The second solution of the Mathieu equation. By Mr E. L. INCE, Trinity College.

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The fact that the Mathieu equation, and, as was proved in the preceding paper, the more general Hill equation, can admit of but one solution of period π or 2π, suggests the problem of determining how far the second solution deviates from periodicity when the first solution has either the period π or the period 2π.

Let the Mathieu equation be written in the form

$$\frac{d^2y}{dx^2} + (a + 16q\cos 2x)\, y = 0,$$

then, if an even solution of period 2π exists, the *characteristic number* a must be a root of the determinantal equation

$$\begin{vmatrix} a-1+8q, & 8q, & 0, & \dots \\ 8q, & a-9, & 8q, & \dots \\ 0, & 8q, & a-25, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0.$$

The equation which determines the characteristic numbers for the odd solutions of period 2π is formed by changing the leading element of the above determinant into $a - 1 - 8q$.

Let a_{2n+1} be the characteristic number for the solution

$$y = ce_{2n+1}(x),$$

then a_{2n+1} is determined accurately to the term in q^{2n+1} if all but the first $2n + 1$ rows and columns of the determinant are ignored, and to a lower term q^{2m} $(m < n)$ if the determinant is stripped of all members except the m rows and columns on each side of the element $a - (2n + 1)^2$.

This shows that if

$$a_{2n+1} = (2n+1)^2 + \sum_{r=1}^{\infty} \alpha_r q^r,$$

then $$\alpha_1 = \alpha_3 = \dots = \alpha_{2n-1} = 0, \quad \alpha_{2n+1} \neq 0.$$

Incidentally it may be verified that, if $p = 2n + 1$,

$$\alpha_2 = \frac{32}{p^2 - 1} \qquad (p \geqslant 3),$$

$$\alpha_4 = \frac{128\,(5p^2 + 7)}{(p^2 - 1)^3\,(p^2 - 4)} \qquad (p \geqslant 5),$$

and so on.

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Now let b_{2n+1} be the characteristic number for the solution

$$y = se_{2n+1}(x),$$

then, since $$b_{2n+1} = (2n+1)^2 + \sum_{r=1}^{\infty} (-1)^r a_r q^r,$$

it follows that $$a_{2n+1} - b_{2n+1} = O(q^{2n+1}).$$

The corresponding theorem for solutions of period π is as follows. If a_{2n} and b_{2n} are the characteristic numbers for

$$ce_{2n}(x) \text{ and } se_{2n}(x)$$

respectively, a_{2n} and b_{2n} are even functions of q. If

$$a_{2n} = 4n^2 + \sum_{r=1}^{\infty} \alpha_{2r} q^{2r},$$

$$b_{2n} = 4n^2 + \sum_{r=1}^{\infty} \beta_{2r} q^{2r},$$

then $$\beta_{2r} = \alpha_{2r} \qquad (r < n),$$

$$\beta_{2n} \neq \alpha_{2n},$$

and consequently $$a_{2n} - b_{2n} = O(q^{2n}).$$

As before, if $p = 2n$,

$$\alpha_2 = \frac{32}{p^2 - 1} \qquad (p \geqslant 2),$$

$$\alpha_3 = \frac{128(5p^2+7)}{(p^2-1)^3(p^2-4)} \qquad (p \geqslant 4),$$

and so on.

The actual development of an even periodic solution in ascending powers of q is of the form

$$ce_p(x) = \cos px + \sum_{r=1}^{\infty} A_r q^r,$$

where

$$A_1 = \frac{2\cos(p+2)x}{p+1} - \frac{2\cos(p-2)x}{p-1} \qquad (p \geqslant 2),$$

$$A_2 = \frac{2\cos(p+4)x}{(p+1)(p+2)} + \frac{2\cos(p-4)x}{(p-1)(p-2)} \qquad (p \geqslant 3),$$

$$A_3 = \frac{4\cos(p+6)x}{3(p+1)(p+2)(p+3)} + \frac{4(p^2+4p+7)\cos(p+2)x}{(p-1)(p+1)^3(p+2)}$$
$$- \frac{4(p^2-4p+7)\cos(p-2)x}{(p+1)(p-1)^3(p-2)} - \frac{4\cos(p-6)x}{3(p-1)(p-2)(p-3)}$$
$$(p \geqslant 4),$$

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and so on. Similarly, the odd solution $se_p(x)$ may be developed in the form

$$se_p(x) = \sin px + \sum_{r=1}^{\infty} B_r q^r,$$

where B_r is derived from A_r, for $r = 1, 2, \ldots, p-1$, by replacing each cosine by the corresponding sine.

Since b_p agrees with a_p up to, but not including, the term in q^p, the second solution corresponding to $ce_p(x)$ will agree with $se_p(x)$ up to and including the term in q^{p-1} at least.

If, therefore, the first solution is

$$y_1 = ce_p(x),$$

the second solution is of the form

$$y_2 = se_p(x) + \kappa x ce_p(x) + \phi(x),$$

where κ and $\phi(x)$ are of the order of q^p and κ is to be so chosen that $\phi(x)$ is odd and purely periodic.

Since y_2 is to satisfy the Mathieu equation for $a = a_p$, and $se_p(x)$ is a solution for $a = b_p$, it follows that $\phi(x)$ must satisfy

$$\phi''(x) + (a_p + 16q\cos 2x)\phi(x) + 2\kappa ce_p'(x) + (a_p - b_p)\, se_p(x) = 0.$$

Let $\kappa_0 q^p$ be the leading term of κ. Then $\phi(x)$ must satisfy

$$\phi''(x) + (a_p + 16q\cos 2x)\phi(x) = 2p\kappa_0 q^p \sin px - (\alpha_p - \beta_p) q^p \sin px + O(q^{p+1}).$$

Since, therefore, $\phi(x)$ is to be purely periodic,

$$\kappa_0 = \frac{\alpha_p - \beta_p}{2p},$$

and consequently $\phi(x) = O(q^{p+1})$.

Thus the periodic part of the second solution agrees with $se_p(x)$ up to and including the term in q^p. The non-periodic part begins with a term in q^p.

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