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TILINGS OF HILBERT SPACES

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Abstract. It is shown that a separable Hilbert space can be covered by non-overlapping closed convex sets C_i with outer radii uniformly bounded from above and inner radii uniformly bounded from below. This answers a question originating from the work of Klee.

§1. *Introduction.* A cover $\{C_i\}$ of a metric space is said to be a tiling if $C_i \cap C_j$ has empty interior whenever $i \neq j$. The problem of existence of non-trivial tilings of infinite-dimensional Hilbert spaces by closed convex sets was considered by Klee in [8]. He showed that some non-separable Hilbert spaces H contain a subset D such that the Voronoi cells, i.e. the closed convex sets

$$C_d = \{x : \operatorname{dist}(x, d) = \operatorname{dist}(x, D)\},$$

form a tiling with the property that for suitable constants $r, R > 0$, $B(d, r) \subset C_d \subset B(d, R)$. However, the results of [2] imply that no such sets D exist in infinite-dimensional separable Hilbert spaces. This and several other results indicate that non-trivial tilings of such spaces by convex sets must be rather complicated, if they exist at all. For a more detailed description of these results see [3]; here we just mention that in [1] it is proved that such tilings cannot be locally finite, and this result is further refined substantially in [5–7]. In the opposite direction, the important paper [4] presented tilings of any normed space by (non-uniformly) bounded convex sets with uniformly bounded inner radii, as well as tilings by uniformly bounded convex sets. Nevertheless, as pointed out in [3], the problem of tiling an infinite-dimensional separable Hilbert space (or any other reflexive space) by closed convex sets whose outer radii are uniformly bounded from above and whose inner radii are uniformly bounded from below remained open.

Here we solve the above problem by establishing the following theorem.

THEOREM 1. *Every separable Hilbert space H has a tiling by closed convex sets whose outer radii are uniformly bounded from above and whose inner radii are uniformly bounded from below.*

Our construction is based on two sources. As one might expect, we use the standard tiling of finite-dimensional Hilbert spaces by Voronoi cells; this is recalled in Lemma 2. Perhaps less expected is the use, in Lemma 5, of a

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slight modification of the tiling from [4] of the whole space by closed convex sets with unbounded outer radii. We tile the space by closed convex sets, say H_i, such that the set H_0 containing 0 is bounded and contains the ball $B(0, r_0)$, while any other H_i is unbounded but such that its intersection with the cylinder $\{x : \|x - \langle x, e\rangle e\| \le r_0\}$ (where e is a fixed unit vector) contains a ball of radius r independent of i. The main technical point that makes the later part of the argument work is that r_0 is simultaneously the radius of the ball contained in H_0 and the radius of the cylinder inside which we may still find balls witnessing that the H_i have inner radii uniformly bounded from below.

In the main part of the proof of Theorem 1, we use Voronoi cells to tile the finite-dimensional spaces V_k spanned by the vectors $e_0, \dots, e_{k-1}$ from an orthonormal basis and the restriction of the tiling of $V_k^\perp$ from Lemma 5 to the cylinders $\{w \in V_k^\perp : \|x - \langle x, e_k\rangle e_k\| \le R\}$, where R is large enough (to contain zero's set from the tiling of V_k). A simple formula, in which the dual role of R plays a similar part to the dual role of r_0 mentioned above, combines these data into the required tiling of the Hilbert space.

In §3 we point out that, similarly to its predecessor in [4], a variant of Lemma 5 holds in general Banach spaces. This leads to a tiling result for all separable Banach spaces: these spaces can be tiled by closed convex sets C_i such that for each i there is a finite-dimensional subspace V_i with $B(V_i, r) \subset C_i \subset B(V_i, R)$, where r and R are absolute constants. While in the proof of Theorem 1 we gave an argument that could be used to get a reasonable estimate for the radii, here we just use coarse bounds. Thus the proof of the general statement is perhaps slightly simpler than that of Theorem 1. Our arguments and discussions incorporate simplifications and other observations due to Joram Lindenstrauss and Vladimir Fonf, which include, among others, the possibility of using simple tilings as in Figure 2 and that the class of spaces to which the methods described in this note apply encompasses substantially more than just Hilbert spaces.

Most of the notation used in this note should be self-explanatory. The closed ball centred at x with radius r is denoted by $B(x, r)$. As an extension of this notation, we write $B(E, r) = \{x : \operatorname{dist}(x, E) \le r\}$ for any non-empty set E. Although we sometimes work in several spaces, it will usually be clear which space the notation $B(x, r)$ is referring to; however, if we feel that this may cause confusion, we will write $B_W(x, r)$ for a ball in the space W.

§2. *Tilings of separable Hilbert spaces.* For completeness, we first recall that tilings of finite-dimensional Hilbert spaces with uniform bounds on the inner and outer radii are produced by Voronoi cells.

LEMMA 2. *For any finite-dimensional Hilbert space V and any $r > 0$ there exists $S \subset V$ such that the closed convex sets*

$$D_d = \{v \in V : \|v - d\| \le \|v - s\| \text{ for every } s \in S\},$$

where d runs through S, form a tiling of V with $B(d, r) \subset D_d \subset B(d, 2r)$ for every $d \in S$.

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Proof. Let $S \subset V$ be a maximal set such that $\|s - t\| > 2r$ whenever s and t are different points of S. Clearly, the sets D_d ($d \in S$) are closed and convex since the inequality $\|v - d\| \le \|v - s\|$ is equivalent to v belonging to the half-space $\langle v, 2(s - d)\rangle \le \|s\|^2 - \|d\|^2$. Also, $D_d \supset B(d, r)$ and $D_d \subset B(d, 2r)$, with the latter holding because for any $v \in V$ with $\|v - d\| > 2r$ we can find $s \in S$ with $\|v - s\| \le 2r$, so that $v \notin D_d$. If $s, t \in S$ with $s \ne t$, then $D_s \cap D_t$ is contained in the hyperplane $\{v : \langle v, 2(t - s)\rangle = \|t\|^2 - \|s\|^2\}$, so it has empty interior. Finally, $V = \bigcup_{d \in S} D_d$ since for every $v \in V$ the set $\{s \in S : \|v - s\| \le 2r\}$ is non-empty and finite, so there is $d \in S$ with $\|v - d\| = \min_{s \in S} \|v - s\|$, implying that $v \in D_d$. □

We will combine the tilings of finite-dimensional spaces with tilings of infinite-dimensional cylinders that will be defined with the help of a special, but very simple, tiling of the plane described in the following lemma. We will denote by $D(p, r)$ the closed disc in the Euclidean plane, with centre p and radius r.

LEMMA 3. *With suitable constants $r_0, r, \delta > 0$, we can tile the plane by closed convex sets $U_0, \dots, U_K$ that satisfy the following.*

(i) *U_0 is bounded and contains $D(0, r_0)$.*

(ii) *For each $k \ge 1$ there is $p_k = (a_k, b_k) \in U_k$ such that:*

  (a) *$D(p_k, r) \subset \{(x, y) \in U_k : |y| \le r_0\}$;*

  (b) *whenever $|t| \le \delta |b_k|$, we have $D((a_k, t), r) \subset U_0$.*

Proof. The proof should be clear from Figure 1. We choose a particular tiling which will lead to a reasonable estimate of the ratio of inner and outer radii in our main result. (However, no attempt has been made to optimize this estimate.) We define

$$U_0 = \{(x, y) : |5x \pm 12y| \le 13\}, \qquad U_1 = \{(x, y) : x, y \ge 0,\ 5x + 12y \ge 13\},$$
$$U_2 = \{(x, y) : (-x, y) \in U_1\}, \qquad U_3 = \{(x, y) : (-x, -y) \in U_1\}$$

and, finally,

$$U_4 = \{(x, y) : (-x, -y) \in U_1\}.$$

Then (i) holds with $r_0 = 1$. We show that (ii) holds with $r = 6/25$ and $\delta = 6/19$.

By symmetry, it suffices to verify the assertion for $k = 1$ only. Let $p_1 = (7/5, 19/25)$. As in Figure 1, the ball $D(p_1, r)$ lies in U_1 and is below the line $y = r_0$, guaranteeing (a). If $|t| \le 6/25$, the point $(7/5, t)$ belongs to the segment J joining $(7/5, 6/25)$ and $(7/5, -6/25)$. Hence the ball $D((7/5, t), r)$ lies below the ball $D((7/5, 6/25), 6/25)$ and above the ball $D((7/5, -6/25), 6/25)$. Since both these balls are contained in U_0, as indicated in Figure 1, we see that (b) holds with $\delta = (6/25)/(19/25) = 6/19$. □

For the rest of this section we fix a tiling $U_0, \dots, U_k$ of the plane and constants r_0, r and δ with the properties given in Lemma 3. Notice that $0 < r \le r_0 < \infty$ and $0 < \delta < 1$. We define $R_1 \ge r_0$ so that $R_1 \ge r + |a_k|$ for each $k = 1, \dots, K$. We also choose $R_0 \ge r_0$ so that $x^2 + y^2/\delta^2 \le R_0^2$ whenever $(x, y) \in U_0$ and $|x| \le R_1$, and find $S \ge 0$ so that $b_k^2 + (x - a_k)^2 - 2b_k y \le S$

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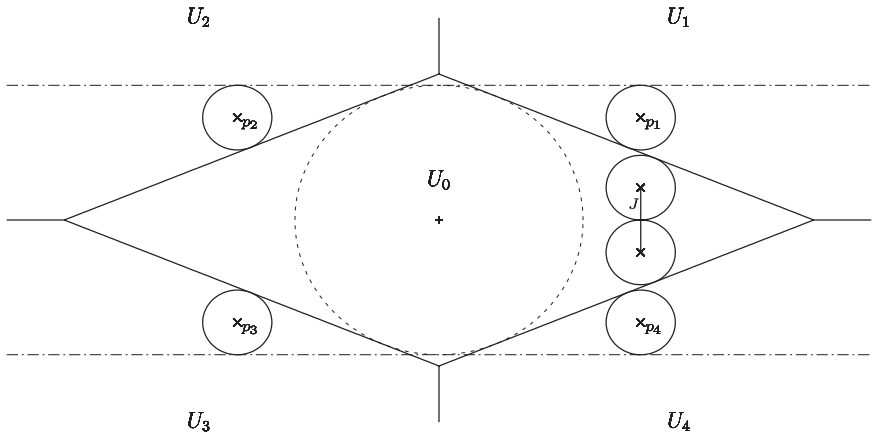


Figure 1: A suitable partition of the plane.

whenever $k = 1, \dots, K$, $(x, y) \in U_k$, $|x| \le R_1$ and $|y| \le R_0$. Finally, we let $R = \sqrt{R_0^2 + S + 4r^2}$. We are now ready to state a more precise version of Theorem 1 that will be proved in this section.

THEOREM 4. *Every separable Hilbert space H has a tiling by closed convex sets C_i such that for each i there is c_i with $B(c_i, r) \subset C_i \subset B(c_i, R)$. In particular, we can construct such a tiling with $R < 15r$.*

As mentioned in §1, the first step in the proof of Theorem 4 is the construction of a special (unbounded) tiling of the Hilbert space.

LEMMA 5. *Let e be a unit vector in an infinite-dimensional separable Hilbert space W. Then W admits a tiling by closed convex sets H_j, $j = 0, 1, 2, \dots$, such that with suitable points $h_j \in H_j$ we have:*

(i) $B(0, r_0) \subset H_0 \subset B(0, R_0)$;
(ii) $\|h_j - \langle e, h_j\rangle e\| \le r_0 - r$;
(iii) $B(h_j, r) \subset H_j$; *and*
(iv) $\|h - h_j\|^2 \le \max(R_0^2, S + \|h - \langle e, h\rangle e\|^2)$ *for every* $h \in H_j$.

Proof. It clearly suffices to tile only the slab $T = \{w \in W : |\langle w, e\rangle| \le R_1\}$; the required tiling of T will be obtained by reordering the sets $H_j + 2kR_1 e$ where $j = 0, 1, 2, \dots$ and $k = 0, \pm 1, \pm 2, \dots$.

Let $\mathcal{U} = \{u_0, u_1, \dots\}$ be a maximal subset of the unit sphere of $e^\perp$ such that $|\langle u_i, u_j\rangle| < \delta$ whenever $i \ne j$. (Of course, this last condition is equivalent to requiring that $\|u_i \pm u_j\| > \sqrt{2 - 2\delta}$ for $i \ne j$.) For each j, let $\pi_j : W \to \mathbb{R}^2$ be the linear map defined by $\pi_j w = (\langle e, w\rangle, \langle u_j, w\rangle)$. We show that a tiling of T with the required properties may be defined, up to an obvious reordering into a sequence, by

$$H_0 = T \cap \bigcap_{j=0}^{\infty} \pi_j^{-1}(U_0)$$

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and, for $j = 0, 1, \ldots$ and $k = 1, \ldots, K$,

$$H_j^k = T \cap \pi_j^{-1}(U_k) \cap \bigcap_{i=1}^{j-1} \pi_i^{-1}(U_0).$$

Clearly, H_0 and H_j^k are closed convex sets covering T, and the intersection of any two of these sets has empty interior since it is contained in the preimage under some π_j of the intersection of two different sets from the tiling U_k of $\mathbb{R}^2$.

Since $r_0 \le R_1$, $B(0, r_0) \subset T$. Moreover, $\|\pi_j\| = 1$ and $D(0, r_0) \subset U_0$ imply $B(0, r_0) \subset \pi_j^{-1}(U_0)$, and so $B(0, r_0) \subset H_0$. To show the second inclusion in (i), let $w \in H_0$. Write $w = ae + bu$ where u is a unit vector from $e^\perp$, and use the maximality of $\mathcal{U}$ to find i such that $|\langle u, u_i\rangle| \ge \delta$. Write $\pi_i w = (x, y)$ and notice that $w \in H_0$ implies $(x, y) \in U_0$ and $|x| \le R_1$. Since $|y| = |b||\langle u_i, u\rangle| \ge \delta|b|$, we get $\|w\|^2 = a^2 + b^2 \le x^2 + y^2/\delta^2 \le R_0^2$, which implies that $\|w\| \le R_0$ as required.

For H_0, the statements (ii)–(iv) hold with $h_0 = 0$. For $j = 0, 1, \ldots$ and $k = 1, \ldots, K$, let $h_j^k = a_k e + b_k u_j$. Recalling from Lemma 3 that

$$D((a_k, b_k), r) \subset \{(x, y) \in \mathbb{R}^2 : |y| \le r_0\},$$

we infer that

$$\|h_j^k - \langle e, h_j^k\rangle e\| = |b_k| \le r_0 - r,$$

which gives (ii). Since $|\langle e, h_j^k\rangle| = |a_k| \le R_1 - r$, we see that $B(h_j^k, r) \subset T$. We also have

$$\pi_j(B(h_j^k, r)) \subset D(p_k, r) \subset U_k$$

and, for $i < j$, $\pi_i h_j^k = (a_k, t)$ where t satisfies $|t| = |b_k\langle u_i, u_j\rangle| \le \delta|b_k|$. Hence

$$\pi_i(B(h_j^k, r)) \subset D(\pi_i h_j^k, r) \subset U_0,$$

and we conclude that $B(h_j^k, r) \subset H_j^k$.

To prove the last statement, let $h \in H_j^k$. Then

$$\begin{aligned}\|h - h_j^k\|^2 &= \|(h - \langle e, h\rangle e) + ((\langle e, h\rangle - a_k)e - b_k u_j)\|^2 \\ &= \|h - \langle e, h\rangle e\|^2 + b_k^2 + (\langle h, e\rangle - a_k)^2 - 2b_k\langle h, u_j\rangle \\ &\le \|h - \langle e, h\rangle e\|^2 + S,\end{aligned}$$

since the point $(\langle h, e\rangle, \langle h, u_j\rangle)$ satisfies the conditions from the definition of S. □

Proof of Theorem 4. As pointed out in Lemma 2, the finite-dimensional case is well known. Hence we assume that H is infinite-dimensional and choose an orthonormal basis $e_0, e_1, \ldots$ for it. Let V_k (respectively, W_k) denote the closed linear span of $e_0, \ldots, e_{k-1}$ (respectively, $e_k, e_{k+1}, \ldots$). Denote by P_k and Q_k

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the orthogonal projections of H onto V_k and W_k, respectively. Since we will use balls in various subspaces of H, we will reserve the notation $B(x, r)$ for balls in H and use $B_W(x, r)$ to denote a ball in the subspace W.

By Lemma 2 we find, for each $k = 1, 2, \ldots$, a tiling of the (finite-dimensional) space V_k by convex sets D_i^k, $i = 0, 1, \ldots$, such that for suitable points $d_i^k \in D_i^k$ we have $B_{V_k}(d_i^k, r) \subset D_i^k \subset B_{V_k}(d_i^k, 2r)$. For $k = 0$ we have $V_0 = \{0\}$, and we let $D_0^0 = \{0\}$ and $d_0^0 = 0$. By applying Lemma 5 to the space W_k with $e = e_k$, we find, for each $k = 0, 1, \ldots$, a tiling of W_k by closed convex sets H_j^k, $j = 0, 1, 2, \ldots$, and points $h_j^k \in H_j^k$ such that:

(i) $B_{W_k}(0, r_0) \subset H_0^k \subset B_{W_k}(0, R_0)$;
(ii) $\|h_j^k - \langle e_k, h_j^k\rangle e_k\| \le r_0 - r$;
(iii) $B(h_j^k, r) \subset H_j^k$; and
(iv) $\|h - h_j^k\|^2 \le \max(R_0^2, S + \|h - \langle e_k, h\rangle e_k\|^2)$ for every $h \in H_j^k$.

Let $T_k = \{x \in H : \|Q_{k+1}x\| \le R_0\}$. Let $\mathcal{I}$ be the set of all triples (k, i, j) such that either $k = i = 0$ or $j \ne 0$. The required tiling will be indexed by the triples $(k, i, j) \in \mathcal{I}$ and formed from the sets

$$C_{i,j}^k = T_k \cap P_k^{-1}(D_i^k) \cap Q_k^{-1}(H_j^k) \cap \bigcap_{m=k+1}^{\infty} Q_m^{-1}(H_0^m).$$

It is obvious that the $C_{i,j}^k$ are closed convex subsets of H.

We let $c_{i,j}^k = d_i^k + h_j^k$ and observe that for every $m \ge k + 1$,

$$\|Q_m c_{i,j}^k\| = \|Q_m(h_j^k - \langle e_k, h_j^k\rangle e_k)\| \le \|h_j^k - \langle e_k, h_j^k\rangle e_k\| \le r_0 - r.$$

We shall show that $B(c_{i,j}^k, r) \subset C_{i,j}^k$. Since $\|Q_{k+1}c_{i,j}^k\| \le r_0 - r \le R_0 - r$, we have $B(c_{i,j}^k, r) \subset T_k$. From $P_k c_{i,j}^k = d_i^k$ we immediately see that $P_k(B(c_{i,j}^k, r)) = B_{V_k}(d_i^k, r) \subset D_i^k$. Similarly, from $Q_k c_{i,j}^k = h_j^k$ we get that $Q_k(B(c_{i,j}^k, r)) = B_{W_k}(h_j^k, r) \subset H_j^k$. Finally, for every $m > k$ we use the fact that $\|Q_m c_{i,j}^k\| \le r_0 - r$ to deduce that $Q_m(B(c_{i,j}^k, r)) \subset B(0, r_0) \subset H_0$, and then we see that indeed $B(c_{i,j}^k, r) \subset C_{i,j}^k$.

If $x \in C_{i,j}^k$, then statement (iv) above, the equality

$$\|Q_k x - \langle e_k, x\rangle e_k\|^2 = \|Q_{k+1}x\|^2$$

and the fact that $x \in T_k$ together imply that

$$\begin{aligned}\|x - c_{i,j}^k\|^2 &= \|P_k x - d_i^k\|^2 + \|Q_k x - h_j^k\|^2 \le 4r^2 + \max(R_0^2, S + \|Q_{k+1}x\|^2)\\ &\le 4r^2 + S + R_0^2 = R^2,\end{aligned}$$

which shows that $C_{i,j}^k \subset B(c_{i,j}^k, R)$.

Next, we show that the sets $C_{i,j}^k$, where (k, i, j) runs through $\mathcal{I}$, cover H. To do so, assume $x \in H$. Since $\|Q_m x\| \le r_0$ for all sufficiently large m,

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the condition $B_{W_m}(0, r_0) \subset H_0^m$ allows us to find the smallest index $k \geq 0$ such that $Q_m x \in H_0^m$ for all $m > k$. Since $Q_{k+1}x \in H_0^{k+1} \subset B_{W_{k+1}}(0, R)$, we have $x \in T_k$. Also, since the sets H_j^k cover W_k, there is some $j \in \{0, 1, \ldots\}$ such that $Q_k x \in H_j^k$. To accommodate the slight difference between the $k = 0$ and $k > 0$ cases, we will treat these two situations separately.

- If $k = 0$, then $i = 0$ and $P_k^{-1}(D_i^k) = H$, and we see that $(k, i, j) \in \mathcal{I}$ and $x \in C_{i,j}^k$.
- If $k > 0$, we find $i \in \{0, 1, \ldots\}$ so that $P_k x \in D_i^k$. Observing that the minimality of k implies that $Q_k x \notin H_0^k$, we infer that $j \neq 0$ and so $(k, i, j) \in \mathcal{I}$ and $x \in C_{i,j}^k$.

It remains to show that the sets $C_{i,j}^k$ have disjoint interiors. For this purpose, assume that (k, i, j) and (n, p, q) are different triples from $\mathcal{I}$. Clearly, it suffices to consider only the case where $k \leq n$. If $k < n$, then $n > 0$ and hence $(n, p, q) \in \mathcal{I}$ implies that $q > 0$. Therefore $H_0^n \cap H_q^n$ has empty interior (in the space W_n), which shows that $C_{i,j}^k \cap C_{p,q}^n \subset Q_n^{-1}(H_0^n \cap H_q^n)$ has empty interior. Similarly, if $k = n$ and $j \neq q$, then $C_{i,j}^k \cap C_{p,q}^n \subset Q_k^{-1}(H_j^k \cap H_q^k)$ has empty interior since $H_j^k \cap H_q^k$ has empty interior (in the space W_k). Finally, if $k = n$, $j = q$ and $i \neq p$, then $k > 0$ (since otherwise $i = p = 0$) and we use the fact that $D_i^k \cap D_p^k$ has empty interior (in the space V_k) to infer that $C_{i,j}^k \cap C_{p,q}^n \subset P_k^{-1}(D_i^k \cap D_p^k)$ has empty interior.

The estimate $R < 15r$ holds for the particular choices made in the proof of Lemma 3. There we have $r_0 = 1$, $r = 6/25$, $\delta = 6/19$ and $p_k = (\pm 7/5, \pm 19/25)$. By definition, $R_1 = 6/25 + 7/5 = 41/25$. The optimal value of R_0^2 is the maximum of the convex function $x^2 + y^2/\delta^2$ over the convex polygon given by the inequalities $|5x \pm 12y| \leq 13$ and $|x| \leq 41/25$. By symmetry, the maximum has to be attained at one of the points $(0, 13/12)$ and $(41/25, 2/5)$. Hence we may choose R_0 to be the square root of $\max\{(13/12)^2(19/6)^2, (41/25)^2 + (2/5)^2\}$, thus $R_0 = (13/12)(19/6)$. Again by symmetry, the optimal value of S is the maximum of the convex function $(19/25)^2 + (x - 7/5)^2 - 2(19/25)y$ over the convex polygon given by the inequalities $0 \leq x \leq 41/25$, $0 \leq y \leq (13/12)(19/6)$ and $5x + 12y \geq 13$. Because of the negative sign in front of y, the maximum has to be attained at one of the points $(0, 13/12)$ and $(41/25, 2/5)$. It follows that $S = (19/25)^2 + (7/5)^2 - 2(19/25)(13/12)$, and a simple calculation involving fractions of relatively large numbers easily yields $\sqrt{R_0^2 + S + 4r^2} < 18/5$, showing that $R/r < 15$ as claimed. (Numerically, we get a slightly better estimate $R \leq 14.9595r$. A quick numerical experiment with other choices of U_k as in Figure 1 shows that if the constants are chosen so that $r = 0.24355r_0$, the above argument will lead to the slightly better estimate $R \leq 14.6463r$. Note, however, that there is no reason why this should be close to optimum, and in this article we do not try to establish any non-trivial lower bound on R/r.) □

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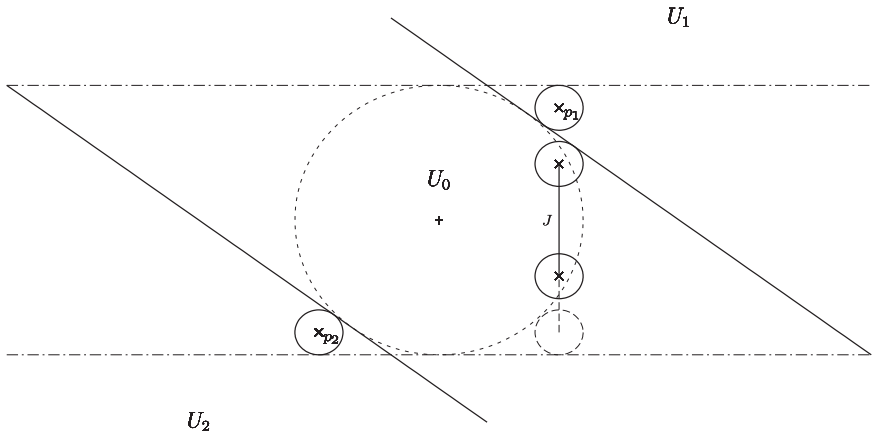


Figure 2: An alternative partition of the plane.

§3. *Simplifications and generalizations.* There are many variants and simplifications of the proof of Lemma 5, especially if one does not care about the bound on the ratio of the radii. For example, instead of the relatively complicated choice of S, we can observe that S is used to estimate an outer radius of a set which we know lies in a bounded cylinder, so we can just estimate by the diameter of the cylinder.

One might also observe that when Lemma 3 was applied, the boundedness of U_0 was used only partially: we merely need that the intersection of U_0 with vertical strips is bounded. This allows us to have $K = 3$ as indicated in Figure 2.

The tiling from Figure 2 has another interesting property that allows the set $\mathcal{U}$ from the proof of Lemma 5 to be maximal only with respect to the condition $\langle u_i, u_j \rangle \le \delta$, which, at least in its equivalent form $\|u_i - u_j\| \ge \sqrt{2 - 2\delta}$, may look more natural. Specifically, the inclusion $B((a_k, t), r) \subset U_0$ in statement (ii)(b) of Lemma 3 holds for all $-(r_0 - r) \le t \le \delta b_k$ when $b_k > 0$ and for all $\delta b_k \le t \le r_0 - r$ when $b_k < 0$. (This is illustrated in Figure 2.) To see that the alternative choice of U works, just observe that this is the bound on t obtained in the penultimate paragraph of the proof of Lemma 5. On the other hand, the radius R for the tiling in Figure 2 is clearly bigger than that for the tiling in Figure 1, thus making the ratio of the radii somewhat larger.

Of course, there is no need to introduce our tilings of the plane at all. In particular, when we use the idea of the tiling from Figure 2, the definition of the sets that form the tiling of the cylinder is very simple, and the properties stated in Lemma 5 can be proved without any reference to U_0, U_1 and U_2.

By noticing that a tiling of a space restricts to a tiling of a dense subset of it, we see that the completeness assumption of Theorem 1 is not needed; we used only the scalar product and separability. Of course, this can also easily be seen via a slight adjustment to the proof. (Similar remarks apply to the generalizations to Banach spaces that we will turn to shortly.)

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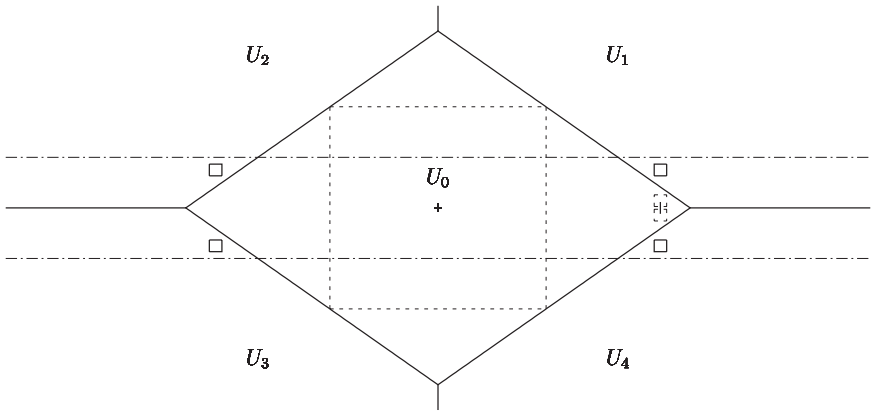


Figure 3: A partition of the plane illustrating Lemma 6.

More interestingly, we remark that, with somewhat worse constants, an appropriate variant of Lemma 5 holds in an arbitrary Banach space X. In this case we need to use a slightly modified tiling of the plane, in which the balls inside U_k are squares. To investigate this in more detail, we shall even replace the ball $B(0, r_0)$ by the r_0-multiple of one set, say D_0, and all other balls by r-multiples of another set, say D. The precise formulation of the conditions that such a tiling has to satisfy is given in the following lemma, but these conditions should be clear from Figure 3, where $D = \{(x, y) : \max(x, y) \le 1\}$ and $D_0 = \{(x, y) : \max(x, y) \le 2\}$.

LEMMA 6. *Given any two bounded sets D_0, $D \subset \mathbb{R}^2$, we can find constants $r_0 \ge r > 0$ and $1 > \delta > 0$ such that the plane can be tiled by closed convex sets $U_0, \ldots, U_K$ which satisfy the following.*

(i) *U_0 is bounded and contains $r_0 D_0$.*
(ii) *For each $k \ge 1$ there is $(a_k, b_k) \in \mathbb{R}^2$ such that:*
  (a) $(a_k, b_k) + rD \subset \{(x, y) \in U_k : |y| \le r_0\}$*; and*
  (b) *whenever $|t| \le \delta |b_k|$, we have $(a_k, t) + rD \subset U_0$.*

To modify the proof of Lemma 5 so that it works in general Banach spaces, we need a suitable replacement for the maximal set $\mathcal{U}$ of unit vectors such that $|\langle u, v\rangle| < \delta$ whenever u and v are different vectors from $\mathcal{U}$. This set has been found in [4, Lemma 2.1]; we will use a simple variant of it.

LEMMA 7. *For any Banach space V and any $0 < \delta < 1$, there are pairs $(v_\alpha, v_\alpha^*) \in V \times V^*$ indexed by ordinals $\alpha < \alpha_0$ such that:*

(i) $\|v_\alpha\| = \|v_\alpha^*\| = \langle v_\alpha^*, v_\alpha\rangle = 1$*;*
(ii) $|\langle v_\alpha^*, v_\beta\rangle| < \delta$ *whenever $\alpha < \beta < \alpha_0$;*
(iii) $\delta\|v\| \le \sup_{\alpha<\alpha_0} |\langle v_\alpha^*, v\rangle|$ *for every $v \in V$.*

Proof. Denote by M the set of all triples $(\alpha, \varphi, \varphi^*)$ where α is an ordinal and $\varphi : \alpha \to X$ and $\varphi^* : \alpha \to X^*$ are such that:

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- $\|\varphi(\beta)\| = \|\varphi^*(\beta)\| = \langle \varphi^*(\beta), \varphi(\beta)\rangle = 1$;
- $|\langle \varphi^*(\beta), \varphi(\gamma)\rangle| < \delta$ whenever $\beta < \gamma < \alpha$.

Then the set M ordered by $(\beta, \psi, \psi^*) \lesssim (\alpha, \varphi, \varphi^*)$ if $\beta \le \alpha$ and $(\psi(\gamma), \psi^*(\gamma)) = (\varphi(\gamma), \varphi^*(\gamma))$ for $\gamma < \beta$ satisfies the assumption of Zorn's lemma. Take its maximal element $(\alpha_0, \varphi_0, \varphi_0^*)$ and write $u_\gamma = \varphi_0(\gamma)$ and $u_\gamma^* = \varphi_0^*(\gamma)$.

The first two assertions of the lemma are obvious. As for the third one, given $v \in V$ of norm one, we can find $v^* \in V^*$ so that $\|v^*\| = \langle v^*, v\rangle = 1$. Since φ_0 cannot be extended by defining $\varphi_0(\alpha_0) = v$ and $\varphi_0^*(\alpha_0) = v^*$, there must be $\beta < \alpha_0$ so that $|\langle \varphi^*(\beta), v\rangle| \ge \delta$, and we are done. □

We are now ready to state and prove a version of Lemma 5 that holds in general Banach spaces. We will use it only for quotient mappings (i.e. with $c = 1$), but it is hoped that the slightly greater generality will give a better picture of what is going on.

LEMMA 8. *Suppose that $Q : W \to V$ is a linear map between two Banach spaces such that $\dim Q^{-1}(0) = 1$ and $Q(B(0, 1/c)) \subset B(0, 1) \subset Q(B(0, c))$ for some $c \ge 1$. Then, with suitable constants $0 < r \le r_0 \le R_0 < \infty$ depending only on c, we can find a tiling of W by closed convex sets H_α, $0 \le \alpha < \alpha_0$ (where α_0 is a suitable ordinal), and points $h_\alpha \in H_\alpha$ such that:*

(i) $B(0, cr_0) \subset H_0 \subset B(0, R_0)$;
(ii) $\|Qh_\alpha\| \le r_0 - cr$;
(iii) $B(h_\alpha, r) \subset H_\alpha$; *and*
(iv) $\|h - h_\alpha\| \le R_0 + 2c\|Qh\|$ *for every* $h \in H_\alpha$.

Proof. We use Lemma 6 with the sets $D = \{(x, y) \in \mathbb{R}^2 : \max(c|x|, |y|) \le c\}$ and $D_0 = cD$ to find the corresponding tiling $U_0, \dots, U_K$ of $\mathbb{R}^2$ together with points $(a_k, b_k) \in \mathbb{R}^2$, $k = 1, \dots, K$, and constants $0 < r \le r_0 < \infty$ and $0 < \delta < 1$. We also let $R_1 = r + cr_0 + \max_{k=1,\dots,K} |a_k|$ and choose $0 < R < \infty$ so that $|y| \le R$ for every $(x, y) \in H_0$ with $|x| \le R_1$. (The condition $|x| \le R_1$ is included only to indicate that we can use the weaker boundedness condition on U_0 as in Figure 2; for bounded U_0 as in Figure 3, the condition is redundant.) Finally, we let $R_0 = \max(R_1 + 2cR/\delta, 2R_1 + 2cr_0)$.

Choose $e \in Q^{-1}(0)$ and $e^* \in V^*$ so that $\|e\| = \|e^*\| = \langle e^*, e\rangle = 1$, and recall that for every $w \in W$ we have

$$\|w\| \le |\langle e^*, w\rangle| + 2c\|Qw\|. \tag{1}$$

This can be seen by finding $\tilde{w} \in Q^{-1}(0)$ such that $\|w - \tilde{w}\| \le c\|Qw\|$ and then estimating as follows:

$$\begin{aligned} \|w\| &\le \|\langle e^*, w\rangle e\| + \|\tilde{w} - \langle e^*, w\rangle e\| + \|w - \tilde{w}\| \\ &= |\langle e^*, w\rangle| + |\langle e^*, \tilde{w} - w\rangle| + \|w - \tilde{w}\| \\ &\le |\langle e^*, w\rangle| + 2\|w - \tilde{w}\| \le |\langle e^*, w\rangle| + 2c\|Qw\|. \end{aligned}$$

Let $v_\alpha \in V$ and $v_\alpha^* \in V^*$ be as in Lemma 7, and write $w_\alpha^* = v_\alpha^* \circ Q$. Also observe that the line $Q^{-1}(u_\alpha)$ meets the kernel of e^*, so we have a $w_\alpha \in W$ such that $Qw_\alpha = u_\alpha$ and $\langle e^*, w_\alpha\rangle = 0$.

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It suffices to tile only the slab $T = \{w \in W : |\langle e^*, w\rangle| \le R_1\}$; the required tiling of W will be obtained by reordering the sets $H_\alpha + 2kR_1 e$ where $\alpha < \alpha_0$ and $k = 0, \pm 1, \pm 2, \ldots$.

Let $\pi_\alpha : W \to \mathbb{R}^2$ be the linear maps defined by $\pi_\alpha w = (\langle e^*, w\rangle, \langle w_\alpha^*, w\rangle)$, and notice that $\pi_\alpha(B(0, 1)) \subset D$. We shall show that a tiling of T with the required properties can be defined, up to an obvious reordering into a (transfinite) sequence, by

$$H_0 = T \cap \bigcap_{\beta<\alpha_0} \pi_\beta^{-1}(U_0)$$

and, for $\alpha < \alpha_0$ and $k = 1, \ldots, K$,

$$H_\alpha^k = T \cap \pi_\alpha^{-1}(U_k) \cap \bigcap_{\beta<\alpha} \pi_\beta^{-1}(U_0).$$

Clearly, H_0 and H_α^k are closed convex sets covering T, and the intersection of any two of these sets has empty interior since it is contained in the preimage under some π_α of the intersection of two different sets from the tiling U_k of $\mathbb{R}^2$.

Since $cr_0 \le R_1$, we have $B(0, cr_0) \subset T$. Since also $\pi_\beta(B(0, cr_0)) \subset r_0 D_0 \subset U_0$ for every β, we infer that $B(0, cr_0) \subset H_0$. If $w \in H_0$, we find $\alpha < \alpha_0$ so that

$$|\langle w_\alpha^*, w\rangle| = |\langle u_\alpha^*, Qw\rangle| \ge \delta \|Qw\|.$$

Then, by (1),

$$\|w\| \le |\langle e^*, w\rangle| + 2c\|Qw\| \le R_1 + 2c|\langle w_\alpha^*, w\rangle|/\delta \le R_1 + 2cR/\delta \le R_0,$$

where the penultimate inequality follows from $\pi_\alpha w \in H_0$ (and also from $|\langle e^*, w\rangle| \le R_1$ if we assume only the weaker condition on U_0). Hence we have $H_0 \subset B(0, R_0)$ as asserted in (i).

For H_0 we let $h_0 = 0$; then, deducing from Lemma 6(ii)(a) that $r_0 \ge cr$, we infer statement (ii). Assertions (iii) and (iv) follow immediately from (i).

For $\alpha < \alpha_0$ and $k = 1, \ldots, K$, we let $h_\alpha^k = a_k e + b_k w_\alpha$. Then (ii) holds since $\|Qh_\alpha^k\| = |b_k|\|u_\alpha\| = |b_k| \le r_0 - cr$. Moreover,

$$\pi_\alpha(B(h_\alpha^k, r)) \subset (a_k, b_k) + rD \subset U_k,$$

and for $\beta < \alpha$ we have $\pi_\beta h_\alpha^k = (a_k, t)$ where t satisfies

$$|t| = |b_k\langle u_\beta^*, u_\alpha\rangle| \le \delta|b_k|.$$

Hence $\pi_\beta(B(h_\alpha^k, r)) \subset (a_k, t) + rD \subset U_0$. Since $|\langle e^*, h_\alpha^k\rangle| = |a_k| \le R_1 - r$, we also have $B(h_\alpha^k, r) \subset T$ and so we see that $B(h_\alpha^k, r) \subset H_\alpha^k$, proving (iii).

To prove (iv), let $h \in H_\alpha^k$. Then $H_\alpha^k \subset T$ implies that $|\langle e^*, h\rangle| \le R_1$, and we infer from estimate (1) and statement (ii) that

$$\begin{aligned}\|h - h_\alpha^k\| &\le |\langle e^*, h\rangle| + |\langle e^*, h_\alpha^k\rangle| + 2c\|Qh\| + 2c\|Qh_\alpha^k\| \\ &\le 2R_1 + 2cr_0 + 2c\|Qh\| \le R_0 + 2c\|Qh\|.\end{aligned}$$

□

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The use of a simpler tiling of the plane of the type indicated in Figure 2 would allow us to use [4, Lemma 2.1] instead of our Lemma 7, because then the absolute values in the last two statements of Lemma 7 would become unnecessary.

The following theorem shows that the first half of the argument proving Theorem 1 leads to a new tiling result for any separable Banach space. In a sense, it reduces the problem of existence of a tiling of such a space by closed convex sets with uniformly bounded radii to existence of such tilings of cylinders around finite-dimensional subspaces.

THEOREM 9. *Let $V_0, V_1, \ldots$ be an increasing sequence of subspaces of a Banach space X such that $\dim V_n = n$ and $\bigcup_n V_n$ is dense in X. Then, for suitable absolute constants $0 < r < R < \infty$, X admits a tiling by closed convex sets E_i such that for each i there are $x_i \in X$ and $n_i \in \mathbb{N}$ with*

$$B(x_i + V_{n_i}, r) \subset E_i \subset B(x_i + V_{n_i}, R).$$

Proof. Let $0 < r \le r_0 \le R_0 < \infty$ be the constants from Lemma 8 with $c = 1$, and let $R = 3R_0$. Let $W_k = X/V_k$ be the quotient space and denote the canonical projections by $Q_k : X \to W_k$ and $Q_{m,k} : W_k \to W_m$ for $m > k$. Notice that the kernel of $Q_{k+1,k}$ coincides with $Q_k(V_{k+1})$ and is a one-dimensional subspace of W_k. Hence, using Lemma 8 in the space W_k with $Q = Q_{k+1,k}$ and $c = 1$, we find for each $k = 0, 1, \ldots$ a tiling of W_k by closed convex sets H_j^k, $j = 0, 1, \ldots$, and points $h_j^k \in H_j^k$ such that:

(i) $B_{W_k}(0, r_0) \subset H_0^k \subset B_{W_k}(0, R_0)$;
(ii) $\|Q_{k+1,k} h_j^k\| \le r_0 - r$;
(iii) $B(h_j^k, r) \subset H_j^k$; and
(iv) $\|h - h_j^k\| \le R_0 + 2\|Q_{k+1,k} h\|$ for every $h \in H_j^k$.

Let $\mathcal{I}$ be the set of all pairs $(k, j) \in \mathbb{N} \times \mathbb{N}$ such that either $k = 0$ or $j \ne 0$. The required tiling will be indexed by $(k, j) \in \mathcal{I}$ and formed from the sets

$$C_j^k = Q_k^{-1}(T_k) \cap Q_k^{-1}(H_j^k) \cap \bigcap_{m=k+1}^{\infty} Q_m^{-1}(H_0^m)$$

where $T_k = \{w \in W_k : \|Q_{k+1,k} w\| \le R_0\}$.

It is obvious that the C_j^k are closed convex subsets of X; we show that they cover X. To do this, assume $x \in X$. Since $\|Q_m x\| = \operatorname{dist}(x, V_m) \le r_0$ for all sufficiently large m, the condition $B_{W_m}(0, r_0) \subset H_0^m$ allows us to find the smallest index $k \ge 0$ such that $Q_m(x) \in H_0^m$ for all $m > k$. Since

$$Q_{k+1,k} Q_k x = Q_{k+1} x \in H_0^{k+1} \subset B_{W_{k+1}}(0, R_0),$$

we have $Q_k x \in T_k$ and so, upon finding $j \in \{0, 1, \ldots\}$ so that $Q_k x \in H_j^k$, we would have $x \in C_j^k$ provided that $(k, j) \in \mathcal{I}$. To see that (k, j) is indeed in $\mathcal{I}$, we treat the two slightly different cases of $k = 0$ and $k > 0$ separately.

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- If $k = 0$, then $(k, j) \in \mathcal{I}$ and $x \in C_j^k$.
- If $k > 0$, observe that the minimality of k implies $Q_k x \notin H_0^k$, and infer that $j \neq 0$ and so $(k, j) \in \mathcal{I}$ and $x \in C_j^k$.

The fact that the sets C_j^k have disjoint interiors is rather obvious. Suppose that (k, j) and (n, q) are different pairs from $\mathcal{I}$ and that $k \leq n$. If $k < n$, then $n > 0$ and hence $(n, q) \in \mathcal{I}$ implies that $q > 0$. Hence $H_0^n \cap H_q^n$ has empty interior (in the space W_n), which shows that $C_j^k \cap C_q^n \subset Q_n^{-1}(H_0^n \cap H_q^n)$ has empty interior. Similarly, if $k = n$ and $j \neq q$, then $C_j^k \cap C_q^n \subset Q_k^{-1}(H_j^k \cap H_q^k)$ has empty interior since $H_j^k \cap H_q^k$ has empty interior (in the space W_k).

Noting that $Q_k^{-1} h_j^k = x_j^k + V_k$ for some $x_j^k \in X$, we can finish the proof of the theorem by showing that

$$B(Q_k^{-1} h_j^k, r) \subset C_j^k \subset B(Q_k^{-1} h_j^k, R). \tag{2}$$

For the second inclusion, we infer from (iv) that $T_k \cap H_j^k \subset B_{W_k}(h_j^k, 3R_0)$ and conclude that

$$C_j^k \subset Q_k^{-1}(T_k \cap H_j^k) \subset Q_k^{-1}(B_{W_k}(h_j^k, 3R_0)) = B(Q_k^{-1} h_j^k, R).$$

To prove the first inclusion of (2), start by noticing that (ii) implies $B_{W_k}(h_j^k, r) \subset T_k$, hence $B(Q_k^{-1} h_j^k, r) \subset Q_k^{-1}(T_k)$. Moreover, by (iii) we have $B_{W_k}(h_j^k, r) \subset H_j^k$, and so $B(Q_k^{-1} h_j^k, r) \subset Q_k^{-1}(H_j^k)$. Hence it remains to show that $Q_m(B(Q_k^{-1} h_j^k, r)) = B_{W_m}(Q_{m,k} h_j^k, r) \subset H_0^m$ for $m > k$. But this was the main point of Lemma 8: since $\|Q_{m,k} h_j^k\| \leq \|Q_{k+1,k} h_j^k\| \leq r_0 - r$ by (ii) and $B_{W_m}(0, r_0) \subset H_0^m$ by (i), we may indeed conclude that $B_{W_m}(Q_{m,k} h_j^k, r) \subset H_0^m$, and the proof is finished. □

The following corollary and Example 11 show that the class of spaces to which the construction of tilings in this note applies encompasses substantially more than just Hilbert spaces. However, it is far from clear to what other spaces this construction might apply.

COROLLARY 10. *Let X be a Banach space containing an increasing sequence $V_0, V_1, \ldots$ of uniformly complemented (in X) subspaces which have* $\dim V_n = n$ *and the property that for each n, V_n admits a tiling by closed convex sets D_i^n such that for suitable $d_i^n \in D_i^n$ and s, $S > 0$ independent of both i and n, we have*

$$B_{V_n}(d_i^n, s) \subset D_i^n \subset B_{V_n}(d_i^n, S).$$

Then X admits a tiling by closed convex sets with inner radii uniformly bounded from below and outer radii uniformly bounded from above.

Proof. Let $P_n : X \to V_n$ be projections with $\sup_n \|P_n\| < \infty$. Then the sets $E_i \cap P_{n_i}^{-1}(D_i^{n_i})$, where E_i and n_i are as in Theorem 9, form the required tiling of X. □

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Example 11. By choosing $t_1, t_2, \ldots$ dense in $[0, 1]$, with $t_1 = 0$ and $t_2 = 1$, and letting V_n, $n \geq 2$, be the space of continuous functions that are affine on every interval not containing any of the $t_1, \ldots, t_n$, we see that V_n is 1-isomorphic to ℓ_∞^n, and so the space $C[0, 1]$ satisfies the assumptions of the previous theorem. Similarly, we can see that any Banach space with $X^* = L_1(\mu)$ satisfies the assumptions of the previous theorem.

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