SELF-CONSISTENT ELECTRODYNAMICS

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ABSTRACT. The idea of direct action between streams is applied to a continuous charged fluid and combined with the new formulation of the electrodynamical laws of motion in terms of conservation of circulation. A simple and rigorous integrated formulation is thus obtained from the Maxwell-Lorentz differential equations, applicable to co-existing positive and negative fluids, as well as vacuum. Exact solutions are obtained, among them one which represents self-consistent, self-maintained flow in a hollow tubular region of infinite axial extent. It is hoped this tube might be bent into a torus and that an electron model will result from merely quantizing the one or two vortices around which this flow-pattern circulates.

1. *Introduction.* (*a*) *Self-consistency and Invariance.* The purpose of this paper is twofold. As a contribution to fundamental electrodynamics, it establishes a simple formulation of the laws governing continuous moving charge distributions and makes a substantial advance towards the construction of an electron model. As a contribution to radio astronomy, it provides the basis for a calculation on spontaneous radiation from electron streams.

These macroscopic and microscopic applications of charge-flow theory both require internal self-consistency of the electromagnetic fields and relativistic accuracy. Accordingly, the laws governing the relativistic, self-consistent streaming of electricity form the subject of the first part of this report and the construction of flow-patterns is carried out in the later sections (§§3–6).

In the astronomical application one can invoke external agents (sunspots, ion-clouds moving under gravity, etc.) for establishing additional fields which will help to maintain the flow. But in fundamental work one would hope for a self-consistent and *self-maintained* solution of all equations, so that the charge moves in equilibrium with its own fields in an otherwise empty space. Hence §§4–6 deal specifically with self-maintained flow patterns.

Quantum effects will not be taken into account in accordance with orthodox quantization rules but in the manner in which they manifest themselves in super-conductivity; rigorous self-consistency will take priority over rigour in the method of quantization. Perhaps some of the difficulties of quantum electrodynamics may eventually be resolved by quantizing *after* self-consistency has been achieved.

Hitherto, quantum electrodynamics has employed iteration of the cycle: field → charge motion → field for approaching self-consistency. This technique is not very successful in practical problems of interacting distributed charges, unless one starts, like Hartree (10), not from vacuum, but from an *almost* self-consistent field. Child (6) and Langmuir (14) discarded iteration for their diode calculation, Thomas (20) for his atomic calculation which provided Hartree's first approximation.

However, relativistic and magnetic effects could be ignored in these early applications of self-consistent field theory, whereas we aim at greater rigour here. Fortunately, the

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new simple formulation of the relativistic equations of motion of distributed charge by Dirac (8, 9) and the author (5) offers a convenient means of making one half of the self-consistency cycle Lorentz-invariant. In this paper the cycle is completed rigorously by identifying the potentials in the New Electrodynamics with the retarded potentials created by the streaming charge itself. An unexpectedly simple and convincing formulation of the Maxwell-Lorentz laws results which lends itself readily to the construction of solutions and to the imposition of a quantum condition.

(*b*) *Assumptions and limits of applicability.* The simplicity of the laws is due, primarily, to the assumption that electricity is continuous and that there are no point particles; it is the author's belief that the modern infinities might be avoided if the picture of a point particle were abandoned and the observed uncertainty of position interpreted literally as a 'smear'. In the continuous picture one ignores, of course, collision effects; there *are* no collisions. Hence viscosity disappears, too. Owing to the long tail of the Coulomb force, collision effects between point particles are very complicated and have often been ignored (successfully) in large-scale applications (discharges, auroras, etc.). On the small scale, we have the encouraging phenomenon of superconductivity which denies viscosity, even against a background of a positive stationary lattice.

A simplification which will be introduced here at an early stage is the assumption that there is no interpenetration of streams, of the same species or of different species, so that there is only one streamline at each point in space-time. This is done partly for convenience, but calculations by Hartree and Nicholson (11) on magnetrons showed a remarkable reluctance of electron streams to cross over. An enforced crossing-over tended to disentangle itself rapidly; the exclusion of matter by matter, appears, classically, as the natural outcome of Coulomb repulsion.

The extension of the new formulation of the electrodynamical laws to several overlapping species or streams is obvious, but calculations become very complicated in these cases. One could apply the new formulation to Bailey's (1) problem, but the mathematics would hardly become less cumbersome. As regards the radiation problem, it might be argued that even in the case of regions of mixed charge (plasmas) the boundary layers are always of one species and radiation from these is covered by our analysis. On the other hand, one should not try to generalize any results presented here to the field of 'Magneto-Hydrodynamics' or the 'Dynamo Problem'.

The co-existence, without overlapping, of several species, on the other hand, is readily taken into account and a single formula describes co-existing positronic and electronic fluids.

(*c*) *Statement of the law of stream-electrodynamics.* The fundamental law governing fields and charge flow is so simple that it can be presented immediately, with only brief explanations of the terminology. The proof of its validity and sufficiency will be given in § 2.

Under the conditions laid down, the current density four-vector, which will be referred to as the 'flow-vector' for brevity, has the same direction as the unique velocity or momentum vectors, these two being just flow vectors normalized to standard lengths. From the equations of motion Dirac (9) derived a close relationship

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between the four-momentum and the four-vector of the potentials. This relationship was subsequently described more physically by the author (5) in terms of 'vortices' and the concept of 'circulation' (i.e. the line integral around a closed circuit that travels with the fluid).

There exists another tight connexion between the flow-vector and the four-vector of the potentials, which arises from that half of the self-consistency cycle where the fields are derived from the moving charges. For, four-dimensionally, the retarded potentials are just the light-cone totals of the flow-vector components (see § 2 (*b*)). We therefore suppress in our theory all field concepts and look upon the four-vector of the potentials as the vector quantity in which, intuitively, the flow-vector distribution would manifest itself as effective at any point in space-time.

With appropriate definitions of signs and units (see § 2 (*a*)) it is therefore possible to state the Maxwell-Lorentz equations in the following integrated form:

The sum of the local normalized flow-vector and the locally effective flow-vector total maintains constant circulation.

This rigorous formulation of the dynamics of fields and matter (whose charge/mass ratio is the same in magnitude everywhere) embraces positronic and electronic flow without change of sign, as well as possible vacuum gaps. Vectors only are referred to (no tensors), and only vectors directly and physically associated with the flow.

(*d*) *Suggestions for quantization: superconductivity.* Without introducing Lagrangians and Hamiltonians, the law leaves the door open for quantization directly, in an obvious place, as it contains the circulation as a constant of integration. Restricting the circulation to integral values was suggested by the author (5) as a method of quantization on somewhat aesthetic grounds: it suggests itself dimensionally and raises 'vortex ribbons' from the realm of mathematical fiction to that of physical reality. It would mean a return to the old Bohr-Sommerfeld method of quantization, in a relativistically invariant manner, and to determinism; the 'hidden variables' referred to by Bohm (2) are, more or less, the variables associated with the microscopic (subelectronic) fluid motion.

Whether such a method of quantization is correct and sufficient has not been explored further except in the following sense: If the extrapolation of our macroscopic law into the infinitesimal subelectronic field is allowed, and if vortices occur in quanta, one would hope to see the free electron appear as the continuous charge distribution around a single elementary vortex (or two such vortices; see § 6). The object of this paper is, partly, to make plausible the existence of such a flow pattern, although a rigorous existence proof has not yet been found.

If one succeeds in constructing the actual pattern, one would check, as a test of this hypothesis on quantization, whether, under given external fields, the pattern as a whole moves (in some approximation) in accordance with the Dirac equations. In particular, the fine-structure constant would have to appear automatically. All 'higher order' effects should result from looking at the electron model more microscopically than one does in the point particle model.

If all elementary vortices are 'frozen' out of a fluid one is left with the irrotational state of super-conductivity described by London (16). Although the suggested method

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of quantization was conceived independently of London's theory, one may interpret it as an extrapolation of that theory into fundamental, four-dimensional electrodynamics. London observes that the superconducting state represents a special (non-vortical) case of the general (vortical) flow of a non-viscous electronic liquid and attributes this restriction of generality to quantum effects. He also finds that circulation in multiply-connected superconductors, termed 'fluxoid' by him, ought to occur in integer multiples of h. What is proposed here amounts to postulating that on a small enough scale all flow is superconducting, but that vortical flow occurs in macroscopic electrodynamics because large numbers of 'fluxoids' of strength h give the impression of continuous vorticity.

(*e*) *The nature of the solutions.* These speculations on a method of quantization do not form the subject or object of this paper. Their only influence on the work presented is to give priority to irrotational solutions of the Maxwell-Lorentz equations, or solutions where there are at most one or two isolated vortices. However, zero vorticity was also

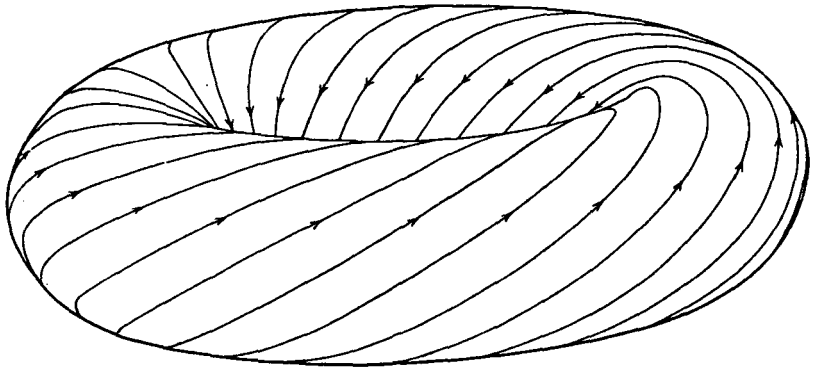


Fig. 1. Schematic illustration of flow pattern envisaged as the finite version of the calculated infinite tubular flow pattern.

imposed because of its frequent occurrence in practical problems (valves), and, principally, because of its mathematical convenience. A case of distributed vorticity has actually been calculated but seems of no physical significance (it will be reported elsewhere).

The classical superconductor solution appears in the first place (§3 (*b*)), then that of the 'slipping stream' analysed non-relativistically in an earlier paper by the author (3) and also by Macfarlane (17), now in its relativistically correct version (§3 (*c*)). These solutions have essentially planar symmetry and are unbounded in two spatial directions. They are not self-maintained (§4 (*a*)).

Cylindrical symmetry (§4 (*b*), (*c*)) leads to a self-maintained, though singly infinite, solution in which the charge confines itself to a hollow tubular region and the flow is peripheral or helical. Velocity components, densities and fields depend upon the distance from the axis. The flow is electronic out to a certain cylindrical layer and positronic outside, but the effective line charge measured in the external vacuum is negative. At the boundary between electronic and positronic flow the fields and the density pass through infinity in an analytically regular manner, the velocity passes

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through that of light without reversing. There is no violation of the fundamental equations at this well-behaved infinity (as there would be for a point-charge). If 'total charge' had a meaning in our system it could be argued that the finite effective charge was obtained by 'subtracting two infinities'. The solution, though non-vortical, has circulation and envelops an axial vortex filament situated in its empty interior (§5).

(*f*) *Further developments.* Attempts to construct rigorous *fully* bounded self-maintained solutions have so far led to great mathematical difficulties (there is non-linearity). But we have no evidence against their existence (see §6), and it seems likely that the tubular solution can be bent (by a perturbation method) into a finite torus as illustrated schematically in Fig. 1. This flow pattern will exhibit an effective charge *and* an effective magnetic moment. It envelopes two vortices, a circular one entirely hidden from the outside by the ring charge, and another along the ring axis. If both are quantized, unique values of size, shape, effective charge, effective magnetic moment, effective mass and angular momentum (under the action of external fields) *may* result, tempting one to identify the ring with the electron.

The radiation problem arising from the astronomical application has actually been dealt with by a perturbation method. The result (that radiation from a slightly disturbed planar slipping stream will not occur spontaneously) was reported by the author (4) in 1951 at a conference on the dynamics of ionized media. The details of the perturbation calculation will be given elsewhere. At this point, it might be added that radiation from undisturbed stationary flow patterns does not occur because we assume charge to be continuous; individual electrons, of course, would radiate from circular or helical orbits.

The stability test, another perturbation calculation, will have to be carried out eventually for the self-maintained tubular solution. In the planar slipping stream one finds instability to low-frequency perturbations, but the tubular solution seems so severely tied by the condition of self-maintenance that the situation may well be different there.

2. *Mathematical formulation of the laws of self-consistent electrodynamics.* (*a*) *Notation and units.* We use four-vectors except where specified, and we gear the unit of time to that of length so that the velocity of light is unity. The fourth component is imaginary; the position vector $\mathbf{r}$ is (x, y, z, it). The associated nabla operator $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, -i\frac{\partial}{\partial t}\right)$ is denoted by $\diamondsuit$, following Page & Adams (18). The scalar product of a four-vector with itself may be referred to as the square of the 'length' or 'magnitude' $(x^2+y^2+z^2-t^2)$, the negative scalar product $(t^2-x^2-y^2-z^2)$ as the square of the 'interval'. This is real for time-like vectors. Among the latter, we can distinguish between 'future' vectors and 'past' vectors, according to the sign of the time-component.

The four-velocity $\mathbf{U}$ is a vector of unit interval; its components will be indicated by capitals: $\mathbf{U} = (X, Y, Z, iT)$. It has to be such that X/T, Y/T, Z/T are the non-relativistic velocity components in any given Lorentz frame. These specifications do not lay down the sign of the components of $\mathbf{U}$. It is customary to let $\mathbf{U}$ be a future vector, but we shall here depart from this custom as regards electronic charge. $\mathbf{U}$ is to

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be a *past* vector for electronic flow, a *future* vector for positronic flow. Then **U** has the same sense as the current density four-vector.

No units of length will be specified in general. All our solutions will be open to 'scaling' by choosing any arbitrary length unit. But in anticipation of dealing mainly with electronic and positronic flow, we use mc^2/e as an electromagnetic unit wherever possible; 511 kV. is our unit of potential (1703 e.s.u.), 17·03 kiloamp. our unit of current, 1·703 kilogauss-cm. our unit of magnetic potential (= 1355 ampere-turns), 1703 e.s.u. our unit for line-charge density (= one electronic charge per classical electron radius), 1703 e.m.u. our unit for magnetic moment per unit area ($= 2\pi/137$ Bohr magnetons per disk of electronic radius). These definitions only involve the charge/mass ratio for the fluid, not the individual values of charge and mass.

(*b*) *Flow vector and effective flow vector.* A charged fluid represents a vector field of the time-like four-vector of current density, **J**. This is a future vector for positive charge, a past vector for negative charge. Owing to conservation of charge it is divergence-free: $\diamondsuit . \mathbf{J} = 0$. Its 'interval' we denote N, the magnitude of the charge density, measured in a Lorentz frame travelling with the fluid. We have $\mathbf{J} = N\mathbf{U}$ and with our convention for the sense of **U**, we have positive N always.

Conventionally, **J** is the *charge* current density, but in a single stream fluid it may be taken to describe the flow of any indestructible quantity associated with the fluid, such as its mass. We shall therefore (and for brevity) refer to **J** as the 'flow vector'; the concept of charge need not be introduced for its definition. The identification of **J** as the charge current density is really a matter of observing the electromagnetic effects of the fluid.

Intuitively, we should expect the fluid to cause effects only at points in space-time which lie at zero interval: 'action at *no* distance', rather than 'action at *a* distance'. Hence we construct from $\mathbf{J}(\mathbf{r})$ an 'effective flow vector' $\mathbf{A}(\mathbf{r})$ by weighting with the delta function which singles out zero interval points (the light-cone) and integrating over the entire space-time of the past:

$$\mathbf{A}(\mathbf{r}) = 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{t}\delta((\mathbf{r}-\mathbf{r}')^2)\,\mathbf{J}(\mathbf{r}')\,dx'\,dy'\,dz'\,dt'.$$

The factor 2 is introduced so that **A** becomes the conventional four-vector of the retarded potentials; the $1/r$ term in the familiar formulae arises when the t'-integration is carried out and the rules for the delta function are observed. The past only is integrated over, for the sake of conforming with our experience of causality. The difficulty of having to introduce advanced potentials as well faced Dirac(7) and Wheeler & Feynman(22). Here it is overcome by eliminating the concept of point particles from our theory. The definition of the 'past' is, in general, not Lorentz-invariant, but the integral is invariant as it only collects from the invariant light-cone.

(*c*) *Gauge transformations, generalized momentum.* In the formulation of the laws of electrodynamics there will be no reference to fields, but the field tensor can readily be obtained from **A** by forming the dyadic tensor $\diamondsuit\mathbf{A}$ and subtracting its transpose:

$$\text{Field tensor} = \diamondsuit\mathbf{A} - (\diamondsuit\mathbf{A})_{\text{transpose}}. \tag{1}$$

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Scalar multiplication with $\diamondsuit$ from the left yields $-4\pi\mathbf{J}$ according to Maxwell's equations:

$$\diamondsuit^2\mathbf{A} - \diamondsuit(\diamondsuit.\mathbf{A}) = -4\pi\mathbf{J}. \tag{2}$$

The second term on the left is usually omitted because the condition $\diamondsuit.\mathbf{A} = 0$ is satisfied by the retarded potentials. However, we shall later want to replace **A** by a vector potential which differs from it by the gradient of an unrestricted scalar. Such an unrestricted 'gauge transformation' will not spoil (1) or (2) but may render the divergence of the potential non-zero.

Among the potentials obtainable from any given potentials by gauge transformation the retarded potentials have preferential status in that they have greater physical significance, being identifiable as the components of the effective flow-vector.

In a continuous fluid one cannot define a momentum vector in the usual sense as there are no individual particles. But we can introduce a 'specific momentum vector' which describes the fluid momentum per unit charge. In our system of units, this is exactly the vector **U** for positive fluid. Similarly, we have the specific generalized momentum vector (which includes the potential with the kinetic momentum); this is $\mathbf{U}+\mathbf{A}$ in our units, for positive charge.

Thanks to our sign convention for the **U**-vector of electronic charge, we can use the same combination, $\mathbf{U}+\mathbf{A}$, for the specific generalized momentum vector of negative charge, although in this case the sense of the vector is opposite to the conventional sense of the momentum vector for electrons. In terms of flow-vectors, we can say that $\mathbf{U}+\mathbf{A}$ is the sum of two flow vectors, one of them (**U**) normalized to unit interval, the other (**A**) the aggregate effective flow-vector.

(*d*) *The law of motion.* I have shown that circulation of generalized momentum is conserved in the flow of electricity (5). This followed from the equations of motion of the fluid. I shall now derive the equations of motion from this conservation law so that the latter can *replace* the equations. (Newton began the formulation of the mechanical laws of motion with the momentum conservation law and placed the law for the acceleration second!)

The conservation of momentum circulation led to the possibility of extending the labelling of vortex filaments by Clebsch variables along the fluid stream lines so that these variables, ξ and η, became the Dirac variables. The representation of generalized momentum in the form

$$\mathbf{U}+\mathbf{A} = \xi\diamondsuit\eta + \diamondsuit\chi, \tag{3}$$

where χ is an unrestricted scalar, follows from this construction, without reference to the laws of motion other than through the circulation theorem. ξ and η are, by construction, constant on the stream lines which must be contained in the time-like surfaces $\xi = \text{const.}$, $\eta = \text{const.}$ Hence $\diamondsuit\xi$ and $\diamondsuit\eta$ are normal to **U**, and if we choose a Lorentz frame which coincides, at a particular point under consideration, with the rest-frame of the fluid there, $\diamondsuit\xi$ and $\diamondsuit\eta$ will be pure space-vectors there.

From (3) and (1) we now obtain the field tensor in the form

$$-\diamondsuit\mathbf{U} + (\diamondsuit\mathbf{U})_{\text{transpose}} + \diamondsuit\xi\diamondsuit\eta - \diamondsuit\eta\diamondsuit\xi,$$

and the last two terms in this sum of four dyads, associated with ξ and η, may be referred to as the 'vortex field tensor'. In the Lorentz frame which follows the fluid this tensor is purely magnetic, in fact we have the three-vector equation

$$H_{\text{vortex}} = \nabla\xi \wedge \nabla\eta. \tag{4}$$

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As the fluid is at rest relative to this magnetic field, it will not be accelerated by it. Hence the vortex field tensor is always ineffective, i.e. it does not produce accelerations. Only the first two terms in the dyad sum above need to be taken into account. The force per unit charge is obtained from the field tensor by multiplying from the left with $-\mathbf{U}$. This leads to $(\mathbf{U}.\Diamond)\,\mathbf{U} - \frac{1}{2}\Diamond(\mathbf{U}.\mathbf{U})$. The second term vanishes because $\mathbf{U}$ has constant interval, the first term represents the rate of change of $\mathbf{U}$, following the charge. With our definition of signs and units, this agrees with the force per unit charge and we have verified that the equations of motion apply.

We may now formulate the laws of motion in the way suggested in the introduction:

The sum of the normalized flow-vector and the effective flow-vector maintains constant circulation.

The formulation includes the field equations. These need not be recorded independently of the charge by which the fields are created and through which they manifest themselves. The vacuum may be ignored altogether in our representation.

One can calculate a retarded $\mathbf{A}$ in any vacuum which adjoins charged fluids, and it is often convenient to continue the vector $\mathbf{U}$ into the vacuum, but it need not be subjected to a normalizing condition there as it does not represent a velocity there. Le Couteur (15) has shown how this could be done if it were necessary.

3. *Construction of solutions.* (*a*) *Equations for non-vortical flow.* When there are no vortices in the fluid, or when the vortices occur in isolated filaments (or 'ribbons'), so that ξ and η are discontinuous variables, we can ignore the term $\xi\Diamond\eta$ in (3). We could then attempt to solve the integral equation

$$\mathbf{J} + 2N\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{t}\mathbf{J}(\mathbf{r}')\,\delta((\mathbf{r}-\mathbf{r}')^2)\,dx'\,dy'\,dz'\,dt' = N\Diamond\chi \tag{5}$$

together with $\Diamond.\mathbf{J} = 0$ and $N^2 = -\mathbf{J}.\mathbf{J}$. (We have to multiply (3) through by N so that the equation remains sensible and valid *in vacuo*.) This integral equation has actually been used in some studies of the problem of charge flow.

The alternative is to use the fact that $-\mathbf{U}$ results from $\mathbf{A}$ by a gauge transformation and to substitute in (2), leading to

$$\Diamond^2\mathbf{U} - \Diamond(\Diamond.\mathbf{U}) = 4\pi N\mathbf{U}. \tag{6}$$

The condition of zero divergence for $N\mathbf{U}$ follows automatically from (6), but $\mathbf{U}.\mathbf{U} = -1$ must be imposed wherever N does not vanish. As (6) is a differential equation, boundary conditions must be set. It is not necessary to set these conditions at the boundary of the fluid, as we can proceed into the adjoining vacuum by continuing to solve (6) there with $N = 0$ and dropping the normalization of $\mathbf{U}$. Conditions need only be given for the distant fields or potentials in this case.

If a solution is self-maintained, distant fields must vanish and a gauge transformation must be found which will convert $-\mathbf{U}$ into a potential that vanishes like $1/r$ and therefore agrees with the retarded $\mathbf{A}$ (by uniqueness). If the solution is not self-maintained, the distant fields must accord with whatever practical conditions are applied externally. In the integral equation, externally imposed fields must be introduced by additional terms (as the integral equation is non-linear, the expression 'inhomogeneous terms' would hardly be appropriate).

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(*b*) *Constant density: 'superconductor' solution.* In (6), N is an inverse area, dimensionally. In fact, $(\pi/N)^{\frac{1}{2}}$, always real by definition, is the free-space wave-length λ corresponding to the 'plasma' or 'critical ionospheric penetration' frequency associated with the density N (see Stratton (19)). In a stream consisting of discrete electrons, $(4\pi N)^{-\frac{1}{2}}$, i.e. $\lambda/2\pi$, is $\frac{1}{6}$ mm. when there are 10^{15} electrons per c.c., 5 km. when there is one electron per c.c. A universal change of the scale of N in any solution can always be made, provided the scale of length is changed accordingly.

It is not possible to lay down an arbitrary distribution N throughout space and then solve (6) as a linear equation for $\mathbf{U}$, for there would then be five equations for the four components of $\mathbf{U}$, including the normalization. However, much progress can be made by starting with plausible assumptions concerning N.

For instance, experience gained from the application of Jeffreys's method ('WKB method') to the similar Schrödinger equation makes one hope that N may be treated, in the first instance, as quasi-constant. This will be sound provided the change of the 'plasma' wave-length is small per wave-length displacement in the fluid. As N does not pass through zero, the well-known difficulties of the Jeffreys method need not be feared. (But we shall see later that N may readily pass through infinity.)

For constant N we can conclude $\diamondsuit . \mathbf{U} = 0$ and (6) becomes the familiar 'meson equation'. The solution $\mathbf{U} = \mathbf{U}_1 \exp(\mathbf{k}.\mathbf{r})$ springs to mind, with constant space-like $\mathbf{k}$ such that $\mathbf{k}.\mathbf{k} = 4\pi N$ and constant $\mathbf{U}_1$ normal to $\mathbf{k}$ because of the vanishing divergence. We may choose $\lambda/2\pi$ as unit of length so that $\mathbf{k}$ becomes a unit vector.

The space-components of this solution represent the classical superconductor solution: flow in the direction $\mathbf{U}_1$ with a transverse velocity gradient in the direction $\mathbf{k}$ (F. London (16)). Such a transverse velocity gradient is a common feature of all solutions found so far and is the consequence of the presence of a magnetic field under non-vortical conditions. The existence of this type of flow under appropriate conditions makes one hope that viscosity effects really *can* be absent.

In a superconductor the fourth component of $\mathbf{U}$ follows laws which differ from those laid down here, owing to the presence of the stationary positive background. For this reason the non-linear condition $\mathbf{U}.\mathbf{U} = -1$ does not make itself felt. Although the assumption of zero vorticity underlies the superconductor theory as well as the solutions presented hereafter, we refrain from calling all non-vortical flow 'superconducting', owing to this difference in the fourth component.

(*c*) *Slipping-stream solution.* As we wish to satisfy the condition $\mathbf{U}.\mathbf{U} = -1$, we must superpose at least two solutions of the superconducting type

$$\mathbf{U} = \mathbf{U}_1 \exp(\mathbf{k}_1.\mathbf{r}) + \mathbf{U}_2 \exp(\mathbf{k}_2.\mathbf{r}), \tag{7}$$

and only if $\mathbf{k}_2 = -\mathbf{k}_1$ do we obtain a constant term in the expression for $\mathbf{U}.\mathbf{U}$. Then we may rewrite $\mathbf{U}$ in the form

$$\mathbf{U} = \mathbf{i} \sinh \mathbf{k}.\mathbf{r} + \mathbf{l} \cosh \mathbf{k}.\mathbf{r}$$

with the conditions $\mathbf{i}.\mathbf{k} = 0$, $\mathbf{l}.\mathbf{k} = 0$ arising from $\diamondsuit . \mathbf{U} = 0$ and $\mathbf{l}.\mathbf{i} = 0$, $\mathbf{i}.\mathbf{i} = 1$, $\mathbf{l}.\mathbf{l} = -1$ arising from $\mathbf{U}.\mathbf{U} = -1$. Hence we can identify $\mathbf{i}$, $\mathbf{k}$ and $\mathbf{l}$ with the x-, z- and t-axes respectively, leading to

$$\mathbf{U} = (\sinh z, 0, 0, i \cosh z). \tag{8}$$

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Both (6) and $\mathbf{U}.\mathbf{U} = -1$ are now satisfied, hence the assumption $N =$ constant is justified.

This solution represents the relativistic adaptation of the 'slipping stream' studied earlier by the author (3) and Macfarlane (17), or the 'single stream steady state' in planar magnetrons. The electric field is in the z-direction and is of magnitude $\sinh z$ in units of 511 kV. per length $\lambda/2\pi$, the magnetic field is in the y-direction and is of magnitude $\cosh z$ in units of 1703 gauss-cm. per length $\lambda/2\pi$.

The solution may be broken off on any surface made up from generators parallel to the x-axis. If one limits the solution to the space between two planes $z = z_1$ and $z = z_2$, one has to postulate constant crossed fields $E_z = \sinh z_1$, $H_y = \cosh z_1$ and $E_z = \sinh z_2$, $H_y = \cosh z_2$ respectively in the vacuum on either side. These external fields must be assumed to be established by other distant charges and currents. The magnetic field is always non-zero on both sides of the streaming layer; the electric field can, at best, vanish on one side only. Hence the solution can never be self-maintained, although it is internally self-consistent. The retarded potentials are infinite as the solution is unbounded in two space-dimensions. But this does not render the solution useless from the point of view of, say, astronomical or technical applications.

It has not yet been possible to construct any stationary, non-vortical flow pattern with constant density other than (8). Addition of further terms in (7) has always led to contradiction with the normalization condition for $\mathbf{U}$. The author would be glad of a non-existence proof or a gegenbeispiel. Only by introducing the time as a variable (i.e. letting further $\mathbf{k}$-vectors in (7) have time-components in a Lorentz frame where $\mathbf{k}_1$ and $\mathbf{k}_2$ have none) was it possible to obtain the somewhat freakish exact solutions:

$$\mathbf{U} = (\sinh z, 0, 0, i\cosh z) - 2\epsilon e^z(\epsilon, \cos(y-t), \sin(y-t), -i\epsilon), \tag{9a}$$

$$\mathbf{U} = (\sinh z, 0, 0, i\cosh z)\cosh\epsilon + (0, \cos t, \sin t, 0)\sinh\epsilon \tag{9b}$$

where ϵ is in each case an arbitrary constant.

The perturbation of (8) does not lead to any important new features when constant density is maintained and solutions (9) are typical examples of a perturbation when ϵ is assumed small. When density fluctuations are admitted, interesting results arise which are of importance in the astronomical and technical applications.

The very specific flow pattern (8) was derived from the fundamental equation by a series of almost inevitable steps which make one think that (8), although not unique, is highly representative. It is, in fact, typical of all known solutions locally, but the chief restricting factor has been the assumption of constant density which is *seriously* violated in all the interesting alternatives to (8). The density tends to develop singularities quite readily, thus invalidating the assumptions underlying Jeffreys's approximation. For example, in Langmuir's diode solution the density is singular at the emitting surface and hence this well-known example of charge flow has not appeared in the present scheme. A relativistic adaptation of Langmuir's solution has, in fact, been resurrected from equations (6) but will be reported elsewhere.

4. *Solutions with special symmetries.* (*a*) *Planar flow patterns.* One respect in which (8) seems to lack generality is that all components of $\mathbf{U}$ vary in the same spatial

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direction. There is no *a priori* reason for expecting the gradients of all four components of **U** to have the same direction, even locally.

On the other hand, solutions such as (9*a*) and (9*b*) are discredited by their very lack of any simplifying feature, and one doubts their physical importance. Quite apart from the mathematical convenience of reducing the number of variables, we attach greater significance to solutions with obvious symmetries.

In the first place, solutions not depending on the time (after application of a Lorentz transformation, if necessary) are of special interest. In this case we obtain $\mathbf{H} = -\operatorname{curl}\mathbf{U}^*$ and $\mathbf{E} = \operatorname{grad} T$, where $\mathbf{U}^*$ is the three-vector (X, Y, Z), and Maxwell's equations yield directly

$$\left.\begin{aligned} -\operatorname{curl}\operatorname{curl}\mathbf{U}^* &= 4\pi N\mathbf{U}^* \\ \operatorname{div}\operatorname{grad} T &= 4\pi NT \end{aligned}\right\} \quad (T = \sqrt{(1+U^{*2})}), \tag{10}$$

as can also be obtained from (6) directly by equating the time-component of $\diamondsuit$ to zero.

As a further simplifying feature, let us introduce 'planar' symmetry, meaning that there should be general dependence on one Cartesian coordinate only, as there is in (8), but let us not prevent N from varying with this coordinate, which we take to be z, as in (8).

Then the z-component of equations (10) yields $NZ = 0$ so that $Z = 0$ in charge-occupied space. For the remaining components of **U** one then obtains

$$X''/X = Y''/Y = T''/T = 4\pi N, \tag{11}$$

where primes indicate differentiation with respect to z. Hence X, Y and T obey the same linear differential equation, whatever the variation (if any) of N. They are therefore interrelated by a universal linear relationship so that **U** lies in a plane (which must be time-like). We chose the (x,t)-plane for this plane.

We introduce the normalizing condition $X^2+1 = T^2$ by putting $T = \cosh\Psi$, $X = \sinh\Psi$ and derive an equation for the function $\Psi(z)$ from the Wronskian of the solutions X and T which must be constant:

$$X'T - T'X = \Psi' = \text{constant}.$$

Hence Ψ is linear in z and we have been led back to solution (8) for which the density is constant. We must assume other symmetries in order to get away from (8).

(*b*) *Cylindrical flow patterns.* The next stage after planar symmetry is cylindrical symmetry. In cylindrical polars, s (distance from the axis), ϕ (azimuth), and z (distance along the axis) we let $\mathbf{U}^*$ have the components (S, Φ, Z), and we restrict all variations to the coordinate s. Then the curl in (10) has no s-component and $S = 0$ follows (cf. argument for $Z = 0$ above). Further, we derive

$$-\mathbf{H} = \operatorname{curl}(0, \Phi, Z) = \left(0, -\frac{dZ}{ds}, \frac{d\Phi}{ds} + \frac{\Phi}{s}\right)$$

and

$$-\operatorname{curl}\operatorname{curl}\mathbf{U}^* = \left(0, \frac{1}{s}\frac{d}{ds}\left(s\frac{d\Phi}{ds}\right) - \frac{\Phi}{s^2}, \frac{1}{s}\frac{d}{ds}\left(s\frac{dZ}{ds}\right)\right).$$

On introducing $\sigma = \log(s/s_0)$, where s_0 is an arbitrary constant length, and using primes for differentiation with respect to σ, we obtain from (10)

$$T''/T = Z''/Z = (\Phi''/\Phi) - 1 = 4\pi Ns^2. \tag{12}$$

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The flow lines will be helical. The pitch of the helix may, or may not, depend on s. We consider here, in the first place, the case of infinite pitch, i.e. pure z-motion. When $\Phi = 0$, the problem of finding solutions T and Z is mathematically the same as that of the planar problem: one obtains

$$T = \cosh \kappa(\sigma - \sigma_0), \quad Z = \sinh \kappa(\sigma - \sigma_0), \quad 4\pi N s^2 = \kappa^2,$$

where κ and σ_0 are arbitrary constants. The constant σ_0 can be absorbed in the definition of s_0. As is usual in potential theory, under cylindrical symmetry the logarithm of the distance from the axis replaces the linear distance in the corresponding planar problem.

The density is no longer constant but varies inversely as s^2. But locally, there is still strong resemblance to the planar slipping stream solution. We observe that the flow reverses direction at $s = s_0$ (having put $\sigma_0 = 0$) and that it becomes fast both near the axis and a long way away. The fields are

$$sE_s = T' = \kappa \sinh \kappa\sigma, \quad sH_\phi = Z' = \kappa \cosh \kappa\sigma.$$

If the solution is broken off at some finite radius, sE_s and sH_ϕ must remain constant in the adjoining vacuum.

Beyond an external boundary of the stream one gets realistic fields, viz. those due to a line charge of the species that is flowing and a current travelling in the sense of the outer flow where $\sigma > 0$. However, internally the fields are opposed to those which would be created only by charge of the sign and current of the sense that is actually flowing there. Hence the solution must be broken off at some internal boundary and an internal device must be postulated which will create the internal electric and magnetic fields required by the flow; the solution cannot be self-maintained. We have here an example of a solution of the differential equation leading to fields that are not compatible with the retarded potentials. If the solution were not broken off internally but continued right into the axis where it becomes singular, the differential equation would apply everywhere (except at $s = 0$), but the integral equation would not, unless a singular source term were added.

(*c*) *Peripheral flow*. The case where $Z = 0$, i.e. where the pitch of the helical stream lines becomes zero and the flow is purely peripheral may in certain cases be achieved from that of non-zero pitch by an axial Lorentz transformation. But in that case the velocity component Z would have to be independent of s, i.e. there would have to be no axial slip. Now the general case of combined axial and peripheral slip has also been calculated but the condition that the flow should be self-maintained can only be satisfied when the slip is purely peripheral and this case only is presented here.

We have to eliminate $4\pi N s^2$ between T and Φ from (12):

$$\frac{T''}{T} \equiv \frac{d(T')^2}{d(T^2)} = \frac{\Phi''}{\Phi} - 1 \equiv \frac{d(\Phi')^2}{d(\Phi^2)} - 1, \tag{13}$$

which we combine with $T^2 = \Phi^2 + 1$ by using $d(T^2) = d(\Phi^2)$ in (13) and integrating:

$$\Phi'^2 - T'^2 = T^2 - \text{const.}$$

It is found that only the case where the constant exceeds unity leads to self-maintained solutions, and it is convenient to introduce the constant in the form $\operatorname{cosec}^2 \alpha$ where α

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is a constant angle between 0 and $\pi/2$. Substitution of $\Phi' = TT'/\Phi$ in the last equation then leads to

$$T'^2 = (T^2 - 1)(T^2 - \operatorname{cosec}^2\alpha), \tag{14}$$

the solution of which is a Jacobi elliptic function

$$T = \operatorname{cosec}\alpha\, ns(\sigma \operatorname{cosec}\alpha), \tag{15}$$

associated with a function Φ such that

$$\Phi/T = dn(\sigma \operatorname{cosec}\alpha). \tag{16}$$

The behaviour of these somewhat unfamiliar functions is best illustrated by introducing first the 'Jacobi amplitude' $\psi(\sigma)$ from the standard elliptic integral

$$\sigma = \sin\alpha F(\alpha, \psi) = \sin\alpha \int_0^{\psi} d\psi^* (1 - \sin^2\alpha \sin^2\psi^*)^{-\frac{1}{2}} \tag{17}$$

tabulated by Jahnke and Emde (12). In fact, we have

$$\psi'^2 = \operatorname{cosec}^2\alpha - \sin^2\psi \quad (\psi' > 0)$$

showing that ψ' oscillates between the two positive extremes $\cot\alpha$ and $\operatorname{cosec}\alpha$ but, on the average, ψ increases steadily with σ. (See Fig. 2 where $\alpha = \frac{1}{4}\pi$ so that ψ' oscillates between 1 and $\sqrt{2}$ and the mean slope of ψ versus σ over a period is 1·20). One may think of the ψ-scale as a slight distortion of the σ-scale.

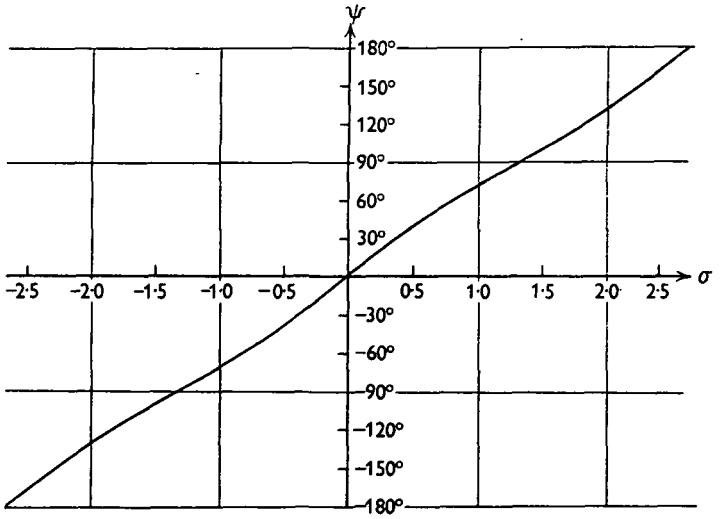


Fig. 2. Jacobi amplitude ψ versus σ $(= \log s/s_0)$.

In terms of $\psi(\sigma)$ we have

$$T = \operatorname{cosec}\alpha \operatorname{cosec}\psi, \quad \Phi/T = \sqrt{(1 - \sin^2\alpha \sin^2\psi)}. \tag{18}$$

As Φ/T does not cross the axis, we must retain the same sign for this quantity throughout; we choose it positive. A plot of T, Φ and Φ/T is seen in Fig. 3. There is periodicity of period 2π in ψ and a corresponding periodicity (of period 5·244 when $\alpha = \frac{1}{4}\pi$) in σ. The flow is in the same sense always and the non-relativistic velocity reaches the velocity of light every half period.

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Our sign convention allows us to interpret the solution as indicating that the flow changes from electronic to positronic every half-period. The infinity of T or Φ has no direct physical interpretation, but the passage through infinity is analytic and fully understood. If T is an 'energy', one might suspect some relation to the 'infinite self-energy' of point particles, but this is highly speculative.

The fields are radially electric and axially magnetic. From the gradient of T and the expression for curl U* in polars (see above) we derive

$$sE_s = T' = -\psi' \operatorname{cosec}\alpha \operatorname{cosec}\psi \cot\psi \tag{19}$$

$$= -\Phi T \cos\psi,$$

$$-sH_z = \Phi + \Phi' = \Phi + T'T/\Phi. \tag{20}$$

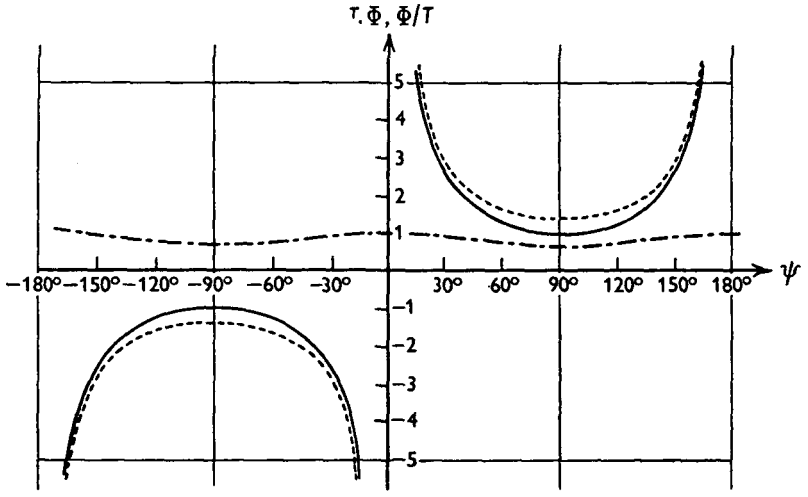


Fig. 3. Components of U and peripheral velocity.
T, ·····; Φ, ——; Φ/T, ·—·—· versus ψ.

These fields are plotted in Fig. 4. They become quadratically infinite every half-period. In Fig. 5 we have shown the ratio $-H_z/E_s$ (which will be found useful) and the density (multiplied by s^2)

$$4\pi N s^2 = T''/T = \operatorname{cosec}^2\alpha\,(2\operatorname{cosec}^2\psi - 1) - 1,$$

which also becomes quadratically infinite at half-period intervals. These infinities in fields and densities are, at first sight, rather disturbing. But they are analytic and well-understood infinities and we shall see that they do not manifest themselves externally.

(*d*) *Boundaries of the peripheral flow pattern.* E_s vanishes every half-period and so does H_z. Any zero of E_s may be chosen as an internal boundary, any zero of H_z as an external boundary. The flow is then fully self-consistent and self-maintained; externally an electric field such as that due to a line charge is observed, internally we find the uniform magnetic field of a solenoid. There is always at least one singular layer between an (internal) zero of E and the next (external) zero of H. There may be arbitrarily many singularities.

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The zeros of E are at odd multiples of $\frac{1}{2}\pi$ in the variable ψ. Those of H can be more readily found by studying the ratio $-H/E$. We obtain

$$-\frac{H}{E}=\frac{\Phi'}{T'}+\frac{\Phi}{T'}=\frac{T}{\Phi}+\frac{\Phi}{T'}. \tag{21}$$

As Φ is an odd function of ψ whereas Φ' and T' are even, we find

$$\left(-\frac{H}{E}\right)_{\sigma}\left(-\frac{H}{E}\right)_{-\sigma}=\frac{\Phi'^2}{T'^2}-\frac{\Phi^2}{T'^2}=1-\frac{T'^2-\Phi'^2+\Phi^2}{T'^2} \tag{22}$$

$$=1-\cot^2\alpha/T'^2.$$

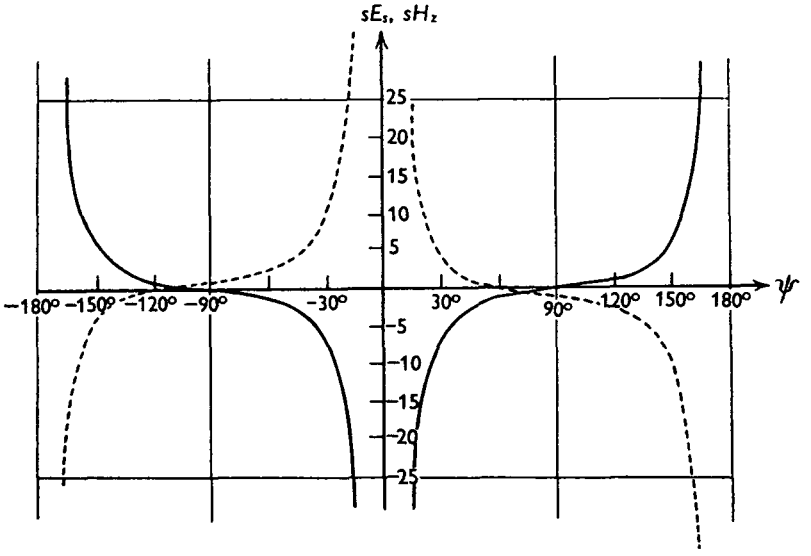


Fig. 4. Electric and magnetic fields. sE_s, ——; sH_z, ····· versus ψ.

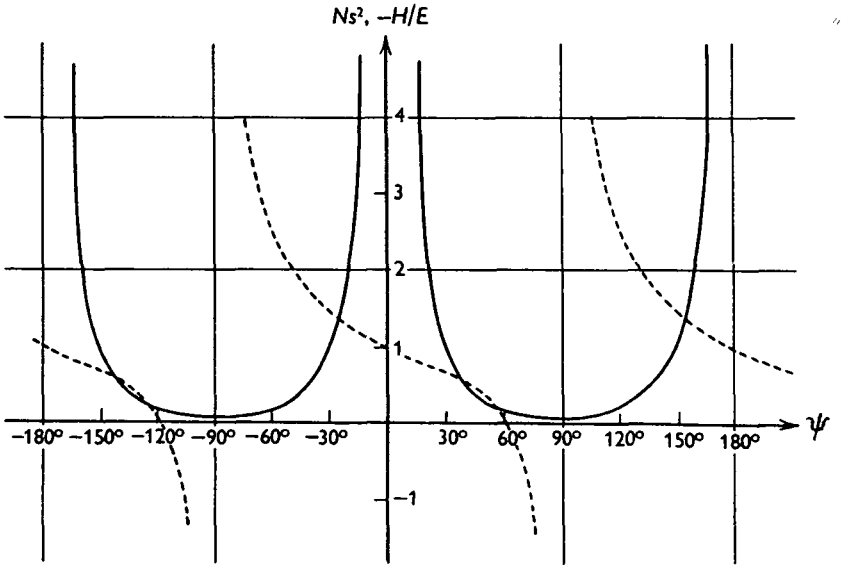


Fig. 5. Density and field ratio. Ns^2, ——; $-H/E$, ····· versus ψ.

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Zeros of H must be zeros of this last product as well, giving $sE_s = T' = \pm \cot\alpha$ as the condition for a zero of H. This is a quadratic equation for $\sin^2\psi$ and in our example ($\alpha = \frac{1}{4}\pi$) we obtain $\psi = 61°$, $\sigma = 0{\cdot}822$ for the first zero to the right of the origin. The first zero of E to the left of the origin is at $\psi = -\frac{1}{2}\pi$, $\sigma = -1{\cdot}311$. Hence the minimum σ – range of a self-maintained flow pattern is 2·133, giving a minimum ratio of outer to inner tube radii of 8·44.

Externally, sE_s must be constant and have the value $\cot\alpha$, representing the field of an electronic line charge $\frac{1}{2}\cot\alpha$, in the units given above. The sign of the line charge is that of the species in the *inner* part of the tube, showing that the external positronic flow acts as a coating only which holds the electronic base together and is, in turn, held by it. The internal magnetic field has the constant value $-\Phi_i/s_i$ where suffix i refers to the internal boundary of the charged region ($\Phi' = 0$ at $\psi = \frac{1}{2}\pi$). We find $\Phi_i = \cot\alpha$ showing that the magnetic potential changes axially by $(L/s_i)\cot\alpha$ units in a length L of the infinite tube. This means that a current of strength $\cot\alpha/4\pi s_i$ per unit length is effectively circulating around the inside of the tube. This value is that obtained if one pictures the effective external charge to be circulating around the inner circumference at the velocity of light. But on careful consideration, one finds that the direction would have to be opposite to the general rotation in the stream. The effective internal current can also be said to be associated with an effective internal magnetic moment per unit length of tube whose value is $\frac{1}{4}s_i\cot\alpha$, or $\frac{1}{2}s_i$ times the charge per unit length.

It was thought at first that the singular layer in the solution presented might be 'cut out' by interposing a finite layer of vacuum between the electronic and the positronic portion of the flow pattern. This is found impossible when one considers the expressions for $-H/E$ given above. The quantity T/Φ in (21) is always positive and greater than unity whereas Φ/T' is positive in odd quadrants, negative in even quadrants of ψ. Hence $-H/E$ exceeds unity in odd quadrants (see Fig. 5). According to (22), it must be reduced to something less than unity when multiplied by the $-H/E$ value of a corresponding even quadrant. Hence $-H/E$ is less than unity in even quadrants. In vacuo, this ratio increases with radius (H_z and sE_s are constant) whereas we should want it to drop from above to below unity in passing from the inside of a singularity to the outside: one cannot 'jump' the singularity.

The solution which we have constructed is self-consistent and self-maintained. It contains, apart from an arbitrary scale factor, only the parameter α which, as we have seen, does not enter in some of the fundamental ratios of the effective features of the flow pattern. The case $\alpha = \frac{1}{2}\pi$ was observed by J. Iliffe to lead to rational expressions in s for all the quantities describing the flow. The internal and external radii go to zero and infinity respectively in this case and although the density decreases like s^{-4}, the field seen at large distances decreases like s^{-3}, so that within the medium, however tenuous, one never observes an effective line charge. Apart from the parameter α, a further parameter appears in the solutions for the case of both axial and peripheral slip.

(*e*) *The retarded potentials.* In the previous section we have chosen boundaries by reference to practical experience of what fields are possible outside or inside the

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boundaries, *in vacuo*. We can obtain the same boundary conditions by constructing the retarded potentials (i.e. satisfying, effectively, the integral equation (5)).

One can obtain the retarded potentials directly by integration of **J** or $N\mathbf{U}$ and one finds that no singular contributions to the integrals arise from the infinities. However, one cannot carry out the integration for the time component (to be denoted iV) of **A** over the infinite length of the tube; one has to work with some finite length, large compared with the dimensions of the region in which the potentials are to be used.

The potentials thus obtained are the same as those which one finds by the following deduction: **A** differs from $-\mathbf{U}$ by a gradient $\diamondsuit\chi$. As **J** has neither s- not z- components, there will be no such components in **A**. The components are absent in **U** and χ can therefore be a function of ϕ and t only. Further, $\partial\chi/\partial\phi$ and $\partial\chi/\partial t$ do not depend on ϕ or t, so that χ must be of the form

$$\chi = \chi_0 + (\phi/\phi_0) + (t/t_0),$$

where χ_0, ϕ_0 and t_0 are constants. Hence we get

$$\mathbf{U} + \mathbf{A} = (0, 1/s\phi_0, 0, -i/t_0)$$

or

$$\Phi + \mathbf{A}_\phi = 1/s\phi_0, \quad T + V = -1/t_0.$$

If the flow is broken off at any radius s_1, internally or externally, we continue the potential $-\mathbf{U}$ into the vacuum smoothly, by solving (6) with $N = 0$ and omitting the normalization condition. We find that T must be linear and Φ exponential in σ, and from the condition of smoothness one gets

$$T = T_1 + T_1'(\sigma - \sigma_1),$$
$$\Phi = \Phi_1 \cosh(\sigma - \sigma_1) + \Phi_1' \sinh(\sigma - \sigma_1).$$

The vacuum potential $-\mathbf{U}$ so constructed differs from the retarded **A** by the same gauge as inside the stream-occupied region. Hence we get

$$V = -T_1' \log s/s_1 - T_1 - 1/t_0,$$

$$A_\phi = -\tfrac{1}{2}(\Phi_1 + \Phi_1')\frac{s}{s_1} - \tfrac{1}{2}(\Phi_1 - \Phi_1')\frac{s_1}{s} + \frac{1}{s\phi_0}.$$

The retarded potentials must be continuous everywhere. Hence if suffix 1 refers to the internal boundary and the axis is in the range of applicability of the formulae given, we must have $T_1' = 0$ and $\frac{1}{2}(\Phi_1 - \Phi_1')s_1 = 1/\phi_0$. Hence the internal electric field vanishes (as we arranged in the preceding section) and $\dfrac{1}{\phi_0} = \frac{1}{2}\Phi_i s_i$. ($\Phi_1' = 0$ follows from $T_1' = 0$.)

If suffix 1 is taken to refer to the external boundary, we obtain

$$-\tfrac{1}{2}(\Phi_1 + \Phi_1')/s_1 = H_1 = 0$$

from the consideration that $A_\phi \to 0$ at large distances. Hence there is no external magnetic field and the condition applied in the previous section is justified from first principles (i.e. the fundamental law stated above). As regards V, we cannot expect this potential to tend to zero at large distances as long as we assume our tube to be of infinite extent. However, if one goes through the procedure of calculating this retarded

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potential for a finite, though very long tube of length L by integration of NT (the fourth component of $\mathbf{J}$), one finds a condition for t_0, viz.

$$\frac{1}{t_0} = -T_1' \log(L/s_1) - T_1.$$

But even without this procedure, one can see directly that the external potential is that due to a line charge of strength $\frac{1}{2}T_1'$ which must be $\frac{1}{2}\cot\alpha$ as shown in the preceding section.

5. *Vorticity and circulation.* The solutions presented so far do not possess any continuous vorticity. This need not be regarded as an arbitrary feature but can be shown to be the consequence of their particular symmetries.

In the first place these solutions represent stationary flow patterns and in any stationary flow pattern the variables ξ and η will be independent of the time by construction. Hence the vortex field tensor will be purely magnetic and we can use equation (4). As the vortex field is ineffective (or because $\Diamond\xi$ and $\Diamond\eta$ are normal to $\mathbf{U}$), H_{vortex} must be along $\mathbf{U}^*$. For the total magnetic field we then obtain

$$\mathbf{H} = -\operatorname{curl}\mathbf{U}^* + \mu\mathbf{U}^*,$$

where μ is a scalar which may vary in space. The equations to be solved in the vortical case are therefore

$$-\operatorname{curl}\operatorname{curl}\mathbf{U}^* + \operatorname{curl}\mu\mathbf{U}^* = 4\pi N\mathbf{U}^*, \qquad \operatorname{div}\operatorname{grad}\sqrt{(1+\mathbf{U}^{*2})} = 4\pi N\sqrt{(1+\mathbf{U}^{*2})}$$

provided the stream lines are stationary.

Let us now consider possible peripheral flow-patterns, vortical or otherwise, where the velocity, always in the ϕ-direction, does not depend on the variable ϕ but may depend on s or z. Then $\operatorname{curl}\mu\mathbf{U}^*$ has no ϕ-component whereas $-\operatorname{curl}\operatorname{curl}\mathbf{U}^*$ and $4\pi N\mathbf{U}^*$ are purely in the ϕ-direction. We therefore conclude $\operatorname{curl}\mu\mathbf{U}^* = 0$ and the problem is reduced to the non-vortical one. The vortex field, if it exists, has neither cause nor effect in the charged fluid. It may be ignored when the flow pattern is determined.

In the case of our tubular solution, the vortex field would have to be a peripheral magnetic field varying inversely as s (it is a vacuum field), and it could only be set up by an axial current introduced artificially inside the tube. Hence it has no place in a self-maintained solution.

An exactly similar argument can be used to eliminate any possible significant vorticity from the axial flow solution or from the planar solution with unidirectional flow. The argument does not apply when the flow may be mixed axial and peripheral or when, in the planar case, the direction of flow, though always transverse to the z-axis, may vary with z. A vortical solution of the latter type has been calculated and will be reported elsewhere.

Absence of vorticity does not mean absence of circulation. In fact, the vector $\Diamond\chi$ for our tubular solution possesses circulation, termed 'fluxoid' by F. London (16), of amount $2\pi/\phi_0$ about the z-axis which is the only vortex filament in this solution; constancy of the t-component of $\Diamond\chi$ ensures that no circulation arises from deforming the path around which circulation is calculated into the time-direction.

For the value of the circulation about the isolated vortex we obtain further

$$(1/2\pi)\text{ times circulation} = 1/\phi_0 = \tfrac{1}{2}\Phi_i s_i = \tfrac{1}{2}s_i \cot\alpha,$$

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and this equals (s_i/L) times the effective charge carried by a length L of the tube. It is obvious that any condition which correct axial termination of our solution would impose on L/s_i, together with a quantum condition on vortices, would fix the value of the smallest possible effective charge, whatever the value of α.

6. *Other geometries.* There is no straightforward way of passing from cylindrical to spherical symmetry. Dependence of, say, the ϕ-component of **U** on distance r from the origin alone, without dependence on the co-latitude θ, leads to difficulties at the poles, i.e. on the z-axis. Even for pure ϕ-motion, there must be dependence on both r and θ which renders the problem extremely difficult.

It can be shown that any pattern of pure ϕ-motion which is bounded axially as well as radially must contain a mixture of positronic and electronic flow, as did the tubular pattern, if it is to be self-maintained. For it has been pointed out by C. O. Hines that the circular stream which is furthest along in the z-direction (i.e. the uppermost or lowermost stream) will always be repelled further in the case where only charge of one species flows. The electrostatic repulsion always wins over the magnetic attraction which acts between like currents, however fast the motion.

If one still retains the idea of pure ϕ-motion one is therefore led to terminating our mixed electronic and positronic tubular solution axially in one way or another. It seems unlikely that the layer of separation between positronic and electronic flow may be bared to the vacuum at the ends. Moreover, such a solution would have a magnetic moment of the wrong sign, if we may extrapolate our calculations for the infinite tube. If the singular layer is allowed to close in on the axis towards the ends, one may expect great difficulties at the point where the layer meets the axis. One feels that with pure ϕ-motion, the axis has to be kept free of charge altogether.

Most likely, the singular layer will have to be shaped into a doubly connected surface, i.e. a torus. With pure ϕ-motion, and a toroidal singular layer, one then gets a picture which represents more nearly our torus of pure *axial* flow, but bent around to be joined on to itself. If we tried to calculate this flow pattern by perturbing the axial flow solution, we would have to start with a solution which is not self-maintained. One therefore feels that there is little hope for the perturbed solution to be self-maintained.

Perhaps one ought to abandon the idea of pure ϕ-motion and try to bend the self-maintained tube with *peripheral* flow into a torus. If the tube has originally only peripheral flow, the result will be a typical smoke-ring flow pattern. No magnetic moment would be observed then. But if a tubular solution of mixed peripheral and axial flow is perturbed, the pattern which is schematically illustrated in Fig. 1 results and this pattern will have a magnetic moment and, perhaps, also an axial vortex in addition to the one which is present inside the tube. If both vortices are quantized to their smallest fundamental value the pattern (which has only few free parameters) may well turn out unique, even in size.

As regards mass and angular momentum of the flow pattern as a whole, one cannot expect to get finite and reasonable answers from just integrating densities of these quantities over the volume of the charge distribution, in a non-relativistic manner. It will be necessary, instead, to expose the charge assembly to given weak external

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electric and magnetic fields and calculate the mean acceleration and the mean precession of it. This is a straightforward perturbation problem, once the unperturbed finite pattern is known. We then obtain an effective inertia and an effective angular momentum of the charge distribution. Whether the ratio between effective charge and mass turns out the same as the original charge mass ratio assumed, cannot yet be said. If not, one will have to renormalize.

As regards the angular momentum, it may be hoped that the calculations of Kramers (13) and Thomas (21) may be applied which predict the anomalous gyromagnetic moment on relativistic, non-quantum arguments.

7. *Conclusion.* From the calculations which have been presented here one gets the impression that rigorous solutions of the Maxwell-Lorentz equations (unquantized) are not plentiful. A few reasonable conditions, in particular the requirements that there should be no vorticity and that the stream lines of the charge should be stationary, limit the choice severely; the persistence with which the slipping stream solution turned up is characteristic. Very few degrees of freedom exist.

The solutions are, in general, not complicated. Yet they are not obvious and often contain an element of unexpectedness. The condition that a solution should be self-maintained as well as self-consistent restricts one's choice even more and one cannot expect any very simple flow geometry to succeed. Whatever may be the fully self-consistent and fully bounded, self-maintained flow pattern, it is likely to be unique or almost unique.

For this reason alone, it is not surprising that a classical (i.e. relativistic but non-quantum) electron model has never be constructed. But lack of optimism in this direction is also due to the early failures of distributions of *static* charge which never stay together. Only dynamic equilibrium seems to be possible, so that magnetic forces can come into play. Success depends, then, on abandoning the point particle picture and looking for a static distribution of stream lines rather than of charged particles.

Even so, one knows that magnetic attraction between moving charges of like sign is always less than Coulomb repulsion, although there can be balancing, component by component, under certain geometrical arrangements. It is doubtful, therefore, whether even dynamic equilibrium can exist with only positive, or only negative, charge present. However, this has not been proved in general.

The only way in which we were able to get nearer to the construction of an electron model at all was by admitting charge of both signs in the same configuration, and admitting infinities in the transition as well. Let us hope that no further complications (e.g. overlapping streams or continuous vorticity) will have to be introduced. An element of optimism is certainly needed in this search for an electron model, and the author hopes that the work presented will inspire some optimism.

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