m\,b_m-\psi_{m,n}\,R_m\,x_m))^q\right)^{1/q} \\ &\quad +\left(\sum_{m=-\infty}^{\infty}(2^{-\theta m}K(2^m, Tc_m))^q\right)^{1/q} \\ &\leqslant \left(\sum_{m=-\infty}^{\infty}(\sigma\psi_{m,n})^q\right)^{1/q}+\left(\sum_{m=-\infty}^{\infty}(2\eta(d_m+h_m)\,\|T\|_{\bar{A},\bar{B}})^q\right)^{1/q} \\ &\leqslant 4\sigma\eta\max\{s,t\}\,(2+\delta)+4\eta(2+\delta)\,\|T\|_{\bar{A},\bar{B}}. \end{aligned}$$

Here δ is any positive number and $\sigma > \sup\{\gamma_{\mathscr{I}}(T_m): m\in\mathbb{Z}\}$. Hence we derive

$$\gamma_{\mathscr{I}}(jT) \leqslant 8(\eta\sup\{\gamma_{\mathscr{I}}(T_m): m\in\mathbb{Z}\}\max\{s,t\}+\eta\|T\|_{\bar{A},\bar{B}}).$$

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Let us now distinguish two cases. If $0 < \theta \leqslant 1/2$, so $\Theta = \min\{\theta, 1-\theta\} = \theta$, we choose

$$\eta = (\sup\{\gamma_{\mathscr{I}}(T_m) : m \in \mathbb{Z}\}/\|T\|_{\bar{A},\bar{B}})^{\theta}.$$

This number is less than or equal to 1 by (2.2). Then taking the infimum over all $t \geqslant \eta^{-1/\theta}$ and $s \geqslant \eta^{-1/(1-\theta)}$, we conclude that

$$\gamma_{\mathscr{I}}(jT) \leqslant 16 \sup\{\gamma_{\mathscr{I}}(T_m)^{\theta} : m \in \mathbb{Z}\} \|T\|^{1-\theta}_{\bar{A},\bar{B}}.$$

In the remaining case where $1/2 \leqslant \theta < 1$, so $\Theta = 1-\theta$, we take instead

$$\eta = (\sup\{\gamma_{\mathscr{I}}(T_m) : m \in \mathbb{Z}\}/\|T\|_{\bar{A},\bar{B}})^{1-\theta}$$

and the outcome is

$$\gamma_{\mathscr{I}}(jT) \leqslant 16 \sup\{\gamma_{\mathscr{I}}(T_m)^{1-\theta} : m \in \mathbb{Z}\} \|T\|^{\theta}_{\bar{A},\bar{B}}.$$

The proof is complete. □

Note that Theorem 2.2 in the special case when $\mathscr{I} = \mathscr{W}$, the ideal of weakly compact operators, gives a quantitative version of Beauzamy's result mentioned in the Introduction.

Next we establish the corresponding result for the measure $\beta_{\mathscr{I}}$.

THEOREM 2.3. *Let $\bar{A}$, $\bar{B}$, G_m, F_m, T and T_m be as before. Let $1 < q < \infty$, $0 < \theta < 1$ and assume that $\mathscr{I}$ is an injective closed operator ideal which satisfies the Σ_q-condition. Then*

$$\beta_{\mathscr{I}}([TQ_{\bar{A}_{\theta,q;J}}]_{\ell_1(U_{\bar{A}_{\theta,q;J}}),\bar{B}_{\theta,q;J}}) \leqslant 16 \sup\{\beta_{\mathscr{I}}(T_m)^{\Theta} : m \in \mathbb{Z}\} \|T\|^{1-\Theta}_{\bar{A},\bar{B}}$$

where $\Theta = \min\{\theta, 1-\theta\}$.

Proof. Let A_k° be the closure of $A_0 \cap A_1$ in A_k $(k = 0, 1)$, let $\bar{A}^{\circ} = (A_0^{\circ}, A_1^{\circ})$ and define B_k° and $\bar{B}^{\circ}$ similarly. It is clear that

$$G_m^{\circ} = (A_0^{\circ} \cap A_1^{\circ}, J(2^m, \cdot\,; \bar{A}^{\circ})) = G_m$$

while

$$F_m^{\circ} = (B_0^{\circ} + B_1^{\circ}, K(2^m, \cdot\,; \bar{B}^{\circ})) \hookrightarrow F_m$$

and if $b \in B_0 \cap B_1$, then

$$K(2^m, b; \bar{B}^{\circ}) = K(2^m, b; \bar{B}).$$

Moreover

$$\|T\|_{\bar{A}^{\circ},\bar{B}^{\circ}} \leqslant \|T\|_{\bar{A},\bar{B}}, \quad (A_0^{\circ}, A_1^{\circ})_{\theta,q;J} = (A_0, A_1)_{\theta,q;J},$$

$$(B_0^{\circ}, B_1^{\circ})_{\theta,q;J} = (B_0, B_1)_{\theta,q;J}, \text{ and it is not hard to check that}$$

$$\beta_{\mathscr{I}}([T_m]_{2^{-\theta m}G_m^{\circ}, 2^{-\theta m}F_m^{\circ}}) = \beta_{\mathscr{I}}([T_m]_{2^{-\theta m}G_m, 2^{-\theta m}F_m}).$$

Therefore, without loss of generality we may assume that $A_0 \cap A_1$ is dense in A_k and that $B_0 \cap B_1$ is dense in B_k $(k = 0, 1)$. This will allow us to use duality because under this assumption $\bar{A}^* = (A_0^*, A_1^*)$ and $\bar{B}^* = (B_0^*, B_1^*)$ are Banach couples.

Since the map

$$\pi \in \mathscr{L}(\ell_q(2^{-\theta m}G_m), \bar{A}_{\theta,q;J}) \quad \text{defined by} \quad \pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$$

is a metric surjection, according to (1.2) it is enough to prove the result for $T\pi$.

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Take any $\sigma > \sup\{\beta_{\mathscr{I}}(T_m): m\in\mathbb{Z}\}$ (note that the supremum is finite because $\beta_{\mathscr{I}}(T_m) \leqslant \|T_m\| \leqslant \|T\|_{\bar{A},\bar{B}}$). By the definition of $\beta_{\mathscr{I}}$, for each $m\in\mathbb{Z}$ there exist a Banach space X_m and an operator $R_m\in\mathscr{I}(2^{-\theta m}G_m, X_m)$ such that

$$\|T_m a\|_{2^{-\theta m}F_m} \leqslant \sigma\|a\|_{2^{-\theta m}G_m} + \|R_m a\|_{X_m}.$$

This is equivalent to say (see [**12**, Lemma 1.1])

$$(T_m)^*(U_{(2^{-\theta m}F_m)^*}) \subseteq \sigma U_{(2^{-\theta m}G_m)^*} + R_m^*(U_{X_m^*}).$$

The argument in the proof of Lemma 2.1 can now be repeated to show that there are Banach spaces Z_m and mappings $i_m\in\mathscr{L}(Z_m, X_m^*)$ such that

$$(T_m)^*(U_{(2^{-\theta m}F_m)^*}) \subseteq \sigma U_{(2^{-\theta m}G_m)^*} + R_m^* i_m(U_{Z_m})$$

and

$$\sup\{\|R_m^* i_m\|_{Z_m,(2^{-\theta m}G_m)^*}: m\in\mathbb{Z}\} < \infty. \tag{2.6}$$

Let $1/q + 1/q' = 1$. The duality theorem for the real method yields

$$(A_0, A_1)^*_{\theta,q;J} = (A_0^*, A_1^*)_{\theta,q';K}, \quad (B_0, B_1)^*_{\theta,q;J} = (B_0^*, B_1^*)_{\theta,q';K}.$$

Moreover, it is easy to check that

$$\begin{aligned}(T_m)^*: 2^{\theta m}F_m^* \longrightarrow 2^{\theta m}G_m^* = (T^*)_{-m}: 2^{\theta m}(B_0^*\cap B_1^*, J(2^{-m},\cdot;\bar{B}^*))\\ \longrightarrow 2^{\theta m}(A_0^* + A_1^*, K(2^{-m},\cdot;\bar{A}^*))\end{aligned}$$

and that

$$(\pi: \ell_q(2^{-\theta m}G_m) \longrightarrow \bar{A}_{\theta,q;J})^* = j: \bar{A}^*_{\theta,q';K} \longrightarrow \ell_{q'}(2^{\theta m}(A_0^* + A_1^*, K(2^{-m},\cdot;\bar{A}^*))).$$

So by the same arguments as in the first part of the proof of Theorem 2.2 we obtain

$$\pi^* T^*(U_{\bar{B}^*_{\theta,q';J}}) \subseteq \lambda U_{(\ell_q(2^{-\theta m}G_m))^*} + \mu S(U_{\ell_{q'}(z_m)}). \tag{2.7}$$

Here $\lambda = 4\eta\sigma\max\{s,t\}(2+\delta) + 4\eta(2+\delta)\|T\|_{\bar{A},\bar{B}}$ with η, s and t as before, $\mu = 4(2+\delta)\eta\max\{s,t\}$ and

$$S\in\mathscr{L}(\ell_{q'}(Z_m), \ell_{q'}(2^{\theta m}(A_0^* + A_1^*, K(2^{-m},\cdot;\bar{A}^*))))$$

is the operator defined by $S(z_m) = (R_m^* i_m z_m)$.

If $x_m\in X_m$, let $\widehat{x_m}$ be the image of x_m in X_m^{**} by the canonical embedding. Consider the operator $R: \ell_q(2^{-\theta m}G_m) \longrightarrow \ell_q(Z_m^*)$ given by $R(u_m) = \mu(i_m^*(\widehat{R_m u_m}))$. By (2.6), R is bounded. In fact, R belongs to $\mathscr{I}$. This follows from the Σ_q-condition taking into account that $R_m\in\mathscr{I}(2^{-\theta m}G_m, X_m)$.

Now take any $u = (u_m)\in\ell_q(2^{-\theta m}G_m)$ and let $f\in U_{\bar{B}^*_{\theta,q;J}}$ such that

$$\|T\pi u\|_{\bar{B}_{\theta,q;J}} = |\langle f, T\pi u\rangle|.$$

According to (2.7) there exist $h\in U_{(\ell_q(2^{-\theta m}G_m))^*}$ and $g = (g_m)\in U_{\ell_{q'}(z_m)}$ so that

$$\begin{aligned}\|T\pi u\|_{\bar{B}_{\theta,q;J}} &= |\langle \pi^* T^* f, u\rangle| \leqslant \lambda|\langle h, u\rangle| + \mu|\langle Sg, u\rangle|\\ &\leqslant \lambda\|u\|_{\ell_q(2^{-\theta m}G_m)} + \mu|\langle (i_m g_m), (R_m u_m)\rangle|\\ &= \lambda\|u\|_{\ell_q(2^{-\theta m}G_m)} + \mu|(\langle i_m^*(\widehat{R_m u_m})), (g_m)\rangle|\\ &\leqslant \lambda\|u\|_{\ell_q(2^{-\theta m}G_m)} + \|Ru\|_{\ell_q(z_m^*)}.\end{aligned}$$

This implies that

$$\beta_{\mathscr{I}}(T\pi) \leqslant 8(\eta\sup\{\beta_{\mathscr{I}}(T_m): m\in\mathbb{Z}\}\max\{s,t\} + \eta\|T\|_{\bar{A},\bar{B}}).$$

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Finally, distinguishing the cases $0 < \theta \leqslant 1/2$ and $1/2 \leqslant \theta < 1$, and proceeding as at the end of Theorem 2.2, the result follows. □

3. *Optimality*

Assume that we are in the degenerated case $B_0 = B_1 = B$. Then

$$F_m = (B_0 + B_1, K(2^m, \cdot)) = \begin{cases} (B, \|\cdot\|_B) & \text{if } m \geqslant 0 \\ (B, 2^m\|\cdot\|_B) & \text{if } m < 0 \end{cases}$$

and it is not hard to check that

$$\sup\{\gamma_{\mathscr{I}}(T_m) : m \in \mathbb{Z}\} = \gamma_{\mathscr{I}}(T_0) = \gamma_{\mathscr{I}}(T_{A_0 \cap A_1, B}).$$

The present authors have shown in [**8**] that in this case the following inequality holds:

$$\gamma_{\mathscr{I}}(T_{\bar{A}_{\theta,q;K},B}) \leqslant 8\gamma_{\mathscr{I}}(T_{A_0 \cap A_1,B})^{\Theta}\,\|T\|^{1-\Theta}_{\bar{A},B}. \tag{3.1}$$

Moreover, it is allowed that q takes also the values 1 and ∞, that is, $1 \leqslant q \leqslant \infty$, and they do not require that the ideal $\mathscr{I}$ satisfies the Σ_q-condition.

Comparing this result with Theorem 2.2, it is natural to wonder if in the general case where $B_0 \neq B_1$ all assumptions of Theorem 2.2 are necessary for the conclusion. Next we show by means of examples that in fact each one of them is essential for the result.

EXAMPLE 3.1 (the Σ_q-condition). Let $\mathscr{I} = \mathscr{K}$, the ideal of compact operators. This ideal is surjective injective and closed, but it does not satisfy the Σ_q-condition because the identity operator in ℓ_q is not compact. The measure $\gamma_{\mathscr{K}}$ coincides with the (ball) measure of non-compactness.

Take $\bar{A} = \bar{B} = (\ell_q, \ell_q(n))$, where the sequence spaces are defined now over $\mathbb{N}$ and $\ell_q(n) = \{(\xi_n) : (n\xi_n) \in \ell_q\}$. Choose T as the identity operator. Since

$$G_m = \ell_q(n), \quad F_m = \ell_q \quad \text{(equivalent norms)}$$

and $T : \ell_q(n) \to \ell_q$ is the limit of a sequence of finite rank operators, we get $\gamma_{\mathscr{K}}(T_m) = 0$ for each $m \in \mathbb{Z}$. However $\gamma_{\mathscr{K}}(J_{\bar{B}_{\theta,q;K}} T) \neq 0$ because

$$\bar{A}_{\theta,q;K} = \bar{B}_{\theta,q;K} = \ell_q(n^{\theta})$$

and the identity operator of this infinite dimensional space is not compact.

For the remaining examples we take $\mathscr{I} = \mathscr{W}$, the ideal of weakly compact operators, so $\gamma_{\mathscr{W}}$ is the measure of weak non-compactness.

EXAMPLE 3.2 (assumption $1 < q < \infty$). Choose $\bar{A} = \bar{B} = (\ell_{p_0}, \ell_{p_1})$ with $1 < p_0 < p_1 < \infty$, and take again T as the identity operator. Reflexivity of ℓ_{p_k} implies that $\gamma_{\mathscr{W}}(T_m) = 0$ for all $m \in \mathbb{Z}$. Nevertheless,

$$\gamma_{\mathscr{W}}(J_{\bar{B}_{\theta,q;K}} T) \neq 0 \quad \text{for} \quad q = 1 \text{ and } \infty$$

because

$$(\ell_{p_0}, \ell_{p_1})_{\theta,q} = \ell_{p,q} \quad \text{where} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

and the Lorentz spaces $\ell_{p,1}$ and $\ell_{p,\infty}$ are not reflexive.

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EXAMPLE 3.3 (the operator $J_{\bar{B}_{\theta,q;K}}$). In order to show that $J_{\bar{B}_{\theta,q;K}}$ is essential in Theorem 2.2, we give an example close to [**7**, Remark 3.4].

Using [**3**, Theorem 4], one can find a Banach space E and a sequence of operators $(S_k)_{k=1}^{\infty} \subset \mathscr{L}(E, c_0)$ so that

$$\|S_k\| \leqslant 2, \quad \gamma_{\mathscr{W}}(S_k^*) = 1 \quad \text{and} \quad \gamma_{\mathscr{W}}(S_k^{**}) \leqslant \gamma_{\mathscr{W}}(S_k) \leqslant 1/k.$$

Let $R_k = Q_E^* S_k^* \in \mathscr{L}(\ell_1, \ell_\infty(U_E))$. Set $F = Q_E^*(E^*)$ and choose $\bar{A} = (\ell_1, \ell_1)$, $\bar{B} = (F, \ell_\infty(U_E))$.

Since F is a closed subspace of $\ell_\infty(U_E)$, we get

$$\bar{B}_{\theta,q;K} = F \quad \text{(equivalent norms)}.$$

Hence, if we suppose that the estimate of Theorem 2.2 holds without $J_{\bar{B}_{\theta,q;K}}$, we would have

$$\gamma_{\mathscr{W}}(T_{\ell_1,F}) \leqslant C \sup\{\gamma_{\mathscr{W}}(T_m)^{\Theta} : m \in \mathbb{Z}\} \|T\|_{\bar{A},\bar{B}}^{1-\Theta} \tag{3.2}$$

for any $T \in \mathscr{L}(\bar{A}, \bar{B})$.

Choose $T = R_k$. Then

$$\gamma_{\mathscr{W}}(T_{\ell_1,F}) = \gamma_{\mathscr{W}}([Q_E^* S_k^*]_{\ell_1,F}) = \gamma_{\mathscr{W}}(S_k^*) = 1$$

because $Q_E^* : E^* \longrightarrow F$ is a surjective isometry. Moreover,

$$\|T\|_{\bar{A},\bar{B}} = \max\{\|Q_E^* S_k^*\|_{\ell_1,F}, \|Q_E^* S_k^*\|_{\ell_1,\ell_\infty(U_E)}\} \leqslant \|S_k\|_{E,c_0} \leqslant 2,$$

and, since $\gamma_{\mathscr{W}}([R_k]_{\ell_1,\ell_\infty(U_E)}) = \gamma_{\mathscr{W}}(R_k^*)$ (see [**2**, Corollary 5.3]) and $G_m = \max\{1, 2^m\}\ell_1$, we have

$$\sup\{\gamma_{\mathscr{W}}(T_m) : m \in \mathbb{Z}\} = \gamma_{\mathscr{W}}([R_k]_{\ell_1,\ell_\infty(U_E)}) = \gamma_{\mathscr{W}}(R_k^*) = \gamma_{\mathscr{W}}(S_k^{**}) \leqslant 1/k.$$

Consequently, we would derive from (3.2) that

$$1 \leqslant C 2^{1-\Theta}/k^{\Theta}, \quad k = 1, 2, \ldots$$

which is impossible.

EXAMPLE 3.4 (the case $\sup\{\gamma_{\mathscr{I}}(T_m) : m \in \mathbb{Z}\}$). We end the paper by showing that a finite number of $\gamma_{\mathscr{I}}(T_m)$ is not enough to dominate $\gamma_{\mathscr{I}}(J_{\bar{B}_{\theta,q;K}} T)$.

Let $0 < \lambda < 1$, choose $\bar{A} = \bar{B} = (c_0, \lambda c_0)$ and let T be the identity operator. Thus

$$\|T\|_{\bar{A},\bar{B}} = 1, \quad G_m = \max\{1, \lambda 2^m\} c_0, \quad F_m = \min\{1, \lambda 2^m\} c_0$$

and it is easy to check that $(c_0, \lambda c_0)_{\theta,q;K} = \lambda^{\theta} c_0$ with equivalence of norms, where the constants are independent of λ.

Using the extension property of $\ell_\infty(U_{(\lambda^\theta c_0)^*})$ and of ℓ_∞, we obtain

$$\gamma_{\mathscr{W}}(J_{\bar{B}_{\theta,q;K}} T) = \gamma_{\mathscr{W}}(I_{\lambda^\theta c_0, \lambda^\theta \ell_\infty}).$$

So, by [**2**, Theorem 5.1],

$$\gamma_{\mathscr{W}}(J_{\bar{B}_{\theta,q;K}} T) = \gamma_{\mathscr{W}}((I_{\lambda^\theta c_0, \lambda^\theta c_0})^{**}) = 1.$$

On the other hand

$$\gamma_{\mathscr{W}}(T_m) = \gamma_{\mathscr{W}}(I_{\max\{1,\lambda 2^m\}c_0, \min\{1,\lambda 2^m\}c_0}) = \min\left\{\lambda 2^m, \frac{1}{\lambda 2^m}\right\}.$$

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Therefore, if the inequality in Theorem 2.2 would be true with only a finite portion of the numbers $\gamma_{\mathscr{I}}(T_m)$, that is,

$$\gamma_{\mathscr{I}}(J_{\bar{B}_{\theta,q;K}} T) \leqslant C \max\{\gamma_{\mathscr{I}}(T_m)^{\Theta} : -N \leqslant m \leqslant N\} \|T\|_{\bar{A},\bar{B}}^{1-\Theta}$$

it would follow that

$$1 \leqslant C(\lambda 2^N)^{\Theta} \quad \text{for every} \quad \lambda \in \left(0, \frac{1}{2^N}\right)$$

which is impossible.

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References

1. A. G. Aksoy and L. Maligranda, 'Real interpolation and measure of weak noncompactness', *Math. Nachr.* 175 (1995) 5–12.
2. K. Astala, 'On measures of non-compactness and ideal variations in Banach spaces', *Ann. Acad. Sci. Fenn. Math. Diss.* 29 (1980) 1–42.
3. K. Astala and H.-O. Tylli, 'Seminorms related to weak compactness and to Tauberian operators', *Math. Proc. Cambridge Philos. Soc.* 107 (1990) 367–375.
4. B. Beauzamy, *Espaces d'interpolation réels: topologie et géométrie*, Lecture Notes in Mathematics 666 (Springer, 1978).
5. J. Bergh and J. Löfström, *Interpolation spaces. An introduction* (Springer, 1976).
6. F. Cobos, T. Kühn and T. Schonbek, 'One-sided compactness results for Aronszajn-Gagliardo functors', *J. Funct. Anal.* 106 (1992) 274–313.
7. F. Cobos, A. Manzano and A. Martínez, 'Interpolation theory and measures related to operator ideals', *Quart. J. Math.*, to appear.
8. F. Cobos and A. Martínez, 'Remarks on interpolation properties of the measure of weak non-compactness and ideal variations', *Math. Nachr.*, to appear.
9. M. Cwikel, 'Real and complex interpolation and extrapolation of compact operators', *Duke Math. J.* 65 (1992) 333–343.
10. F. S. De Blasi, 'On a property of the unit sphere in a Banach space', *Bull. Math. Soc. Sci. Math. R. S. Roumanie* 21 (1977) 259–262.
11. D. J. H. Garling and S. J. Montgomery-Smith, 'Complemented sub-spaces of spaces obtained by interpolation', *J. London Math. Soc.* (2) 44 (1991) 503–513.
12. M. González, E. Saksman and H.-O. Tylli, 'Representing non-weakly compact operators', *Studia Math.* 113 (1995) 265–282.
13. S. Heinrich, 'Closed operator ideals and interpolation', *J. Funct. Anal.* 35 (1980) 397–411.
14. H. Jarchow, *Locally convex spaces* (Teubner, Stuttgart, 1981).
15. L. Maligranda and A. Quevedo, 'Interpolation of weakly compact operators', *Arch. Math.* 55 (1990) 280–284.
16. M. Mastylo, 'On interpolation of weakly compact operators', *Hokkaido Math. J.* 22 (1993) 105–114.
17. A. Pietsch, *Operator ideals* (North-Holland, Amsterdam, 1980).
18. H. Triebel, *Interpolation theory, function spaces, differential operators* (North-Holland, Amsterdam, 1978).
19. H.-O. Tylli, 'The essential norm of an operator is not self-dual', *Israel J. Math.* 91 (1995) 93–110.

Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad Complutense de Madrid
28040 Madrid
Spain

cobos@eucmax.sim.ucm.es

Departamento de Matemática Aplicada
ETS Ingenieros Industriales
Universidad de Vigo
36200 Vigo
Spain

antonmar@uvigo.es

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