Proc. Camb. Phil. Soc. (1963), **59**, 637
With 13 *text-figures*
Printed in Great Britain

On axially symmetric waves

I. Linearized compressible flow with axial boundary conditions

By J. W. CRAGGS
The University, Leeds, 2

(*Received* 21 *July* 1962)

Abstract. The wave equation with axial symmetry is reduced, by the assumption of dynamic similarity, to an equation in two variables, r/t and θ. A basic solution, having some of the attributes of a source, is introduced, and it is shown that, by use of this solution and a suitable contour integral representation, the solution of any appropriate boundary-value problem may be reduced to the determination of an analytic function, with boundary conditions given by an integral equation.

The method of solution is illustrated by reference to some basic problems in the theory of linearized compressible flow.

Introduction. One of the basic mathematical difficulties of continuum mechanics is the intractability of partial differential equations in three or more independent variables. When the equations are linear this difficulty may be circumvented by a number of methods, all of which depend essentially on beginning with symmetrical problems, for which the number of independent variables is reduced, and then building up the solutions of asymmetric problems by superposition.

In particular, considerable progress in the solution of two-dimensional, unsteady, wave problems has been made by the use of basic solutions with dynamic similarity, in which length and time are involved only in the combination (length ÷ time) with the dimensions of velocity (1–3) (5, 6). It is the purpose of the present paper to show how this technique may be extended to cover problems in axially symmetric motion. In this first paper the method is established, and illustrated by reference to some problems in the motion of a compressible fluid, simplified by the use of the acoustic approximation, in which density variations are assumed to be so small that the linearization of the equations of motion is admissible. In a subsequent paper the method will be applied to problems of elastic waves.

1. *The basic solutions of the wave equation.* Let r, θ, t be the radius and polar angle in spherical polar coordinates, and the time, respectively. Let $\psi(r, \theta, t)$ be a solution of the wave equation

$$\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=0, \tag{1·1}$$

with constant wave velocity c. Then those solutions of the form

$$\psi=\psi(\omega,\theta); \quad \omega=r/t \tag{1·2}$$

are called dynamically similar, and form the subject of this investigation.

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Substitute from (1·2) in (1·1). Then ψ satisfies the equation

$$\omega^2\left(1-\frac{\omega^2}{c^2}\right)\frac{\partial^2\psi}{\partial\omega^2}+2\omega\left(1-\frac{\omega^2}{c^2}\right)\frac{\partial\psi}{\partial\omega}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)=0. \tag{1·3}$$

This equation is of mixed type, hyperbolic, with real characteristic lines*

$$\omega = c\sec(\theta\pm\theta_0) \tag{1·4}$$

for $\omega > c$, and elliptic, with conjugate complex characteristics

$$\omega = c\,\mathrm{sech}\,(\theta_0\pm i\theta) \tag{1·5}$$

for $\omega < c$.

In a polar diagram with coordinates ω, θ, the characteristics (1·4) are tangents to the circle $\omega = c$ (Fig. 1), which it is convenient to call the sonic circle, the parts of the tangent on opposite sides of its contact with the envelope must be considered as characteristics of different families.

The forms of (1·4), (1·5) suggest the substitutions

$$\alpha = \sec^{-1}\omega/c \qquad (\omega > c), \tag{1·6}$$

$$\beta = -\,\mathrm{sech}^{-1}\omega/c \quad (\omega < c), \tag{1·7}$$

where $0 \leqslant \alpha \leqslant \frac{1}{2}\pi$, $-\infty < \beta < 0$. The negative sign in β is chosen so that as $r, \omega \to 0$, $\beta \to \log\omega$. In terms of these substitutions

$$\sin\alpha\frac{\partial}{\partial\alpha}\left(\frac{\partial\psi}{\partial\alpha}\operatorname{cosec}\alpha\right)-\operatorname{cosec}\theta\frac{\partial}{\partial\theta}\left(\frac{\partial\psi}{\partial\theta}\sin\theta\right)=0 \quad (\omega > c), \tag{1·8}$$

$$\sinh\beta\frac{\partial}{\partial\beta}\left(\frac{\partial\psi}{\partial\beta}\,\mathrm{cosech}\,\beta\right)+\operatorname{cosec}\theta\frac{\partial}{\partial\theta}\left(\frac{\partial\psi}{\partial\theta}\sin\theta\right)=0 \quad (\omega < c). \tag{1·9}$$

Consider first equation (1·8). Use new dependent variables

$$u = \frac{\partial\psi}{\partial\alpha}\operatorname{cosec}\alpha, \quad v = \frac{\partial\psi}{\partial\theta}\sin\theta, \tag{1·10}$$

so that

$$\frac{\partial}{\partial\theta}(u\sin\alpha)-\frac{\partial}{\partial\alpha}(v\operatorname{cosec}\theta)=0,$$

i.e.

$$\sin\alpha\frac{\partial u}{\partial\theta}-\operatorname{cosec}\theta\frac{\partial v}{\partial\alpha}=0 \tag{1·11}$$

and

$$\sin\alpha\frac{\partial u}{\partial\alpha}-\operatorname{cosec}\theta\frac{\partial v}{\partial\theta}=0, \tag{1·12}$$

from (1·8). Now write

$$\lambda = \cos(\theta-\alpha), \quad \mu = \cos(\theta+\alpha), \tag{1·13}$$

where, since $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \alpha \leqslant \frac{1}{2}\pi$,

$$\lambda-\mu = 2\sin\alpha\sin\theta \geqslant 0.$$

* These characteristics are of the same form as those of the equation

$$\omega^2(1-\omega^2/c^2)\,(\partial^2\psi/\partial\omega^2)+\omega(1-2\omega^2/c^2)\,(\partial\psi/\partial\omega)+\partial^2\psi/\partial\theta^2=0$$

for two-dimensional flows with dynamic similarity (1).

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Then
$$\frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial \lambda}\sin(\theta-\alpha) - \frac{\partial u}{\partial \mu}\sin(\theta+\alpha),$$
$$\frac{\partial u}{\partial \alpha} = \frac{\partial u}{\partial \lambda}\sin(\theta-\alpha) - \frac{\partial u}{\partial \mu}\sin(\theta+\alpha)$$

so, adding (1·11) and (1·12)
$$\sin\theta\sin\alpha\frac{\partial u}{\partial \mu} - \frac{\partial v}{\partial \mu} = \tfrac{1}{2}(\lambda-\mu)\frac{\partial u}{\partial \mu} - \frac{\partial v}{\partial \mu} = 0 \tag{1·14}$$

and, subtracting
$$\sin\theta\sin\alpha\frac{\partial u}{\partial \lambda} + \frac{\partial v}{\partial \lambda} = \tfrac{1}{2}(\lambda-\mu)\frac{\partial u}{\partial \lambda} - \frac{\partial v}{\partial \lambda} = 0. \tag{1·15}$$

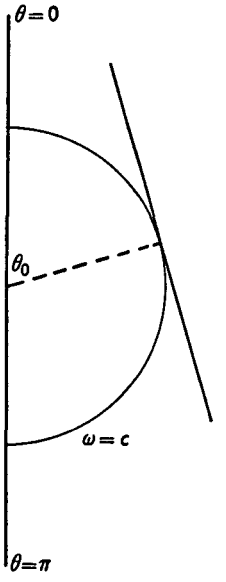


Fig. 1. Characteristic lines.

Equations (1·14), (1·15) are a particular case of the equations of 'Generalized axially symmetric potential theory', as studied by Mackie (4). He gives a method of solution by contour integrals, which is in some respects similar to that used here. (The main difference is that it is easier to solve the boundary-value problems of this paper by using (open) line integrals instead of the closed contour integrals used by Mackie. This destroys some of the flexibility and elegance of his method, but the loss is compensated by the ease with which the effect of discontinuous boundary conditions may be displayed.)

A one-parameter set of solutions of (1·14) and (1·15) is given by
$$u = (\xi-\lambda)^{-\frac{1}{2}}(\xi-\mu)^{-\frac{1}{2}}, \quad v = \tfrac{1}{2}(\lambda+\mu-2\xi)(\xi-\lambda)^{-\frac{1}{2}}(\xi-\mu)^{-\frac{1}{2}}, \tag{1·16}$$
where $\xi > \lambda > \mu$. The corresponding solution of (1·8) is given by
$$\frac{\partial \psi}{\partial \alpha}\operatorname{cosec}\alpha = (\xi^2 - 2\xi\cos\alpha\cos\theta + \cos^2\alpha + \cos^2\theta - 1)^{-\frac{1}{2}},$$
$$\frac{\partial \psi}{\partial \theta}\sin\theta = (\cos\alpha\cos\theta - \xi)(\xi^2 - 2\xi\cos\alpha\cos\theta + \cos^2\alpha + \cos^2\theta - 1)^{-\frac{1}{2}},$$

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which lead to

$$\psi = -\cosh^{-1}\frac{\cos\alpha - \xi\cos\theta}{\sin\theta(1-\xi^2)^{\frac{1}{2}}}, \tag{1·17}$$

$$= -\log\frac{\cos\alpha - \xi\cos\theta + (\xi^2 - 2\xi\cos\alpha\cos\theta + \cos^2\alpha + \cos^2\theta - 1)^{\frac{1}{2}}}{\sin\theta\, e^{-i\frac{1}{2}\pi}(\xi^2-1)^{\frac{1}{2}}}, \tag{1·18}$$

$$= -\cosh^{-1}\frac{(c/\omega) - \xi\cos\theta}{\sin\theta(1-\xi^2)^{\frac{1}{2}}}. \tag{1·19}$$

In interpreting the functions on the right, ξ is to be regarded as the limit as $\eta \to 0+$ of a complex variable $\zeta = \xi + i\eta$ defined in the upper half-plane, and the radicals in (1·18) are taken to be those branches which are positive for large, real, positive ξ.

Note that, although the solution was deduced for $\omega > c$, the form (1·19) is equally applicable for $\omega < c$.

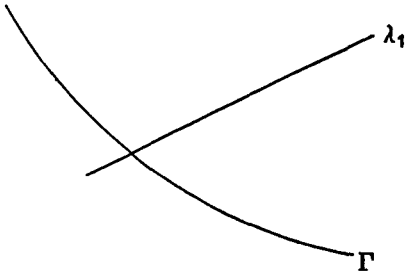


Fig. 2. Boundary conditions for a simple wave.

The solutions to the problems in this series of papers depend on the use of solutions obtained from (1·19) either by integration with respect to a real variable ξ or by integration along a line in the plane of a complex variable $\zeta = \xi + i\eta$ $(\eta \geqslant 0)$.

2. *The solution of boundary-value problems.* It is helpful to recall, at this point, the type of boundary-value problem which is associated with the standard second-order wave equation, and with Laplace's equation.

For the wave equation, three boundary-value problems, and for the elliptic equation one, are of common occurrence.

(i) *The 'simple wave' problem.* For the one-dimension wave equation,

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0,$$

there are simple wave solutions, $\phi = \phi(x-ct)$. Such a solution can be determined when ϕ is constant on a given characteristic, $x - ct = \lambda_1$, and the value of ϕ is given as

$$\phi = \phi_0(s)$$

on a finite open curve, Γ, $x = x(s)$, $x = y(s)$ which intersects that characteristic once, and is nowhere tangent to a characteristic of the same family (Fig. 2). For then it is sufficient to take

$$\phi(x-ct) = \phi_0\{x(s) - ct(s)\}.$$

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(ii) *The Cauchy problem.* When $\partial\phi/\partial x = \phi_0(s)$, $\partial\phi/\partial t = \phi_1(s)$ are given on a finite open curve Γ, which cuts no characteristic line twice, the solution is again easily found in the form

$$\phi = f(x-ct)+g(x+ct)$$

with

$$f'\{x(s)-ct(s)\}+g'\{x(s)+ct(s)\} = \phi_0(s),$$
$$-cf'\{x(s)-ct(s)\}+cg'\{x(s)+ct(s)\} = \phi_1(s).$$

(iii) *The two-characteristic problem.* When ϕ is given on finite portions of two intersecting characteristics, the solution is found by a similar method.

(iv) For an elliptic equation (e.g. Laplace's equation) only the Dirichlet problem need be considered. Here ϕ *or* one of its derivatives is given at each point of a closed curve, and the function ϕ is then determined in the region enclosed by the curve.

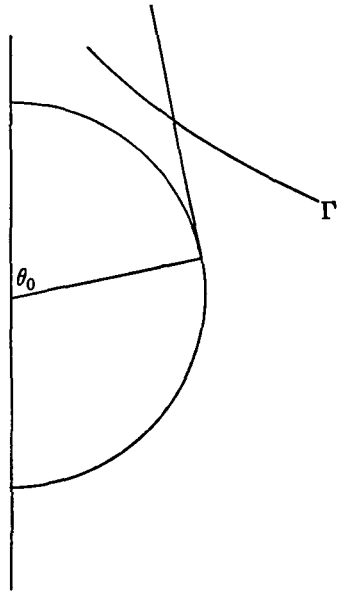


Fig. 3. A simple wave in axial symmetry.

For the mixed-type equation under consideration here each of these problems may be reduced to a corresponding problem in integral equations.

(i) *The 'simple wave' problem.* Let $\psi = \psi(s)$ be given on a curve Γ, $\alpha = \alpha(s)$, $\theta = \theta(s)$, nowhere tangent to a characteristic $\theta-\alpha$ constant, with $\psi = \psi_0$ constant on an intersecting characteristic $\cos(\theta-\alpha) = \cos\theta_0$, where

$$\cos\theta_0 > \cos\{\theta(s)+\alpha(s)\} \quad \text{(Fig. 3).} \tag{2·1}$$

Then the expression

$$\psi = \psi_0 + \int_{\cos\theta_0}^{\cos(\theta-\alpha)} f(\xi)\cosh^{-1}\frac{\cos\alpha-\xi\cos\theta}{\sin\theta(1-\xi^2)^{\frac{1}{2}}}\,d\xi \tag{2·2}$$

satisfies both boundary conditions, for $\cos\theta_0 < \cos(\theta-\alpha)$, provided that

$$\psi(s) = \psi_0 + \int_{\cos\theta_0}^{\cos\{\theta(s)-\alpha(s)\}} f(\xi)\cosh^{-1}\frac{\cos\alpha(s)-\xi\cos\theta(s)}{\sin\theta(s)(1-\xi^2)^{\frac{1}{2}}}\,d\xi. \tag{2·3}$$

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Note that the integrand has a singularity (compare (1·18)) at the upper limit of integration, but the conditions have been so restricted that there is no other singular point in the range of integration.

(ii) *The Cauchy problem.* Suppose $\partial\psi/\partial\alpha$, $\partial\psi/\partial\theta$ given on a curve Γ, $\alpha = \alpha(s)$, $\theta = \theta(s)$, $s_0 < s < s_1$ intersecting no characteristic twice (Fig. 4).

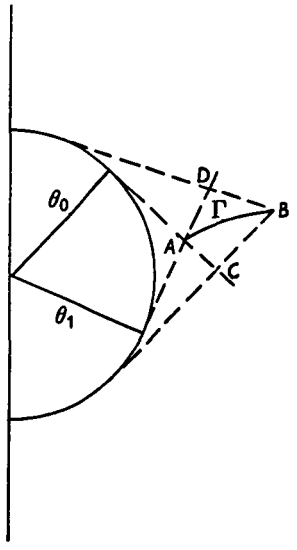


Fig. 4. Cauchy boundary-value problem.

Then a solution valid in an appropriate characteristic quadrilateral $ABCD$ as shown is given by

$$\psi = \int_{\cos\theta_0}^{\cos(\theta-\alpha)} f_1(\xi)\cosh^{-1}\frac{\cos\alpha-\xi\cos\theta}{\sin\theta(1-\xi^2)^{\frac{1}{2}}}d\xi + \int_{\cos(\theta+\alpha)}^{\cos\theta_1} f_2(\xi)\cosh^{-1}\frac{\cos\alpha-\xi\cos\theta}{\sin\theta(1-\xi^2)^{\frac{1}{2}}}d\xi, \tag{2·4}$$

provided that $f_1(\xi)$, $f_2(\xi)$ can be chosen to satisfy the simultaneous integral equations

$$\left(\frac{\partial\psi}{\partial\alpha}\right)_{\alpha(s),\theta(s)} = \int_{\cos\theta_0}^{\cos\{\theta(s)-\alpha(s)\}} \frac{\sin\alpha(s)\,f_1(\xi)\,d\xi}{\{\xi^2-2\xi\cos\alpha(s)\cos\theta(s)+\cos^2\alpha(s)+\cos^2\theta(s)-1\}^{\frac{1}{2}}} + \int_{\cos\{\theta(s)+\alpha(s)\}}^{\cos\theta_1} \frac{\sin\alpha(s)\,f_2(\xi)\,d\xi}{\{\xi^2-2\xi\cos\alpha(s)\cos\theta(s)+\cos^2\alpha(s)+\cos^2\theta(s)-1\}^{\frac{1}{2}}}$$

and

$$\left(\frac{\partial\psi}{\partial\theta}\right)_{\alpha(s),\theta(s)} = \int_{\cos\theta_0}^{\cos\{\theta(s)-\alpha(s)\}} \frac{\{\cos\alpha(s)\cos\theta(s)-\xi\}\operatorname{cosec}\theta(s)\,f_1(\xi)\,d\xi}{\{\xi^2-2\xi\cos\alpha(s)\cos\theta(s)+\cos^2\alpha(s)+\cos^2\theta(s)-1\}^{\frac{1}{2}}} + \int_{\cos\{\theta(s)+\alpha(s)\}}^{\cos\theta_1} \frac{\{\cos\alpha(s)\cos\theta(s)-\xi\}\operatorname{cosec}\theta(s)\,f_2(\xi)\,d\xi}{\{\xi^2-2\xi\cos\alpha(s)\cos\theta(s)+\cos^2\alpha(s)+\cos^2\theta(s)-1\}^{\frac{1}{2}}}.$$

Note that it is not necessary to differentiate at the variable limit of either integral because the integrands in (2·4) are zero at those points, also that differentiation under

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the integral sign needs to be justified by assuming appropriate restrictions on $f_1(\xi)$, $f_2(\xi)$, as usual in integral equation methods.

(iii) The method of solution of the two-characteristic problem is a simple variant on this.

(iv) The solution of a Dirichlet problem in the elliptic region may be reduced to an integral equation in a similar manner.

Suppose that ψ is given on a closed curve Γ_1 ($\omega = \omega(s)$, $\theta = \theta(s)$) in the region $\omega \leqslant c$. Let the curve enclose a region R_1.

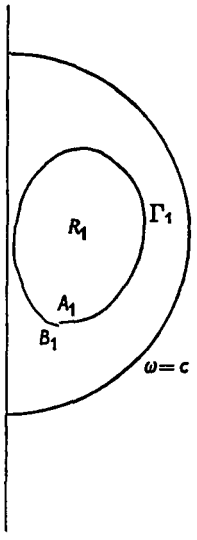


Fig. 5. Dirichlet boundary problem in the ω, θ plane.

Choose a solution

$$\psi = \Re \int_{A,B}^{\cosh(\beta - i\theta)} F(\zeta) \cosh^{-1} \frac{\cosh\beta - \zeta\cos\theta}{\sin\theta(1-\zeta^2)^{\frac{1}{2}}} d\zeta, \tag{2·5}$$

where

(a) $F(\zeta)$ is defined and non-singular in a region, R, of the complex ζ-plane including all points

$$\zeta = \cosh(\beta - i\theta), \tag{2·6}$$

where $\beta = -\operatorname{sech}^{-1}(\omega/c)$, for which β, θ is an interior point of the region R_1 (Fig. 5).

(b) A, B are the points corresponding, in the sense of (2·6), to two adjacent points A_1, B_1 of Γ_1. (These points may be chosen arbitrarily, but for any given problem there is generally one obvious choice which reduces the problem to its simplest terms.)

(c) The path of integration is an open curve Γ, starting at A, keeping the point $\zeta = \cosh(\beta - i\theta)$ on the left, and ending at B, and there is a cut in the plane, from $\zeta = \cosh(\beta - i\theta)$, passing between A and B (Fig. 6). Note that $\beta < 0$, $0 \leqslant \theta \leqslant \pi$ so

$$\eta = \Im(\zeta) \geqslant 0.$$

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In the problems to be discussed, Γ_1 is either the axis $\theta = 0, \theta = \pi$ and the semi-circle $\omega = c$, or the axis $\theta = 0$ or π, the line $\theta = \frac{1}{2}\pi$ and a quarter circle $\omega = c$. Then R is either the upper half of the ζ-plane or one quadrant of it.

The boundary conditions now lead to an integral equation for either the real part or the imaginary part of $F(\zeta)$ at each point of Γ, and once this is solved the determination of $F(\zeta)$ becomes a Dirichlet problem for an analytic function.

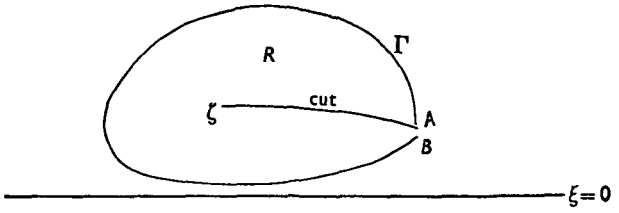


Fig. 6. Dirichlet boundary problem in the ζ-plane.

3. *Sources in an acoustic medium.* The motion of an ideal compressible fluid may be described by a velocity potential ϕ, from which the velocity $\mathbf{q}$ is deduced by

$$\mathbf{q} = \nabla\phi, \tag{3·1}$$

or by a condensation ψ defined by

$$\psi = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}, \tag{3·2}$$

with

$$\nabla\psi = -\frac{1}{c^2}\frac{\partial\mathbf{q}}{\partial t}. \tag{3·3}$$

The density, in the linearized theory, is then

$$\rho(1+\psi)$$

where ρ is the density in equilibrium, and the excess of pressure over the equilibrium value is

$$p = \rho c^2\psi. \tag{3·4}$$

Both ϕ and ψ satisfy the wave equation with wave velocity $c = (dp/d\rho)^{\frac{1}{2}}$,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)(\phi, \psi) = 0. \tag{3·5}$$

Now consider the motion due to sources distributed along the axis of symmetry. Then for a line source of constant strength, the velocity falls off like $(r\sin\theta)^{-1}$ as $r \to \infty$, $\theta \neq 0, \pi$. A dynamically similar solution with

$$\phi = \phi(\omega, \theta) \tag{3·6}$$

will lead to

$$q_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \quad q_r = \frac{\partial\phi}{\partial r} = \frac{1}{r\omega}\frac{\partial\phi}{\partial\omega},$$

and these components also fall off like r^{-1}, so (3·6) is an appropriate form for a source distribution on the axis which has constant strength as $t \to \infty$.

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Consider therefore the following problem.

To find the velocity potential $\phi(\omega, \theta)$ such that

$$\text{(i)} \quad \phi \to 0, \quad \omega \to \infty, \quad 0 < \theta < \pi, \tag{3·7}$$

$$\left.\begin{aligned} \text{(ii)} \quad \sin\theta(\partial\phi/\partial\theta) &\to k(z/t), \qquad \theta \to 0, \quad \omega \to z/t > 0 \\ &\to -k(-z/t), \quad \theta \to \pi, \quad \omega \to -z/t > 0. \end{aligned}\right\} \tag{3·8, 3·9}$$

The condition (i) is appropriate to motion from rest at $t = 0$. The conditions (3·8) (3·9) imply sources on the axis of symmetry (Cartesian coordinate z) such that the mass issuing from a length δz in a time δt is

$$2\pi k(z/t)\,\delta z\,\delta t \quad (t > 0). \tag{3·10}$$

The solution may be found in four stages, corresponding to the regions indicated in Fig. 7.

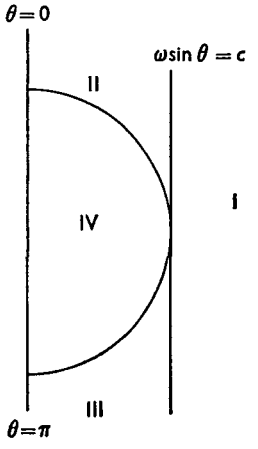


Fig. 7. Division of a meridian plane for a line of sources.

In region I the solution to the Cauchy problem for $\partial\phi/\partial\theta = \partial\phi/\partial\omega = 0$ on a line parallel to the axis at a large distance is

$$\phi \equiv 0, \quad \omega\sin\theta > c. \tag{3·11}$$

In region II, $\omega > c$, $0 < \theta < \sin^{-1}c/\omega$, the solution is a simple wave, of the form

$$\phi = \int_0^{\cos(\theta+\alpha)} f_1(\xi)\cosh^{-1}\frac{\cos\alpha - \xi\cos\theta}{\sin\theta(1-\xi^2)^{\frac{1}{2}}}\,d\xi, \tag{3·12}$$

(compare (2·2)). The boundary condition (3·8) gives

$$\sin\theta\frac{\partial\phi}{\partial\theta} = \int_0^{\cos(\theta+\alpha)} \frac{(\xi - \cos\alpha\cos\theta)\,f_1(\xi)\,d\xi}{(\xi^2 - 2\xi\cos\alpha\cos\theta + \cos^2\alpha + \cos^2\theta - 1)^{\frac{1}{2}}} \to k(\omega) \quad (\theta \to 0), \tag{3·13}$$

where $\sec\alpha = \omega/c$. Now as $\theta \to 0$ the radical becomes $(\xi - \cos\alpha)$ so

$$\begin{aligned} \int_0^{\cos\alpha} f_1(\xi)\,d\xi &= \int_0^{c/\omega} f_1(\xi)\,d\xi \\ &= k(\omega) \end{aligned}$$

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and
$$-(c/\omega^2) f_1(c/\omega) = k'(\omega),$$
$$f_1(\xi) = -(c/\xi^2) k'(c/\xi) \quad (0 < \xi < 1). \tag{3·14}$$

Similarly in region III, $\omega > c$, $\pi > \theta > \pi - \sin^{-1}(c/\omega)$, the solution is the simple wave

$$\phi = \int_{\cos(\theta-\alpha)}^{0} f_2(\xi) \cosh^{-1} \frac{\cos\alpha - \xi\cos\theta}{\sin\theta(1-\xi^2)^{\frac{1}{2}}} d\xi \tag{3·15}$$

and the boundary condition

$$\sin\theta \frac{\partial\phi}{\partial\theta} = \int_{\cos(\theta-\alpha)}^{0} \frac{(\xi - \cos\alpha\cos\theta) f_2(\xi)\, d\xi}{(\xi^2 - 2\xi\cos\alpha\cos\theta + \cos^2\theta + \cos^2\alpha - 1)^{\frac{1}{2}}} \to -k(-\omega) \quad (\theta \to \pi) \tag{3·16}$$

gives
$$\int_{-c/\omega}^{0} f_2(\xi)\, d\xi = -k(-\omega),$$

i.e.
$$f_2(\xi) = -(c/\xi^2) k'(c/\xi) \quad (0 > \xi > -1). \tag{3·17}$$

The solution in region IV, $\omega < c$, must now satisfy boundary conditions (3·8), (3·9) on the axis and be continuous with the solutions (3·12), (3·15) as $\omega \to c$. Take the cut in the ζ plane from 0 to $\cosh(\beta - i\theta)$, and the solution in the form

$$\phi = \tfrac{1}{2}\Re \int_{0+,\,0-}^{\cosh(\beta - i\theta)} F(\zeta) \cosh^{-1} \frac{\cosh\beta - \zeta\cos\theta}{\sin\theta(1-\zeta^2)^{\frac{1}{2}}} d\zeta, \tag{3·18}$$

with the same convention as in (1·18) as to the signs of radicals. The continuity condition as $\omega \to c$, $\beta \to 0$ is then expressed by the two relations

$$\int_0^{\cos\theta} f_1(\xi) \cosh^{-1} \frac{1-\xi\cos\theta}{\sin\theta(1-\xi^2)^{\frac{1}{2}}} d\xi = \tfrac{1}{2}\Re \int_{0+,\,0-}^{\cos\theta} F(\zeta) \cosh^{-1} \frac{1-\zeta\cos\theta}{\sin\theta(1-\zeta^2)^{\frac{1}{2}}} d\zeta$$
$$(0 < \theta < \tfrac{1}{2}\pi) \tag{3·19}$$

and
$$\int_{\cos\theta}^{0} f_1(\xi) \cosh^{-1} \frac{1-\xi\cos\theta}{\sin\theta(1-\xi^2)^{\frac{1}{2}}} d\xi = \tfrac{1}{2}\Re \int_{0+,\,0-}^{\cos\theta} F(\zeta) \cosh^{-1} \frac{1-\zeta\cos\theta}{\sin\theta(1-\zeta^2)^{\frac{1}{2}}} d\zeta$$
$$(\tfrac{1}{2}\pi < \theta < \pi). \tag{3·20}$$

Now in (3·19) the contour integral may be evaluated by using a path of integration, as shown in Fig. 8, consisting of the real axis, from 0 to $\cos\theta$ described twice, once on each side of the cut, together with a loop encircling the singularity P ($\zeta = \cosh(\beta - i\theta)$) but not Q ($\zeta = \cosh(\beta + i\theta)$), as $\beta \to 0$.

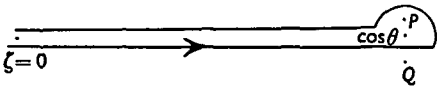


Fig. 8. Contour integral for $\omega = c$, $\theta < \frac{1}{2}\pi$.

Then on the lower side of the cut, $\zeta = \xi_1$ (real)

$$\cosh^{-1} \frac{1-\zeta\cos\theta}{\sin\theta(1-\zeta^2)^{\frac{1}{2}}} \to \log \frac{1-\cos\theta + (\cos\theta - \xi_1)}{\sin\theta(1-\xi_1^2)^{\frac{1}{2}}} = -\cosh^{-1} \frac{1-\xi_1\cos\theta}{\sin\theta(1-\xi_1^2)^{\frac{1}{2}}}$$

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and on the upper side

$$\cosh^{-1}\frac{1-\zeta\cos\theta}{\sin\theta(1-\zeta^2)^{\frac{1}{2}}}\to+\cosh^{-1}\frac{1-\xi_1\cos\theta}{\sin\theta(1-\xi_1^2)^{\frac{1}{2}}}.$$

Then (3·19) is satisfied if, for real ξ_1

$$\Re F(\xi_1)=-f_1(\xi_1)\quad(0<\xi_1<1). \tag{3·21}$$

Similarly use of a contour as shown in Fig. 9 shows that a sufficient condition of continuity is

$$\Re F(\xi_1)=-f_2(\xi_1)\quad(-1<\xi_1<0). \tag{3·22}$$

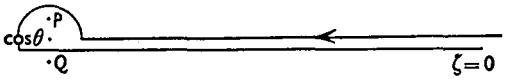


Fig. 9. Contour integral for $\theta\to 0$.

Then from (3·14), (3·17), (3·21) and (3·22)

$$\Re F(\xi_1)=-(c/\xi_1^2)\,k'(c/\xi_1)\quad(-1<\xi_1<1). \tag{3·23}$$

Next use the conditions on the axis, $\theta\to 0$,

$$\sin\theta\frac{\partial\phi}{\partial\theta}=\frac{1}{2}\int_{+0,\,-0}^{\cosh(\beta-i\theta)}\frac{F(\zeta)(\zeta-\cos\theta\cosh\beta)\,d\zeta}{(\zeta^2-2\zeta\cos\theta\cosh\beta+\cos^2\theta+\cosh^2\beta-1)^{\frac{1}{2}}}\to-k(\omega)\quad(\theta\to 0).$$



Fig. 10. Contour integral for $\theta\to 0$.

The appropriate contour is now similar to that in Fig. 8, but indented at $\zeta=1$ (Fig. 10) and the condition reduces to

$$\Re\int_0^{c/\omega}F(\zeta)\,d\zeta=-k(\omega),$$

$$\Re F(\xi_1)=-(c/\xi_1^2)\,k'(c/\xi_1)\quad(1<\xi_1<\infty). \tag{3·24}$$

The condition on $\theta=\pi$ leads similarly to

$$\Re F(\xi_1)=-(c/\xi_1^2)\,k'(c/\xi_1)\quad(-\infty<\xi_1<-1). \tag{3·25}$$

Equations (3·23), (3·24), (3·25) define $\Re F(\xi_1)$ for all real ξ_1.

Assume (for the moment) that

$$F(\zeta)=O(\zeta^{-1-\delta})\quad(\delta>0,\ |\zeta|\to\infty), \tag{3·26}$$

then use of Cauchy's Theorem gives

$$F(\zeta)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\Re\frac{F(\xi_1)\,d\xi_1}{\xi_1-\zeta}\quad(\Im(\zeta)>0), \tag{3·27}$$

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and the solution of the problem is complete. Note that (3·18) implies

$$q_\theta = \frac{\partial \phi}{r\partial\theta} = \Re \frac{1}{2r} \int_{0+,0-}^{\cosh(\beta - i\theta)} \frac{(\zeta - \cos\theta \cosh\beta) F(\zeta)\, d\zeta}{\sin\theta (\zeta^2 - 2\zeta \cos\theta \cosh\beta + \cos^2\theta + \cosh^2\beta - 1)^{\frac{1}{2}}},$$

$$q_r = \frac{\partial \phi}{\partial r} = \Re \frac{-\cosh\beta}{2r} \int_{0+,0-}^{\cosh(\beta - i\theta)} \frac{F(\zeta)\, d\zeta}{(\zeta^2 - 2\zeta \cos\theta \cosh\beta + \cos^2\theta + \cosh^2\beta - 1)^{\frac{1}{2}}}.$$

The condition (3·26) is the condition that the integrals converge as $\cosh\beta \to \infty$, and so implies that the velocity components are not of greater order than those due to a line source, as $r \to 0$. (In fact the condition may be relaxed to $\delta \geqslant 0$, provided that the real part of each integral remains finite as $\cosh\beta \to \infty$. The theorem (3·27) can then be used for $\{F(\zeta) - A\zeta^{-1}\}$, where $F(\zeta) \to A\zeta^{-1}$ as $\zeta \to \infty$.)

4. *The case of a uniform line source.* The solution given in § 3 is a formal solution of the problem, depending as it does on the validity of certain interchanges of limiting operations. There is, however, one special case of some importance where even the formal solution may be questioned. This is the case of a line source of uniform strength over some interval of the axis, with no sources outside the region.

$$\left.\begin{aligned} k(z/t) &= k_0 \quad (a < z/t < b), \\ &= 0 \quad (z/t < a,\ z/t > b). \end{aligned}\right\} \tag{4·1}$$

It is convenient to assume $a \leqslant 0, b \geqslant 0$. The corresponding results for other cases follow similarly. If now $k(z/t)$ is regarded as the difference of two Hamilton step functions, $k_0'(z/t)$ becomes the difference of impulse functions, and (3·12) is replaced by

$$\phi = -k_0 \cosh^{-1} \frac{\cos\alpha - \cos\theta \cos\alpha_0}{\sin\theta \sin\alpha_0}, \quad \cos(\theta + \alpha) > (c/b) = \cos\alpha_0 \tag{4·2}$$

if $b > c$ (and if $b < c$ the potential ϕ is zero for all $\omega > c$, $\theta < \frac{1}{2}\pi$). Similarly (3·15) is replaced by

$$\phi = -k_0 \cosh^{-1} \frac{\cos\alpha - \cos\theta \cos\alpha_1}{\sin\theta \sin\alpha_1}, \quad \cos(\theta - \alpha) < +\cos\alpha_1 = c/a \tag{4·3}$$

for $a < -c$ (and if $a > -c$, $\phi \equiv 0$, $\omega > c$, $\theta > \frac{1}{2}\pi$), where, in both (4·2) and (4·3), the positive value of $\cosh^{-1}$ is taken.

Now the analytic function for which the real part on the real axis corresponds to an impulse function at $\zeta = b$ is $-i(\zeta - b)^{-1}$, so the form for ϕ in $\omega < c$ becomes

$$\phi = \Re \frac{k_0}{2i\pi} \int_{0+,0-}^{\cosh(\beta - i\theta)} \left\{\frac{1}{\zeta - c/a} - \frac{1}{\zeta - c/b}\right\} \cosh^{-1} \frac{\cosh\beta - \zeta\cos\theta}{\sin\theta (1 - \zeta^2)^{\frac{1}{2}}}\, d\zeta, \tag{4·4}$$

with the convention,

$$\cosh^{-1} \frac{\cosh\beta - \zeta\cos\theta}{\sin\theta (1 - \zeta^2)^{\frac{1}{2}}}$$

$$= \log \frac{\cosh\beta - \zeta\cos\theta + (\zeta^2 - 2\zeta\cos\theta \cosh\beta + \cos^2\theta + \cosh^2\beta - 1)^{\frac{1}{2}}}{\sin\theta\, e^{-\frac{1}{2}i\pi} (\zeta^2 - 1)^{\frac{1}{2}}},$$

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using the branches of the radicals which have argument zero for large real positive ζ, and assuming $\Im(\zeta) \to 0+$ for real ζ. The boundary and continuity conditions are now easily checked by using contours like those of Figs. 8, 9, 10 but with indentations at

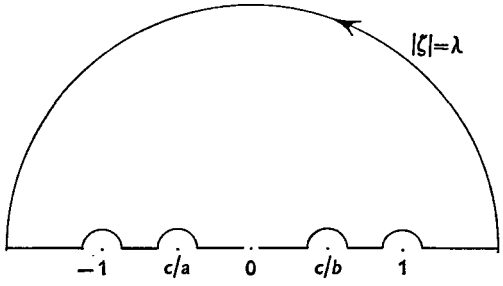


Fig. 11. Contour integral valid for $\omega < c$.

$\zeta = c/b$, $\zeta = c/a$. Alternatively, (4·4) may be evaluated by using a contour (Fig. 11) consisting of the real axis, indented at $\zeta = c/b$, $\zeta = c/a$, $\zeta = \pm 1$, together with a semi-circle of large radius λ. Then the contributions to ϕ from the residues at c/b, c/a are

$$\frac{k_0}{2}\log\frac{\cosh\beta-(c/b)\cos\theta+\{(c/b)^2-2(c/b)\cos\theta\cosh\beta+\cos^2\theta+\cosh^2\beta-1\}^{\frac{1}{2}}}{\sin\theta\,|1-(c/b)^2|^{\frac{1}{2}}}$$

and

$$-\frac{k_0}{2}\log\frac{\cosh\beta-(c/a)\cos\theta+\{(c/a)^2-2(c/a)\cos\theta\cosh\beta+\cos^2\theta+\cosh^2\beta-1\}^{\frac{1}{2}}}{\sin\theta\,|1-(c/a)^2|^{\frac{1}{2}}},$$

and the integrals 1 to λ and $-\lambda$ to 1 give

$$\frac{k_0}{4}\log\left|\frac{1-c/b}{1+c/b}\right|\cdot\left|\frac{1+c/a}{1-c/a}\right|$$

(where the modulus signs are included to ensure that the resulting solution is valid for all a, b for which $b > a$). There is no contribution from the semi-circle, as $\lambda \to \infty$, so finally

$$\phi = -\frac{k_0}{2}\log\left\{\frac{\cosh\beta-(c/b)\cos\theta+\{(c/b)^2-2(c/b)\cos\theta\cosh\beta+\cos^2\theta+\sinh^2\beta\}^{\frac{1}{2}}}{\cosh\beta-(c/a)\cos\theta+\{(c/a)^2-2(c/a)\cos\theta\cosh\beta+\cos^2\theta+\sinh^2\beta\}^{\frac{1}{2}}}\times\left|\frac{1+(c/a)}{1+(c/b)}\right|\right\}, \quad (4\cdot5)$$

for $\omega < c$.

To check that this agrees with (4·2), (4·3) try, for example, $\beta = 0$, $\cos\theta > c/b > 1$. Then (4·5) gives

$$-\frac{k_0}{2}\log\frac{1-(c/b)\cos\theta+\cos\theta-c/b}{1-(c/a)\cos\theta-\{\cos\theta-c/a\}}\,\frac{1+c/a}{1+c/b}=\frac{k_0}{2}\log\frac{1+\cos\theta}{1-\cos\theta}\,\frac{1-c/b}{1+c/b}$$

$$=-k_0\log\frac{1+\cos\theta}{\sin\theta}\left(\frac{1-c/b}{1+c/b}\right)^{\frac{1}{2}}.$$

But from (4·2) $$\phi = -k_0\log\frac{1-(c/b)\cos\theta+\cos\theta-c/b}{\sin\theta\,(1-c^2/b^2)^{\frac{1}{2}}}$$

and the two results agree.

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Note that if now $a \to -\infty$, $b \to \infty$, (4·2), (4·3) and (4·5) are comprehended by the single formula

$$\phi = k_0 \cosh^{-1} \frac{ct}{r \sin\theta} \quad (r \sin\theta < ct) \tag{4·6}$$

and, as one would expect, the potential depends only on the cylindrical polar radius $r \sin\theta$.

5. *The sudden motion of an infinite cone.* An important problem with some resemblance to that of § 4 is that of the motion of a thin cone in the direction of its axis.

Suppose a thin cone, of semi-angle δ, initially at rest in a fluid at rest, commences to move with velocity V in the direction of its axis. Take the origin of spherical polar coordinates at the initial position of the vertex, then it is reasonable to expect a solution corresponding to sources on the axis. Use z as the Cartesian coordinate in the direction of motion. Then the thickness of the cone at z is $2(Vt-z)\tan\delta$, and the normal velocity of the surface is $V \sin\delta$. The boundary condition on the cone is therefore

$$q_\theta \cos(\theta+\delta) + q_r \sin(\theta+\delta) = V \sin\delta, \tag{5·1}$$

at

$$r \sin\theta = (Vt-z)\tan\delta = (Vt - r\cos\theta)\tan\delta. \tag{5·2}$$

This condition may be replaced by

$$r \sin\theta q_\theta \to V\delta^2(Vt-r) \quad (\theta \to 0,\ r < Vt), \tag{5·3}$$

$$r \sin\theta q_\theta \to -V\delta^2(Vt+r) \quad (\theta \to \pi,\ r > 0), \tag{5·4}$$

for sufficiently small δ. Now differentiate these conditions with respect to t; then

$$r \sin\theta \frac{\partial q_\theta}{\partial t} \to \pm V^2\delta^2, \quad \theta \to \frac{\pi}{2} \mp \frac{\pi}{2}; \tag{5·5}$$

but

$$\frac{\partial q_\theta}{\partial t} = -\frac{c^2}{r}\frac{\partial \psi}{\partial \theta},$$

where ψ is the condensation, so (5·5) is equivalent to

$$\sin\theta \frac{\partial \psi}{\partial \theta} \to \mp \frac{V^2}{c^2}\delta^2, \quad \theta \to \frac{\pi}{2} \mp \frac{\pi}{2}. \tag{5·6}$$

The analogy between (5·6) and (3·8), (3·9), and the fact that ϕ, ψ satisfy the same partial differential equation, implies that the solution for ψ is given by replacing ϕ by ψ in (4·2), (4·3), (4·6) and taking $k_0 = (-V^2\delta^2/c^2)$, $b = V$ and $a \to -\infty$. Then

$$\psi = +\frac{V^2\delta^2}{c^2} \cosh^{-1} \frac{\cos\alpha - (c/V)\cos\theta}{\sin\theta\,(1-c^2/V^2)^{\frac{1}{2}}} \quad (\omega > c,\ \cos(\theta+\alpha) > c/V), \tag{5·7}$$

$$\psi = +\frac{V^2\delta^2}{c^2} \cosh^{-1} \frac{\cos\alpha}{\sin\theta} \quad (\omega > c,\ \cos(\theta-\alpha) < 0) \tag{5·8}$$

and

$$\psi = +\frac{V^2\delta^2}{2c^2} \log \frac{\cosh\beta - (c/v)\cos\theta + |(c^2/V^2) - 2(c/V)\cos\theta\cosh\beta + \cos^2\theta + \sinh^2\beta|^{\frac{1}{2}}}{\{\cosh\beta + |\cos^2\theta + \sinh^2\beta|^{\frac{1}{2}}\}(1+c/v)} \tag{5·9}$$

for $\omega < c$.

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The most important result is that for the excess pressure on the cone. Now (5·7), (5·9), (5·8) give

$$\psi \to \frac{V^2\delta^2}{c^2}\log\frac{2c(V-\omega)}{\omega\sin\theta(V^2-c^2)^{\frac{1}{2}}} \quad (c<\omega<V,\ \theta\to 0), \tag{5·10}$$

$$\psi \to \frac{V^2\delta^2}{2c^2}\log\frac{4c^2(V-\omega)}{\omega^2(V+c)\sin^2\theta} \quad (0<\omega<c,\ \theta\to 0), \tag{5·11}$$

$$\psi \to \frac{V^2\delta^2}{2c^2}\log\frac{4c^2(V+\omega)}{\omega^2(V+c)\sin^2\theta} \quad (0<\omega<c,\ \theta\to \pi), \tag{5·12}$$

$$\psi \to \frac{V^2\delta^2}{c^2}\log\frac{2c}{\omega\sin\theta} \quad (\omega>c,\ \theta\to\pi). \tag{5·13}$$

Use a Cartesian coordinate z on the axis as before, then the dominant term in each of (5·10) to (5·13) is

$$-(V^2\delta^2)\log\delta \quad (z<Vt) \tag{5·14}$$

and if the next term is included in the formula for the pressure,* $p=\rho c^2\psi$, on the cone the complete result is

$$\left.\begin{aligned} p-\rho V^2\delta^2(\log\delta^{-1}) &= \rho V^2\delta^2\log\{2c(V^2-c^2)^{-\frac{1}{2}}\} && (ct<z<Vt),\\ &= \rho V^2\delta^2\log\{2c(V-z/t)^{-\frac{1}{2}}(V+c)^{-\frac{1}{2}}\} && (-ct<z<ct),\\ &= \rho V^2\delta^2\log\{2c(V-z/t)^{-1}\} && (z<-ct). \end{aligned}\right\} \tag{5·15}$$

Equation (5·15) is now correct, neglecting terms $O(\delta^3)$.

6. *The sudden motion of an axisymmetric body.* The results of § 5 may easily be extended to give the flow due to sudden motion from rest of a thin axially symmetric rigid body of any given shape, provided that the velocity is parallel to the axis of symmetry.

Use cylindrical polar coordinates, z, ϖ, and write the solution expressed by (5·7), (5·8), (5·9) as

$$\psi = V^2\delta^2\psi_0\left(\frac{z}{t},\frac{\varpi}{t},V\right), \tag{6·1}$$

corresponding to a boundary condition

$$\left.\begin{aligned} \sin\theta\frac{\partial\psi}{\partial\theta} &\to \mp\frac{V^2}{c^2}\delta^2 && (\theta\to 0,\pi;\ z<Vt),\\ &\to 0 && (\theta\to 0,\ z>Vt). \end{aligned}\right\} \tag{6·2}$$

Now consider a rigid body with radius

$$\varpi = \epsilon\varpi_0(\tau) \tag{6·3}$$

* Note that the solution for ψ was obtained from boundary conditions on a derivative of ψ, together with the result $\psi\to 0$, $\omega\to\infty$, $\theta\neq\pi$. However, this last condition was used in the form $\psi=0$ at the border of the disturbed region. To verify the expressions (5·15) completely it is necessary to show that the normal velocity of the fluid tends to zero at the edge of the disturbance, and so, that no jump in pressure is generated. This may be done without difficulty.

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at a distance τ behind the nose. Let the motion commence, with velocity V, from the origin at time $t = 0$ (Fig. 12). Then the corresponding boundary condition is

$$\left.\begin{aligned} \sin\theta\frac{\partial\psi}{\partial\theta} &\to \mp\frac{\epsilon^2}{c^2}V^2\frac{\partial}{\partial(Vt)}\{\varpi_0(Vt-z)\,\varpi_0'(Vt-z)\} \quad (\theta\to 0,\ \pi;\ z<Vt),\\ &\to 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad (\theta\to 0,\ z>Vt), \end{aligned}\right\} \tag{6·4}$$

say

$$\sin\theta\frac{\partial\psi}{\partial\theta} = \mp\frac{\epsilon^2V^2}{c^2}k(Vt-z).$$

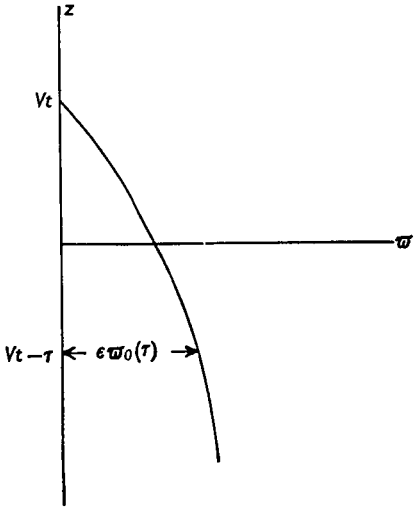


Fig. 12. Motion of an axisymmetric body.

Define $H(\xi)$ as the step-function

$$H(\xi) = \Re\lim_{\eta\to 0+}\frac{1}{\pi i}\log(\xi+i\eta). \tag{6·5}$$

Then

$$\begin{aligned} k(Vt-z) &= \int_{-\infty}^{Vt-z}k'(\lambda)\,d\lambda\\ &= \int_{-\infty}^{\infty}H(z-Vt+\lambda)\,k'(\lambda)\,d\lambda. \end{aligned} \tag{6·6}$$

But (6·2) may be written

$$\sin\theta\frac{\partial\psi}{\partial\theta} \to \mp\frac{V^2\delta^2}{c^2}H(z-Vt)$$

and (6·4) corresponds to

$$\sin\theta\frac{\partial\psi}{\partial\theta} \to \mp\frac{\epsilon^2V^2}{c^2}\int_{-\infty}^{\infty}H(z-\lambda+Vt)\,k'(\lambda)\,d\lambda.$$

So the corresponding condensation is

$$\psi = \epsilon^2V^2\int_{-\infty}^{\infty}k'(\lambda)\,\psi_0\left(\frac{z+\lambda}{t},\ \frac{\varpi}{t},\ V\right)d\lambda, \tag{6·7}$$

where from (6·4)

$$k'(\lambda) = \frac{d^2}{d\lambda^2}(\varpi_0(\lambda)\,\varpi_0'(\lambda)). \tag{6·8}$$

7. *Further generalizations.* It is not possible to obtain solutions for the motion due to accelerated bodies by direct integration from the results of §§ 5 and 6, because a variable velocity $V(t)$ changes the whole form of the solutions found. However it is

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possible to obtain approximate solutions by a step-by-step procedure. It is sufficient to consider the case in which the initial velocity, V_0, changes to a new value V_1 suddenly at $t = t_1$, when the nose is at $z = z_1 = V_0 t_1$. Then the boundary condition on the velocity is

$$r \sin\theta q_\theta \to \pm V_0(z - V_0 t)\, H(z - V_0 t) \quad (\theta = 0, \pi;\ t < t_1) \tag{7·1}$$

and $$r \sin\theta q_\theta \to \pm V_1\{(z - z_1) - V_1(t - t_1)\} H\{(z - z_1) - V_1(t - t_1)\} \quad (\theta = 0, \pi;\ t > t_1). \tag{7·2}$$

But (7·2) could equally well be written as

$$\pm V_0(z - V_0 t)\, H(z - V_0 t)$$
$$\pm V_1\{(z - z_1) - V_1(t - t_1)\} H\{(z - z_1) - V_1(t - t_1)\} - \{z - z_1 - V_0(t - t_1)\} \times H\{(z - z_1) - V_0(t - t_1)\} \quad (t > t), \tag{7·3}$$

since $z_1 = V_0 t$. The solution for ψ is then

$$\psi = \delta^2 V_0^2 \psi_0\left(\frac{z}{t}, \frac{\varpi}{t}, V_0\right) \quad (t < t_1) \tag{7·4}$$

and

$$\psi = \delta^2 V_0^2 \psi_0\left(\frac{z}{t}, \frac{\varpi}{t}, V_0\right) - \delta^2 V_0^2 \psi_0\left(\frac{z - z_1}{t - t_1}, \frac{\varpi}{t - t_1}, V_0\right) + \delta^2 V_1^2 \psi_0\left(\frac{z - z_1}{t - t_1}, \frac{\varpi}{t - t_1}, V_1\right) \quad (t > t_1), \tag{7·5}$$

and the extension to a step-by-step calculation for an accelerated projectile is obvious.

A second obvious generalization is to a sudden change in velocity of a projectile which has been moving steadily at a supersonic speed.

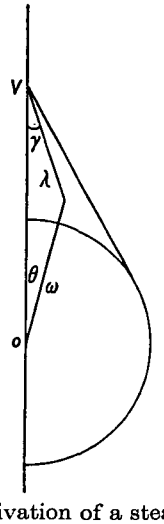


Fig. 13. Derivation of a steady solution.

The solution for steady motion may be obtained by a change of variable in (5·7). Set

$$\omega \cos\theta = (V - \lambda \cos\gamma), \quad \omega \sin\theta = \lambda \sin\gamma$$

(Fig. 13). Then (5·7) is

$$\frac{V^2\delta^2}{c^2} \cosh^{-1} \frac{V - \omega\cos\theta}{\omega \sin\theta (V^2/c^2 - 1)^{\frac{1}{2}}} = \frac{V^2\delta^2}{c^2} \cosh^{-1} \frac{\cot\gamma}{(V^2/c^2 - 1)^{\frac{1}{2}}} \quad (\gamma < \sin^{-1} c/V). \tag{7·6}$$

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It is now only necessary to solve the problem of § 5 with the new boundary conditions

$$\psi \to \frac{V^2\delta^2}{c^2}\cosh^{-1}\frac{\cot(\pi-\theta)}{(V^2/c^2-1)^{\frac{1}{2}}} \quad (\omega\to\infty,\ \theta > \pi-\cot^{-1}(V^2/c^2-1)^{\frac{1}{2}}) \tag{7·7}$$

and

$$\left.\begin{aligned} \sin\theta\frac{\partial\psi}{\partial\theta} &\to \mp\frac{V_1^2\delta^2}{c^2} && (\theta\to 0,\ \pi;\ z < V_1 t),\\ &\to 0 && (\theta\to 0,\ z > V_1 t), \end{aligned}\right\} \tag{7·8}$$

where V_1 is the new velocity and the origin is the position of the vertex at the change in velocity (compare § 9).

REFERENCES

(1) CRAGGS, J. W. *Proc. Roy. Soc. London, Ser. A*, **237** (1956), 372.
(2) CRAGGS, J. W. *J. Fluid Mech.* **3** (1957), 176.
(3) CRAGGS, J. W. *Proc. Cambridge Philos. Soc.* **56** (1960), 269.
(4) MACKIE, A. G. *J. Rat. Mech. and Anal.* **4** (1955), 733.
(5) MILES, J. W. *Quart. Appl. Math.* **18** (1960), 37.
(6) PAPADOPOULOS, V. M. *Proc. Roy. Soc. London, Ser. A*, **255** (1960), 538; 550.

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