RANDOM WALK AND ELECTRIC CURRENTS IN NETWORKS

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ABSTRACT. Let G be a locally finite connected graph and c be a positive real-valued function defined on its edges. Let $D(\xi)$ denote the sum of the values of c on the edges incident with a vertex ξ. A particle starts at some vertex α and performs an infinite random walk

$$\alpha = \xi_0, \lambda_1, \xi_1, \lambda_2, \xi_2, \lambda_3, \ldots$$

in which (i) the ξ_j are vertices of G, (ii) λ_j is an edge joining ξ_{j-1} to ξ_j ($j = 1, 2, 3, \ldots$), (iii) if λ is any edge incident with ξ_j, then

$$\text{prob}\,(\lambda_{j+1} = \lambda) = c(\lambda)/D(\xi_j) \quad (j = 0, 1, 2, \ldots).$$

Let v be a set of vertices of G such that the complementary set of vertices is finite and includes α. A geometrical characterization is given of the probability (τ, say) that the particle will visit some element of v without first returning to α. An essentially equivalent problem is obtained by regarding G as an electrical network and $c(\lambda)$ as the conductance of an edge λ; the current flowing through the network from α to v when an external agency maintains α at potential 1 and all elements of v at potential 0 is found to be $\tau D(\alpha)$.

A necessary and sufficient condition (of a geometrical character) for the particle to be certain to return to α is obtained; and, as an application, a new proof is given of a conjecture of Gillis (3) regarding centrally biased random walk on an n-dimensional lattice.

1. *Introduction.* A *graph* G consists (for the purposes of this paper) of two disjoint sets $V(G)$, $E(G)$, together with two mappings t, h of $E(G)$ into $V(G)$. The elements of $V(G)$ are called *vertices* of G and those of $E(G)$ are called *edges* of G. If λ is an edge of G, λt and λh are called the *tail* and *head* of λ respectively, and λ is said to *join* them. Two vertices of G are *adjacent* if they are joined by at least one edge of G. In this paper, the letter G, without further introduction, will always denote a graph.

A vertex ξ and edge λ of G are *incident* if $\xi = \lambda t$ or $\xi = \lambda h$ or both. The set of those edges of G which are incident with ξ will be denoted by $\iota(\xi)$. A graph is *locally finite* if no vertex is incident with an infinite number of edges. A graph is *trivial* if it has no edge. A *G-conductance* is a positive real-valued function defined on the edges of G. If c is a G-conductance and $\xi \in V(G)$, then $D_c(\xi)$ will denote the sum of the values of c on the edges incident with ξ. A *path* of G is a finite sequence

$$\xi_0, \lambda_1, \xi_1, \lambda_2, \xi_2, \lambda_3, \ldots, \lambda_n, \xi_n \quad (n \geqslant 0),$$

in which (i) the ξ_j are vertices of G, (ii) the λ_j are edges of G, and (iii) λ_j joins ξ_{j-1} to ξ_j ($j = 1, 2, \ldots, n$). The *length* of this path is defined to be n. A path is said to *connect* its first and last terms. A *$\xi\eta$-path* of G is a path of G whose first and last terms are ξ and η respectively. G is *connected* if every pair of its vertices are connected by a path of G. We define a *network* to be an ordered pair $[G, c]$ such that G is a non-trivial locally finite connected graph and c is a G-conductance. A network $[G, c]$ will be called *finite* or *infinite* according as G is a finite or infinite graph respectively.

Let $[G, c]$ be a network and α be a vertex of G. Then a *$[G, c]$-walk* starting at α is a non-terminating random walk

$$\xi_0, \lambda_1, \xi_1, \lambda_2, \xi_2, \lambda_3, \ldots, \tag{1}$$

in which (i) the ξ_j are vertices of G, (ii) the λ_j are edges of G, (iii) $\xi_0 = \alpha$, (iv) λ_j joins

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ξ_{j-1} and ξ_j $(j = 1, 2, 3, \ldots)$, and (v) for any vertex ξ and edge λ of G which are incident with one another,

$$\text{prob}\,(\lambda_j = \lambda \mid \xi_{j-1} = \xi, I) = c(\lambda)/D_c(\xi) \quad (j = 1, 2, 3, \ldots),$$

where I is any information provided about the ξ_k, $1 \leqslant k < j-1$ and the λ_k, $1 \leqslant k \leqslant j-1$ (so that the walker's route between α and ξ_{j-1} does not affect the probability). The vertex α is *c-recurrent* if a $[G, c]$-walk which starts at α is certain to re-visit α. Since G is connected and locally finite, $V(G)$ is clearly finite or enumerable—for a detailed proof of this, see (7), p. 79. From this fact and the connectedness of G, it follows, by the theorem on p. 355 of (1), that all or none of the vertices of G are c-recurrent; the network $[G, c]$ will be called *recurrent* or *non-recurrent* according as the vertices of G are or are not c-recurrent respectively. It is (as we shall see) easily proved that a finite network is recurrent; and our main objective is to determine which infinite networks are also recurrent. We are, of course, considering only a very special kind of random walk: if the vertices of G are enumerated and p_{rs} denotes the probability of a step to the sth vertex when the walker is at the rth, the matrix $[p_{rs}]$ has the special property that it can be made symmetric by multiplying each row by a suitable constant.

If v is a subset of $V(G)$, $\bar{v}$ will denote the set $V(G) - v$; and v will be called a *co-finite* subset of $V(G)$ if $\bar{v}$ is finite. If u, v are subsets of $V(G)$, $u \circ v$ will denote the set of those edges of G which join elements of u to elements of v, i.e. of those edges λ such that either $\lambda t \in u$ and $\lambda h \in v$ or $\lambda h \in u$ and $\lambda t \in v$. If v is a subset of $V(G)$ and α is an element of $\bar{v}$, an *αv-constriction* of G is a finite sequence $y_0, y_1, \ldots, y_n$ $(n \geqslant 1)$ of disjoint subsets of $V(G)$ such that $\alpha \in y_0$, $v \subset y_n$, $y_0 \cup y_1 \cup \ldots \cup y_n = V(G)$ and $y_j \circ y_k = \emptyset$ whenever $|j-k| \geqslant 2$.

If S is a set and f is a function defined on its elements, $f.S$ will denote $\sum_{x \in S} f(x)$.

We shall use the term *system* to mean an ordered quadruple $[G, c, \alpha, v]$ such that $[G, c]$ is a network, α is a vertex of G, v is a non-empty co-finite subset of $V(G)$ and $\alpha \in \bar{v}$. If $S = [G, c, \alpha, v]$ is a system and C denotes an αv-constriction $y_0, y_1, \ldots, y_n$ of G, then $\Phi_c(C)$ will denote

$$\sum_{j=1}^{n} \frac{1}{c.(y_{j-1} \circ y_j)}.$$

(Since G is connected, no denominator $c.(y_{j-1} \circ y_j)$ vanishes.) Furthermore, $M(S)$ will denote the maximum of $\Phi_c(C)$ over all αv-constrictions C of G.

A set will sometimes be denoted by an enumeration of its elements between braces, e.g. $\{a_1, a_2, \ldots, a_n\}$. If A, B are sets, $A \times B$ will denote the set of all ordered pairs $[a, b]$, where $a \in A$ and $b \in B$. The set of all real numbers in the open interval $(0, 1)$ will be denoted by $\mathscr{I}$.

When, to avoid ambiguity, it is necessary to specify the graph to which a graph-theoretical symbol relates, the letter denoting the graph will be attached to the symbol in some convenient way; e.g. if ξ is a common vertex of two graphs G and H, $\iota_H(\xi)$ denotes the set of those edges of H which are incident in H with ξ. We shall, however, make the convention that, in any context where one of the graphs under consideration is denoted by the letter G, all graph-theoretical symbols relate to G unless the contrary is indicated, e.g. $\iota(\xi)$ if otherwise ambiguous means $\iota_G(\xi)$.

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Let X be a subset of $E(G)\times\mathscr{I}$. Then, for any $\lambda\in E(G)$, $X[\lambda]$ will denote the set of those values of θ for which $[\lambda,\theta]\in X$, and $X(\lambda)$ will denote $X[\lambda]\cup\{0,1\}$. We shall call X *locally finite* if $X[\lambda]$ is finite for every λ. If X is a locally finite subset of $E(G)\times\mathscr{I}$, $\lambda\in E(G)$ and the elements of $X(\lambda)$, arranged in ascending order of magnitude, are

$$0=\theta_0<\theta_1<\theta_2<\ldots<\theta_n=1,$$

then $\Sigma(X,\lambda)$ will denote the set of ordered triples

$$\{[\lambda,\theta_0,\theta_1],\ [\lambda,\theta_1,\theta_2],\ \ldots,\ [\lambda,\theta_{n-1},\theta_n]\}.$$

A *refinement* of G is a graph H such that, for some locally finite subset X of $E(G)\times\mathscr{I}$,

(i) $V(H)=V(G)\cup X$,

(ii) $E(H)=\bigcup_{\lambda\in E(G)}\Sigma(X,\lambda)$, and

(iii) if $[\lambda,\theta,\phi]\in E(H)$, then

$$[\lambda,\theta,\phi]t_H=\begin{cases}\lambda t & \text{if } \theta=0,\\ [\lambda,\theta] & \text{if } \theta>0,\end{cases}$$

$$[\lambda,\theta,\phi]h_H=\begin{cases}\lambda h & \text{if } \phi=1,\\ [\lambda,\phi] & \text{if } \phi<1.\end{cases}$$

An edge λ of G will be called *undivided in* H if $X[\lambda]=\emptyset$. (Roughly speaking, we think of a refinement of G as being obtained from G by inserting a finite number of new vertices in the interior of each edge. If $\lambda t=A$ and $\lambda h=B$, then λ is thought of as the line AB and $[\lambda,\theta]$ as the point P of λ such that $AP=\theta AB$. Moreover, $[\lambda,\theta,\phi]$ is thought of as the segment PQ of λ such that $AP=\theta AB$ and $AQ=\phi AB$.) If H is a refinement of G and c is a G-conductance, the H-conductance d *corresponding* to c is defined by the rule that $\quad d([\lambda,\theta,\phi])=c(\lambda)/(\phi-\theta)$

for every $[\lambda,\theta,\phi]\in E(H)$. A *refinement* of a network $[G,c]$ is a network $[H,d]$ such that H is a refinement of G and d is the H-conductance corresponding to c. A *refinement* of a system $[G,c,\alpha,v]$ is a system $[H,d,\alpha,v]$ such that $[H,d]$ is a refinement of $[G,c]$ and every element of $v\circ v$ is undivided in H. The set of all refinements of a system S will be denoted by $\mathfrak{R}(S)$.

The set of all positive integers will be denoted by P, and the set of all positive integers less than or equal to a non-negative integer j by P_j. (Thus $P_0=\emptyset$.)

If $S=[G,c,\alpha,v]$ is a system, we define $\tau(S)$ (the *transmission probability* of S) to be the probability that a $[G,c]$-walk which starts at α will visit an element of v without first returning to α, i.e. in the notation of (1),

$$\tau(S)\equiv\text{prob}\,(\exists j\in P,\xi_j\in v \text{ and } \xi_k\neq\alpha \text{ for every } k\in P_{j-1}\,|\,\xi_0=\alpha).$$

In § 2, we prove

THEOREM 1. *For any system* $S=[G,c,\alpha,v]$,

$$1/\tau(S)=D_c(\alpha)\max_{T\in\mathfrak{R}(S)}M(T).$$

It is found that an essentially equivalent problem is obtained by regarding G as an electrical network in which the conductance of an edge λ (i.e. the reciprocal of its electrical resistance) is $c(\lambda)$. Then the *conductance* of the system may be defined to be the current flowing through G from α to v when an external agency maintains α at potential 1

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and all elements of v at potential 0. (It is here perhaps most natural to think of the case in which v has only a single element.) The *resistance* of S is defined to be the reciprocal of its conductance. It will be found that the resistance of S is equal to $1/D_c(\alpha)\tau(S)$, or, equivalently, by Theorem 1, to the maximum, over refinements T of S, of $M(T)$. Note that, when $[G, c]$ is regarded as an electrical network, a refinement of $[G, c]$ represents essentially the same electrical network, but with each wire regarded as being subdivided into a finite number of wires connected in series.

A *constriction* of an infinite graph G is an infinite sequence $y_0, y_1, y_2, \ldots$ of disjoint non-empty finite subsets of $V(G)$ such that $y_0 \cup y_1 \cup y_2 \cup \ldots = V(G)$ and $y_j \circ y_k = \emptyset$ whenever $|j-k| \geqslant 2$. If $[G, c]$ is an infinite network, a constriction $y_0, y_1, y_2, \ldots$ of G is *c-slowly-widening* if

$$\sum_{j=1}^{\infty} \frac{1}{c.(y_{j-1} \circ y_j)}$$

is divergent. (No denominator $c.(y_{j-1} \circ y_j)$ vanishes, since G is connected and the y_j are non-empty.) The network $[G, c]$ is *slowly-widening* if G has a c-slowly-widening constriction. The main result of this paper is

THEOREM 2. *An infinite network is recurrent if and only if it has a slowly-widening refinement.*

The result that a recurrent infinite network has a slowly-widening refinement is (as we shall see) virtually contained in a theorem of Foster (2) on Markov chains of a more general kind. (It could alternatively be deduced from Theorem 1, but this would take more trouble.) A converse of Foster's result was obtained by Kendall (6), but this does not seem to imply, in any simple way, that an infinite network having a slowly-widening refinement is recurrent, a result which we deduce from Theorem 1. (Foster's and Kendall's results, as applied to random walk in networks, amount to saying that an infinite network is recurrent if and only if it has a slowly-widening refinement of a certain special kind.)

As an application of Theorem 2, we give a new proof of a conjecture of Gillis (3) regarding centrally biased random walk on an n-dimensional lattice. This conjecture was first proved by Prof. P. Erdős, by sharpening the argument of (3). (Prof. Erdős's proof is unpublished.)

2. *Proof of Theorem* 1. Let f, g be functions defined on the elements of a set S and let k be a constant. Then the statements $f = g$ on S, $f = k$ on S, $f \geqslant k$ on S will mean that, for every $x \in S$, $f(x) = g(x)$, $f(x) = k$ or $f(x) \geqslant k$ respectively.

A *G-potential* is a real-valued function defined on the vertices of G, and a *G-current* is a real-valued function defined on the edges of G. (In neither case is the function restricted to positive values.) A G-potential ϕ is *local* if $\phi = 0$ on some co-finite subset of $V(G)$. If $\xi \in V(G)$, we shall make the definitions

$$\begin{aligned}
\gamma(\xi) &\equiv \{\lambda \mid \lambda \in E(G), \lambda t = \xi\},\\
\delta(\xi) &\equiv \{\lambda \mid \lambda \in E(G), \lambda h = \xi\},\\
\overline{\gamma}(\xi) &\equiv \{\lambda \mid \lambda \in E(G), \lambda t = \xi, \lambda h \neq \xi\},\\
\overline{\delta}(\xi) &\equiv \{\lambda \mid \lambda \in E(G), \lambda h = \xi, \lambda t \neq \xi\},\\
\pi(\xi) &\equiv \{\xi\} \circ \{\xi\} = \{\lambda \mid \lambda \in E(G), \lambda t = \lambda h = \xi\}.
\end{aligned}$$

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If v is a subset of $V(G)$, we make the definitions

$$\bar{\gamma}(v) \equiv \{\lambda \mid \lambda \in E(G),\ \lambda t \in v,\ \lambda h \notin v\},$$
$$\bar{\delta}(v) \equiv \{\lambda \mid \lambda \in E(G),\ \lambda h \in v,\ \lambda t \notin v\}.$$

If G is locally finite and i is a G-current, then

(i) for any vertex ξ of G, $f_i(\xi)$ will denote $i.\gamma(\xi) - i.\delta(\xi)$;

(ii) for any finite subset v of $V(G)$, $F_i(v)$ will denote $i.\bar{\gamma}(v) - i.\bar{\delta}(v)$.

(Thus, if we think of $i(\lambda)$ as denoting, for each edge λ, the current flowing in λ, measured in the direction from λt to λh, then $f_i(\xi)$ is the total current flowing out of ξ and $F_i(v)$ is the total current flowing from v to $\bar{v}$.) If c is a G-conductance and ϕ is a G-potential, then $I_{c\phi}$ will denote the G-current defined by

$$I_{c\phi}(\lambda) = [\phi(\lambda t) - \phi(\lambda h)]\, c(\lambda) \quad (\lambda \in E(G)).$$

(Thus, if we think of G as an electrical network in which each edge λ has conductance $c(\lambda)$ and each vertex ξ is maintained at potential $\phi(\xi)$, then $I_{c\phi}(\lambda)$ is the current flowing in λ, measured in the direction from λt to λh.) The expressions $f_{I_{c\phi}}(\xi)$, $F_{I_{c\phi}}(v)$ will be abbreviated to $f_{c\phi}(\xi)$, $F_{c\phi}(v)$ respectively for notational convenience.

LEMMA 1. *Let G be locally finite, i be a G-current and ϕ be a local G-potential. Then*

$$\sum_{\xi \in V(G)} \phi(\xi) f_i(\xi) = \sum_{\lambda \in E(G)} [\phi(\lambda t) - \phi(\lambda h)]\, i(\lambda).$$

Proof. It is clear that

$$\sum_{\xi \in V(G)} \phi(\xi)\, i.\gamma(\xi) = \sum_{\lambda \in E(G)} \phi(\lambda t)\, i(\lambda), \tag{2}$$

$$\sum_{\xi \in V(G)} \phi(\xi)\, i.\delta(\xi) = \sum_{\lambda \in E(G)} \phi(\lambda h)\, i(\lambda). \tag{3}$$

The result follows by subtraction of (3) from (2).

COROLLARY 1 A. *If $[G, c]$ is a network and ϕ is a local G-potential, then*

$$\sum_{\xi \in V(G)} \phi(\xi)\, f_{c\phi}(\xi) = \sum_{\lambda \in E(G)} [\phi(\lambda t) - \phi(\lambda h)]^2\, c(\lambda).$$

COROLLARY 1 B. *If G is locally finite, i is a G-current and v is a finite subset of $V(G)$, then $F_i(v) = f_i.v$.*

Proof. Apply Lemma 1, taking $\phi(\xi)$ to be 1 if $\xi \in v$ and 0 if $\xi \in \bar{v}$.

Definition. If a vertex ξ and edge λ of a graph are incident, then $\xi \otimes \lambda$ will denote the vertex to which λ joins ξ.

LEMMA 2. *Let $[G, c]$ be a network, ϕ be a G-potential and ξ be a vertex of G. Then*

$$f_{c\phi}(\xi) = \phi(\xi)\, D_c(\xi) - \sum_{\lambda \in \iota(\xi)} \phi(\xi \otimes \lambda)\, c(\lambda).$$

Proof. For every $\lambda \in \iota(\xi)$, write

$$g(\lambda) \equiv [\phi(\xi) - \phi(\xi \otimes \lambda)]\, c(\lambda).$$

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Then it is clear that $g = 0$ on $\pi(\xi)$. Hence we have

$$\phi(\xi) D_c(\xi) - \sum_{\lambda \in \iota(\xi)} \phi(\xi \otimes \lambda) c(\lambda) = g.\iota(\xi) = g.\bar{\gamma}(\xi) + g.\pi(\xi) + g.\bar{\delta}(\xi)$$
$$= g.\bar{\gamma}(\xi) + 2g.\pi(\xi) + g.\bar{\delta}(\xi) = g.\gamma(\xi) + g.\delta(\xi) = I_{c\phi}.\gamma(\xi) - I_{c\phi}.\delta(\xi) = f_{c\phi}(\xi).$$

Definitions. If G is connected, the *distance* between two of its vertices is the minimum of the lengths of the paths connecting them in G, and the *distance* $\partial(\xi, v)$ of a vertex ξ from a non-empty subset v of $V(G)$ is the minimum of the distances of ξ from the elements of v.

Let $[G, c]$ be a network, α be a vertex of G and v be a subset of $V(G)$ which may or may not include α. Then $q(G, c, \alpha, v)$ will denote the probability that a $[G, c]$-walk which starts at α will *thereafter* visit some element of v; and $q_n(G, c, \alpha, v)$ will denote the probability that this will happen after (at least one and) at most n steps. In other words, in the notation of (1),

$$q(G, c, \alpha, v) \equiv \text{prob}\,(\xi_j \in v \text{ for some } j \in P \mid \xi_0 = \alpha),$$
$$q_n(G, c, \alpha, v) \equiv \text{prob}\,(\xi_j \in v \text{ for some } j \in P_n \mid \xi_0 = \alpha).$$

We shall also write

$$r(G, c, \alpha, v) \equiv 1 - q(G, c, \alpha, v), \quad r_n(G, c, \alpha, v) \equiv 1 - q_n(G, c, \alpha, v).$$

Let $S = [G, c, \alpha, v]$ be a system. For any $\xi \in V(G) - (\{\alpha\} \cup v)$, let $p(\xi)$ be the probability that a $[G, c]$-walk starting at ξ will visit α without first visiting any element of v, i.e. in the notation of (1),

$$p(\xi) \equiv \text{prob}\,(\exists j \in P, \xi_j = \alpha \text{ and } \xi_k \in \bar{v} \text{ for every } k \in P_{j-1} \mid \xi_0 = \xi).$$

Define $p(\alpha)$ to be 1 and $p(\eta)$ to be 0 for every $\eta \in v$. Then the G-potential p will be called the *probability potential* of S. We define $\rho(s)$ (the *reflexion probability* of S) to be the probability that a $[G, c]$-walk starting at α will re-visit α without first visiting any element of v, i.e. in the notation of (1),

$$\rho(S) \equiv \text{prob}\,(\exists j \in P, \xi_j = \alpha \text{ and } \xi_k \in \bar{v} \text{ for every } k \in P_{j-1} \mid \xi_0 = \alpha).$$

LEMMA 3. *If $[G, c]$ is a network, α is a vertex of G and v is a non-empty co-finite subset of $V(G)$, then $q(G, c, \alpha, v) = 1$.*

Proof. If $v = V(G)$, the result is trivial; otherwise, let

$$d \equiv \max_{\xi \in \bar{v}} \partial(\xi, v), \quad Q \equiv \min_{\xi \in \bar{v}} q_d(G, c, \xi, v).$$

Then clearly $Q > 0$, and hence we have

$$r_{nd+1}(G, c, \alpha, v) \leqslant (1 - Q)^n \to 0 \quad \text{as} \quad n \to \infty.$$

This proves the lemma.

COROLLARY 3A. *For any system S, $\rho(S) + \tau(S) = 1$.*

COROLLARY 3B. *Every finite network is recurrent.*

(Corollary 3B also follows from the Corollary on p. 355 of (1).)

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LEMMA 4. *Let $S = [G, c, \alpha, v]$ be a system and p be the probability potential of S. Then*
(i) $f_{cp} = 0$ *on* $V(G) - (\{\alpha\} \cup v)$,
(ii) $\tau(S) = f_{cp}(\alpha)/D_c(\alpha)$.

Proof. It is clear from probabilistic considerations that, for every $\xi \in V(G) - (\{\alpha\} \cup v)$,

$$p(\xi) = \sum_{\lambda \in \iota(\xi)} [c(\lambda)/D_c(\xi)]\, p(\xi \otimes \lambda), \tag{4}$$

and that

$$\rho(S) = \sum_{\lambda \in \iota(\alpha)} [c(\lambda)/D_c(\alpha)]\, p(\alpha \otimes \lambda). \tag{5}$$

The result (i) follows from (4) and Lemma 2. Moreover, by the definition of p,

$$1 = p(\alpha). \tag{6}$$

On subtracting (5) from (6), and using Corollary 3 A and Lemma 2, we obtain (ii).

Definition. Let $S = [G, c, \alpha, v]$ be a system. Then a G-potential ϕ is *S-admissible* if it satisfies the conditions

$$\phi(\alpha) = 1,\ \phi = 0 \text{ on } v, \quad f_{c\phi} = 0 \text{ on } V(G) - (\{\alpha\} \cup v).$$

An immediate consequence of the definition of 'probability potential' and Lemma 4 (i) is

COROLLARY 4 A. *The probability potential of a system S is S-admissible.*

We shall show later (Lemma 9) that the probability potential of a system $S = [G, c, \alpha, v]$ is the only S-admissible G-potential; but our theorems on random walk will be proved independently of this result.

LEMMA 5. *If a_j, b_j are real numbers $(j = 1, 2, \ldots, n)$, then* $\sum a_j^2 \sum b_j^2 \geqslant (\sum a_j b_j)^2$.

Proof. See (4), Theorem 7. (Neither of the proofs there given depends on the authors' assumption, introduced at the beginning of the chapter, that the numbers concerned are non-negative.)

LEMMA 6. *Let $S = [G, c, \alpha, v]$ be a system, ϕ be an S-admissible G-potential and C denote an αv-constriction $y_0, y_1, \ldots, y_n$ of G. Then $\Phi_c(C)\, f_{c\phi}(\alpha) \leqslant 1$, and this inequality becomes an equality if ϕ is constant on each y_j.*

Proof. Let $y_0 \cup y_1 \cup \ldots \cup y_r \equiv Y_r\ (r = 0, 1, 2, \ldots, n)$ and $y_{j-1} \circ y_j \equiv \omega_j\ (j = 1, 2, \ldots, n)$. For any $\lambda \in \omega_j$, let $I(\lambda)$ denote $I_{c\phi}(\lambda)$ or $-I_{c\phi}(\lambda)$ according as λt belongs to y_{j-1} or y_j respectively. Then

$$I \,.\, \omega_j = F_{c\phi}(Y_{j-1}) = f_{c\phi} \,.\, Y_{j-1} = f_{c\phi}(\alpha) \quad (j = 1, 2, \ldots, n) \tag{7}$$

by Corollary 1 B and the S-admissibility of ϕ. Using (7), Lemma 5 and Corollary 1 A, we have

$$\Phi_c(C)[f_{c\phi}(\alpha)]^2 = \sum_{j=1}^{n} [(I \,.\, \omega_j)^2/(c \,.\, \omega_j)] \leqslant \sum_{j=1}^{n} \sum_{\lambda \in \omega_j} I(\lambda)^2/c(\lambda) = \sum_{j=1}^{n} \sum_{\lambda \in \omega_j} I_{c\phi}(\lambda)^2/c(\lambda)$$

$$= \sum_{j=1}^{n} \sum_{\lambda \in \omega_j} [\phi(\lambda t) - \phi(\lambda h)]^2\, c(\lambda) \leqslant \sum_{\lambda \in E(G)} [\phi(\lambda t) - \phi(\lambda h)]^2\, c(\lambda) = \sum_{\xi \in V(G)} \phi(\xi)\, f_{c\phi}(\xi) = f_{c\phi}(\alpha)$$

since ϕ is S-admissible. Therefore $\Phi_c(C)\, f_{c\phi}(\alpha) \leqslant 1$.

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If $\phi = \phi_j$ on y_j $(j = 0, 1, 2, \ldots, n)$, where the ϕ_j are constants, then

$$I(\lambda) = (\phi_{j-1} - \phi_j)\, c(\lambda) \quad \text{for each} \quad \lambda \in \omega_j.$$

Therefore, by (7),

$$f_{c\phi}(\alpha)/c.\omega_j = \phi_{j-1} - \phi_j,$$

whence, by summation on j,

$$\Phi_c(C)\, f_{c\phi}(\alpha) = \phi_0 - \phi_n = 1 - 0$$

since $\phi(\alpha) = 1$, $\phi = 0$ on v, $\alpha \in y_0$ and $v \subset y_n$.

Definition. Let H be a refinement of G and ϕ be a G-potential. Then the H-potential ψ *induced* by ϕ is defined by the rules

$$\psi = \phi \text{ on } V(G)$$

$$\psi([\lambda, \theta]) = (1-\theta)\,\phi(\lambda t) + \theta\phi(\lambda h) \text{ for every } [\lambda, \theta] \in V(H) - V(G).$$

LEMMA 7. *Let $[G, c]$ be a network, $[H, d]$ be a refinement of $[G, c]$, ϕ be a G-potential and ψ be the H-potential induced by ϕ. Then*

$${}_H I_{d\psi}([\lambda, \theta, \theta']) = I_{c\phi}(\lambda) \text{ for every } [\lambda, \theta, \theta'] \in E(H).$$

(In accordance with the convention explained in § 1, we attach 'H' to '$I_{d\psi}$' to emphasize that the latter is to be interpreted in H.)

The proof of Lemma 7 may be left to the reader.

COROLLARY 7 A. *In the circumstances of Lemma 7,*

$${}_H f_{d\psi} = f_{c\phi} \text{ on } V(G),$$
$${}_H f_{d\psi} = 0 \text{ on } V(H) - V(G).$$

COROLLARY 7 B. *If $S = [G, c, \alpha, v]$ is a system and $T = [H, d, \alpha, v]$ is a refinement of S, then an S-admissible G-potential induces a T-admissible H-potential.*

This clearly follows from Corollary 7 A.

Definition. If j, k are distinct integers, $B_{jk} (= B_{kj})$ will denote the set of those integers which lie strictly between j and k.

LEMMA 8. *If $S = [G, c, \alpha, v]$ is a system and p is the probability potential of S, then*

$$f_{cp}(\alpha) = 1/\max_{T \in \mathfrak{R}(S)} M(T).$$

Proof. Let $T = [H, d, \alpha, v]$ be any refinement of S, C be an αv-constriction of H and q be the H-potential induced by p. Since p is S-admissible (Corollary 4 A), q is T-admissible (Corollary 7 B); therefore, by Lemma 6 and Corollary 7 A,

$$1/{}_H\Phi_d(C) \geqslant {}_H f_{dq}(\alpha) = f_{cp}(\alpha),$$

and this inequality becomes an equality if q is constant on each term of C. To prove Lemma 8, therefore, it suffices to show that H, C can be so chosen that q is constant on each term of C.

Let

$$1 = p_0 > p_1 > p_2 > \ldots > p_n = 0 \tag{8}$$

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be the distinct values assumed by p on vertices of G. Let X be the subset of $E(G) \times \mathscr{I}$ defined by the rule that, if $p(\lambda t) = p_k$ and $p(\lambda h) = p_l$, then

(i) if $|k-l| = 0$ or 1, $X[\lambda] = \emptyset$;

(ii) if $|k-l| \geqslant 2$, $X[\lambda]$ consists of the numbers

$$\frac{p_j - p_k}{p_l - p_k} \quad (j \in B_{kl}).$$

Let H be the refinement of G whose set of vertices is $V(G) \cup X$. Then, in the circumstances of (ii), we have

$$q([\lambda, (p_j - p_k)/(p_l - p_k)]) = p_j, \tag{9}$$

where q is the H-potential induced by p. (Thus, loosely speaking, H is the refinement of G obtained by 'inserting a new vertex wherever a p-equipotential through a vertex of G interesects an edge of G'.) Since $p = 0 (= p_n)$ on v, each element of $v \circ v$ is undivided in H; therefore, if d is the H-conductance corresponding to c, $[H, d, \alpha, v] \in \mathfrak{R}(S)$. From (9) and the fact that $q = p$ on $V(G)$, it follows that q only assumes values in the sequence (8). Let z_j be the set of those vertices of H on which q assumes the value p_j. Then (since $q = p$ on $V(G)$) $\alpha \in z_0$ and $v \subset z_n$; and, from the manner in which H was constructed, it is not difficult to see that $z_0, z_1, \ldots, z_n$ is an αv-constriction of H. Moreover, q is, by the definition of $z_0, z_1, \ldots, z_n$, constant on each z_j. Lemma 8 is therefore proved.

Theorem 1 follows at once from Lemmas 4 (ii) and 8.

We conclude this section by relating its contents to the theory of electrical networks.

LEMMA 9. *The probability potential of a system $S = [G, c, \alpha, v]$ is the only S-admissible G-potential.*

Proof. Let ϕ and ψ be two S-admissible G-potentials and let $\chi \equiv \phi - \psi$. Then

$$\chi = \phi - \psi = 0 \text{ on } \{\alpha\} \cup v, \tag{10}$$

$$f_{c\chi} = f_{c\phi} - f_{c\psi} = 0 \text{ on } V(G) - (\{\alpha\} \cup v).$$

Therefore
$$0 = \sum_{\xi \in V(G)} \chi(\xi) f_{c\chi}(\xi) = \sum_{\lambda \in E(G)} [\chi(\lambda t) - \chi(\lambda h)]^2 c(\lambda)$$

by Corollary 1 A. Therefore $\chi(\lambda t) = \chi(\lambda h)$ for every $\lambda \in E(G)$. Since G is connected, this implies that χ is constant and hence, by (10), zero on $V(G)$. Therefore there can be at most one S-admissible G-potential. But, by Corollary 4 A, the probability potential of S is S-admissible. The lemma is therefore proved.

Let $S = [G, c, \alpha, v]$ be a system. Regard $[G, c]$ as an electrical network in the sense explained in § 1, and let $\phi(\xi)$ denote the potential at any vertex ξ when an external agency maintains α at potential 1 and all elements of v at potential 0. Then, since the current flowing in the network must satisfy Kirchhoff's First Law at any element of $V(G) - (\{\alpha\} \cup v)$, ϕ must be S-admissible. Therefore, by Lemma 9, ϕ is the probability potential of S. Therefore, by Lemmas 4 (ii) and 8 respectively, the resistance of S is equal to each of the expressions

$$1/D_c(\alpha)\tau(S), \quad \max_{T \in \mathfrak{R}(S)} M(T).$$

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Remark. The author does not know whether the proof of Lemma 9 (establishing, in effect, that the behaviour of an essentially finite electrical network is uniquely determined by Kirchhoff's Laws and certain boundary conditions) has been previously published in the above form, but he obtained it as an adaptation of the argument of, for example, § 186 of (5).

3. *Proof of Theorem* 2.

Definition. If H is a refinement of G, an element of $V(H) - V(G)$ will be called a *division vertex.*

LEMMA 10. *An infinite network which has a slowly-widening refinement is recurrent.*

Proof. Let $[G, c]$ be an infinite network and $[H, d]$ be a slowly-widening refinement of $[G, c]$. Then H has a d-slowly-widening constriction $y_0, y_1, y_2, \dots$. Select any vertex α of G. We will show that α is c-recurrent in G.

Suppose that $\alpha \in y_m$. Then, given any $\epsilon > 0$, we can, since the constriction $y_0, y_1, y_2, \dots$ of H is d-slowly-widening, select an $n\ (\geqslant m)$ such that

$$\sum_{j=m}^{n} 1/d.(y_j \underset{H}{\circ} y_{j+1}) > 1/\epsilon D_c(\alpha). \tag{11}$$

Let Y_j denote $y_0 \cup y_1 \cup \dots \cup y_j$ $(j = 0, 1, 2, \dots)$, and let v be the set of all vertices ξ of G satisfying both of the conditions (i) $\xi \notin Y_n$, (ii) no element of $\iota(\xi)$ is the first component of a division vertex which belongs to Y_n. Then $[G, c, \alpha, v]$ is clearly a system. Let K be the refinement of G such that $V(K)$ is obtained from $V(H)$ by dropping therefrom all division vertices whose first components belong to $v \circ v$, and let e be the K-conductance corresponding to c. Then $[K, e, \alpha, v]$ is a refinement of $[G, c, \alpha, v]$; and it is not difficult to see that

$$Y_m,\ y_{m+1},\ y_{m+2},\ \dots,\ y_n,\ V(K) - Y_n$$

is an αv-constriction (C, say) of K and that ${}_K\Phi_e(C)$ is equal to the left-hand side of (11). Therefore, by (11) and Theorem 1, $\tau([G, c, \alpha, v]) < \epsilon$. Therefore, by Corollary 3A,

$$1 - \epsilon < \rho([G, c, \alpha, v]) \leqslant q(G, c, \alpha, \{\alpha\}).$$

Since ϵ was arbitrary, it follows that $q(G, c, \alpha, \{\alpha\}) = 1$. Therefore $[G, c]$ is recurrent.

We will now deduce the converse of Lemma 10 from Foster's result.

LEMMA 11. *Let $[G, c]$ be a recurrent network and ξ, α be distinct vertices of G. Then a $[G, c]$-walk which starts at ξ will certainly pass through α.*

Proof. Since G is connected, there exists at least one $\alpha\xi$-path of G. Select an $\alpha\xi$-path

$$\alpha = \xi_0,\ \lambda_1,\ \xi_1,\ \lambda_2,\ \dots,\ \lambda_n,\ \xi_n = \xi$$

of minimum length in G. Since this path is of minimum length, $\xi_1, \xi_2, \dots, \xi_n$ are all different from α. Therefore

$$\left[\prod_{j=1}^{n} \frac{c(\lambda_j)}{D_c(\xi_{j-1})}\right] r(G, c, \xi, \{\alpha\}) \leqslant r(G, c, \alpha, \{\alpha\}) = 0$$

since $[G, c]$ is recurrent. Therefore $r(G, c, \xi, \{\alpha\}) = 0$.

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Definitions. A *subgraph* of G is a graph H such that $V(H) \subset V(G)$, $E(H) \subset E(G)$ and every edge λ of H satisfies the conditions $\lambda t_H = \lambda t$, $\lambda h_H = \lambda h$. If $H_1, H_2, \ldots, H_n$ are subgraphs of G, their *union* is the subgraph H of G defined by

$$V(H) = V(H_1) \cup V(H_2) \cup \ldots \cup V(H_n), \quad E(H) = E(H_1) \cup E(H_2) \cup \ldots \cup E(H_n).$$

A vertex α of a connected graph G is a *cut-node* of G if G is the union of two non-trivial subgraphs H_1 and H_2 such that $V(H_1) \cap V(H_2) = \{\alpha\}$, $E(H_1) \cap E(H_2) = \emptyset$.

If f is a function defined on the elements of a set S, the restriction of f to a subset A of S will be denoted by $f \mid A$.

If f is a real-valued function defined on the elements of an infinite set S, then $f \to \infty$ on S (or f tends to ∞ on S) will mean that for every real K, there are at most finitely many $\xi \in S$ such that $f(\xi) \leqslant K$.

LEMMA 12. *Let $[G, c]$ be a recurrent infinite network and α be a vertex of G which is not a cut-node. Then there exists a G-potential x such that* (i) $x(\alpha) = 0$, (ii) $x \to \infty$ *on* $V(G)$, *and* (iii) $f_{cx} \geqslant 0$ *on* $V(G) - \{\alpha\}$.

Proof. Consider a $[G, c]$-walk modified by making the vertex α 'absorbent' in the sense that, once the walker reaches α (or if it is initially there), every subsequent step is from α to α. The resulting process is a Markov chain whose states may be identified with the vertices of G. Since G is locally finite, connected and infinite, $V(G)$ is clearly enumerable ((7), p. 79); let $\eta_0, \eta_1, \eta_2, \ldots$ be an enumeration of $V(G)$ such that $\eta_0 = \alpha$. Let p_{jk} denote the probability of a step to η_k when the walker is at η_j; thus

$$p_{jk} = c.(\{\eta_j\} \circ \{\eta_k\})/D_c(\xi_j) \quad (j \neq 0), \tag{12}$$

$$p_{0k} = 0 \quad (k \neq 0); \qquad p_{00} = 1.$$

Let $[p_{jk}^{(n)}]$ denote the matrix which is the nth power of $[p_{jk}]$. (Thus, when the walker is at η_j, $p_{jk}^{(n)}$ is the probability that it will be at η_k after precisely n further steps.) Let $\pi_{jk}^{(n)} = (p_{jk}^{(1)} + p_{jk}^{(2)} + \ldots + p_{jk}^{(n)})/n$. From Lemma 11 and the fact that α is absorbent, we see that, for each pair (j, k), $p_{jk}^{(n)} \to \delta_{0k}$ and hence also $\pi_{jk}^{(n)} \to \delta_{0k}$ as $n \to \infty$, where δ_{0k} is the Kronecker delta. Our Markov chain is therefore 'non-dissipative', i.e. satisfies condition (4) of (2). We will now show that it also satisfies conditions (5), (6), (7) and (8) of (2), where the set C in these conditions is taken to $\{\alpha\}\,[= \{\eta_0\}]$ and, in condition (8), δ is taken to be 0.

For the above choice of C, (5) of (2) reduces to $p_{00} = 1$, which is true since α is absorbent; and (6) of (2) is satisfied by taking t to be the distance between α and η_i. Since α is not a cut-node, η_i and η_j (where $0 < i < j$) can be connected by a path which does not pass through α, and (7) of (2) is satisfied by taking t to be the length of such a path. Finally, (8) of (2), with $\delta = 0$, is satisfied if the number t_i of positive integers t for which $p_{i0}^{(t)} = 0$ tends to ∞ as $i \to \infty$. But t_i is, for $i > 0$, one less than the distance between η_i and α; and, since G is locally finite, distance from α clearly tends to ∞ on $V(G)$. Hence clearly $t_i \to \infty$ as $i \to \infty$.

Since Conditions (4) to (8) of (2) are satisfied, there exist, by Theorem 2 of (2), real numbers $x_0, x_1, x_2, \ldots$ such that

$$\sum_{k=0}^{\infty} p_{jk} x_k \leqslant x_j \quad (j = 0, 1, 2, \ldots) \tag{13}$$

and

$$x_j \to \infty \quad \text{as} \quad j \to \infty. \tag{14}$$

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Write $x(\eta_j) = x_j$. Then (14) is clearly equivalent to (ii). Moreover, in view of (12), the inequalities (13) for $j > 0$ tell us that

$$x(\xi) \geqslant \sum_{\lambda \in \iota(\xi)} [c(\lambda)/D_c(\xi)]\, x(\xi \otimes \lambda)$$

for every $\xi \neq \alpha$, which, by Lemma 2, is equivalent to (iii). By adding a suitable constant to the value of x on each vertex, x can be made to satisfy (i) also.

COROLLARY 12A. *In Lemma* 12, *the hypothesis that* α *is not a cut-node can be dropped.*

Proof. Let $[G, c]$ be a recurrent infinite network and α be a vertex of G. Then G is clearly the union of a (uniquely determined) finite set of non-trivial connected subgraphs $H_1, H_2, \ldots, H_n$ such that (i) $V(H_j) \cap V(H_k) = \{\alpha\}$ and $E(H_j) \cap E(H_k) = \emptyset$ whenever $j \neq k$, and (ii) α is not a cutnode of any H_j. Since α is c-recurrent in G, it must clearly be $[c \,|\, E(H_j)]$-recurrent in H_j for each j, i.e. each $[H_j, c \,|\, E(H_j)]$ must be recurrent. Applying Lemma 12 to $[H_j, c \,|\, E(H_j)]$ for each infinite H_j, and defining x to be zero on the vertices of each finite H_j, we obtain a definition of x over the whole of $V(G)$ satisfying Conditions (i)–(iii) of Lemma 12.

LEMMA 13. *A recurrent infinite network has a slowly-widening refinement.*

Proof. Let $[G, c]$ be a recurrent infinite network. Select any vertex α of G. Then, by Corollary 12A, there is a G-potential x such that

$$(a)\ \ x \to \infty \text{ on } V(G),$$
$$(b)\ \ f_{cx} \geqslant 0 \text{ on } V(G) - \{\alpha\}.$$

By (a), the distinct values assumed by x on vertices of G can be arranged in an ascending infinite sequence

$$x_0 < x_1 < x_2 < \ldots. \tag{15}$$

We now proceed as in the proof of Lemma 8. Let X be the subset of $E(G) \times \mathscr{I}$ defined by the rule that, if $x(\lambda t) = x_k$ and $x(\lambda h) = x_l$, then

(i) if $|k-l| = 0$ or 1, $X[\lambda] = \emptyset$;

(ii) if $|k-l| \geqslant 2$, $X[\lambda]$ consists of the numbers

$$\frac{x_j - x_k}{x_l - x_k} \quad (j \in B_{kl}).$$

Let H be the refinement of G whose set of vertices is $V(G) \cup X$. Then, in the circumstances of (ii), we have

$$w([\lambda, (x_j - x_k)/(x_l - x_k)]) = x_j,$$

where w is the H-potential induced by x. Since, moreover, $w = x$ on $V(G)$, it follows that w only assumes values in the sequence (15). Let z_j be the set of those vertices of H on which w assumes the value x_j. Since $x \to \infty$ on $V(G)$, it is clear that $w \to \infty$ on $V(H)$; therefore the sets z_j are finite. From this fact and the preceding discussion, it is not difficult to see that $z_0, z_1, z_2, \ldots$ is a constriction of H.

Let d be the H-conductance corresponding to c. Then, by Condition (b) and Corollary 7A,

$$_H f_{dw} \geqslant 0 \text{ on } V(H) - \{\alpha\}. \tag{16}$$

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Let* $\alpha \in z_m$, and let Z_j denote $z_0 \cup z_1 \cup \ldots \cup z_j$ $(j = 0, 1, 2, \ldots)$. Then, for $j \geqslant m$, using Corollary 1 B and (16), we have

$$\begin{aligned}(x_{j+1}-x_j)\,d.(z_j \underset{H}{\circ} z_{j+1}) &= -{}_H F_{dw}(Z_j) = -{}_H f_{dw}.Z_j \leqslant -{}_H f_{dw}.Z_m \\ &= -{}_H F_{dw}(Z_m) = (x_{m+1}-x_m)\,d.(z_m \underset{H}{\circ} z_{m+1}) \equiv K,\end{aligned}$$

say, where K is clearly positive. Therefore, for $n \geqslant m$,

$$\sum_{j=m}^{n} 1/d.(z_j \underset{H}{\circ} z_{j+1}) \geqslant \sum_{j=m}^{n} (x_{j+1}-x_j)/K = (x_{n+1}-x_m)/K \to \infty \quad \text{as} \quad n \to \infty,$$

since $x_{n+1} \to \infty$ by Condition (a). This proves the lemma.

Theorem 2 follows from Lemmas 10 and 13.

4. *Proof of Gillis's conjecture.* Let $\mathbf{m}$ denote the general lattice point $(m_1, m_2, \ldots, m_d)$ in a d-dimensional lattice, L_d denote the set of all such lattice points, and $\mathbf{E}_j$ denote the unit vector parallel to the positive direction of the jth coordinate axis. Let ϵ be a real number such that $|\epsilon| < 1$. Consider a ('centrally biased') random walk starting at the origin, in which the only steps permitted are of the type $\mathbf{m} \to \mathbf{m} \pm \mathbf{E}_j$, with respective probabilities $P_j(\mathbf{m})$, $Q_j(\mathbf{m})$ $(j = 1, 2, \ldots, d)$, where

(i) if $m_j \neq 0$, $P_j(\mathbf{m}) = (1-\epsilon/m_j)/2d$ and $Q_j(\mathbf{m}) = (1+\epsilon/m_j)/2d$;

(ii) if $m_j = 0$, $P_j(\mathbf{m}) = Q_j(\mathbf{m}) = 1/2d$.

Call the origin *recurrent* if it is certain to be re-visited. Gillis (3) proved that the origin is recurrent for $\epsilon > \frac{1}{2} - d^{-1}$ and non-recurrent for $\epsilon < \frac{1}{2} - d^{-1}$, and conjectured that it is recurrent for $\epsilon = \frac{1}{2} - d^{-1}$, a result which, as stated above, has been proved by Erdős. The following is an alternative proof of recurrence for $\epsilon \geqslant \frac{1}{2} - d^{-1}$.

Let $p(0) \equiv \phi(0) \equiv 1$ and, for any positive integer n, let $p(n) \equiv 1 - \epsilon/n$ and

$$\phi(n) \equiv n(n-1-\epsilon)\,(n-2-\epsilon) \ldots (1-\epsilon)/(n+\epsilon)\,(n-1+\epsilon) \ldots (1+\epsilon)$$

(where the numerator is 1 if $n = 1$). Let G be a graph in which $V(G) = L_d$ and two vertices $\mathbf{m}$, $\mathbf{m}'$ are non-adjacent unless $\mathbf{m} - \mathbf{m}' = \pm \mathbf{E}_j$ for some j, in which case they are joined by a single edge. If an edge λ has end-vertices $\mathbf{m}$, $\mathbf{m}'$, of which $\mathbf{m}$ is the nearer to the origin, and if $\mathbf{m} - \mathbf{m}' = \pm \mathbf{E}_j$, define $c(\lambda)$ to be

$$p(|m_j|) \prod_{k=1}^{d} \phi(|m_k|).$$

Then it is easily verified that Gillis's random walk is a $[G, c]$-walk.

Consider the constriction $y_0, y_1, y_2, \ldots$ of G, where y_n is the set of all vertices $\mathbf{m}$ such that the maximum of the moduli of the components of $\mathbf{m}$ is n. (The elements of y_n lie on the boundary of a hypercube with centre at the origin.) We have

$$c.(y_n \circ y_{n+1}) = 2dp(n)\,\phi(n)\left[\sum_{r=-n}^{n} \phi(|r|)\right]^{d-1} < 4d\phi(n)\left[2\sum_{r=0}^{n} \phi(r)\right]^{d-1}.$$

Hence, to prove that $[G, c]$ is slowly-widening and therefore (by Theorem 2) recurrent, it suffices to show that

$$\sum_{n} \left(1/\phi(n)\left[\sum_{r=0}^{n} \phi(r)\right]^{d-1}\right) \tag{17}$$

* It is in fact easy, but unnecessary, to prove that $m = 0$.

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diverges. But, for $r > 0$,

$$\phi(r) = Kr\Gamma(r-\epsilon)/\Gamma(r+1+\epsilon) \sim Kr^{-2\epsilon}, \tag{18}$$

as $r \to \infty$, where $K = \Gamma(1+\epsilon)/\Gamma(1-\epsilon)$. Suppose that $\epsilon \geqslant \frac{1}{2} - d^{-1}$. Then, by (18), $\phi(r) \leqslant Hr^{-2\epsilon} \leqslant Hr^{(2/d)-1}$ for some $H = H(\epsilon)$ and all positive r. Therefore

$$\sum_{r=0}^{n} \phi(r) \leqslant Mn^{2/d}$$

for some $M = M(d, \epsilon)$ and all positive n. Therefore

$$\phi(n)\left[\sum_{r=0}^{n} \phi(r)\right]^{d-1} \leqslant HM^{d-1}n \quad (n > 0),$$

whence (17) diverges and so $[G, c]$ is recurrent.

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Note added in proof. When this paper was written, I did not know that Dr Gillis's conjecture had also been proved by Flatto (L. Flatto. A problem on a random walk. *Quart. J. Math.* (2) 9 (1958), 299–300).

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