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DERIVED FUNCTORS AND HILBERT POLYNOMIALS OVER HYPERSURFACE RINGS

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Abstract Let $(A, \mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. We show that there is a integer $r_I \geq -1$ (depending only on I) such that if M is any non-free maximal Cohen–Macaulay (= MCM) A-module the function $n \to \ell(\operatorname{Tor}_1^A(M, A/I^{n+1}))$ (which is of polynomial type) has degree r_I. Analogous results hold for Hilbert polynomials associated to Ext-functors. Surprisingly, a key ingredient is the classification of thick subcategories of the stable category of MCM A-modules (obtained by Takahashi, see [11, 6.6]).

Keywords: functions of polynomial type; derived functors; hypersurface rings; stable category of MCM modules

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1. Introduction

Let $(A, \mathfrak{m})$ be a non-regular Cohen–Macaulay local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. If N is an A-module of finite length then $\ell(N)$ denotes its length. Let M be a maximal Cohen–Macaulay (= MCM) A-module. The function $t_I(M, n) = \ell(\operatorname{Tor}_1^A(M, A/I^{n+1}))$ is of polynomial type, see [12, Corollary 4] (also see [8, Proposition 17]). Let $t_I^M(z) \in \mathbb{Q}[z]$ be such that $t_I^M(n) = t_I(M, n)$ for all $n \gg 0$. It is easily shown that $\deg t_I^M(z) \leq d-1$. In [8, Theorem 18], we proved that $\deg t_{\mathfrak{m}}^M(z) = d-1$ for any non-free MCM A-module. It was also shown that if I is a parameter ideal then $t_I(M, n) = 0$ for all $n \geq 0$, see [8, Remark 20]. In general, it is a difficult question to determine the degree of $t_I^M(z)$ and the answer is known only for a few classes of ideals and modules, see [6, 3.5] for some examples. The fact that $\deg t_{\mathfrak{m}}^M(z) = d-1$ for non-free MCM has an important consequence in the study of associated graded modules (with respect to $\mathfrak{m}$) of MCM A-modules, see [9].

In this paper, we prove few surprising results. Recall A is said to be a hypersurface ring if its completion $\widehat{A} = Q/(f)$ where $(Q, \mathfrak{n})$ is a regular local ring and $f \in \mathfrak{n}^2$ is non-zero. We set the degree of the zero polynomial to be -1. We show

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Theorem 1.1. *Let $(A, \mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. Then there is an integer $r_I \geq -1$ (depending only on I) such that if M is any non-free MCM A-module then* $\deg t_I^M(z) = r_I$.

1.1.

For the Ext functors, we prove an analogous result. It is known that if M is a finitely generated A-module the function $n \to \ell(\mathrm{Ext}_A^1(M, A/I^{n+1}))$ is of polynomial type say of degree s_I^M, see [12, Corollary 4]. We prove

Theorem 1.2. *Let $(A, \mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. Then there is an integer $s_I \geq -1$ (depending only on I) such that if M is any non-free MCM A-module then $s_I^M = s_I$.*

It is also known that if M is a finitely generated A-module the function $n \to \ell(\mathrm{Ext}_A^{d+1}(A/I^{n+1}, M))$ is of polynomial type say of degree e_I^M, see [12, Theorem 5]. Let $\mathrm{Spec}^0(A) = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$. We prove

Theorem 1.3. *Let $(A, \mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. Then there is an integer $e_I \geq -1$ (depending only on I) such that if M is any non-free MCM A-module free on* $\mathrm{Spec}^0(A)$ *then* $e_I^M = e_I$.

See Remark 4.1 on why in Theorem 1.3 we need to restrict to the case of MCM modules free on $\mathrm{Spec}^0(A)$ while in Theorems 1.1 and 1.2 we do not have such restriction.

Finally we show

Theorem 1.4. *(with hypotheses as in Theorems 1.1 and 1.2) We have $r_I = s_I$.*

1.2.

Let $G_I(A) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ be the associated graded ring of I. Let M be an A-module. Let $G_I(M) = \bigoplus_{n \geq 0} I^n M/I^{n+1}M$ be the associated graded module of M with respect to I, considered as a module over $G_I(A)$. We studied the function $t_I(M, n)$ as it is useful in the study of associated graded modules. The following result shows that the reverse direction is very strong at least for hypersurfaces; i.e. some properties of $G_I(M)$ for some MCM M has a strong impact on r_I.

Theorem 1.5. *Let $(A, \mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. Suppose there exists a non-free MCM A-module M with $G_I(M)$ unmixed and equi-dimensional. Then $r_I = d - 1$ or $r_I = -1$. Furthermore if $r_I = -1$ then* $\mathrm{projdim}\, A/I^n < \infty$ *for all $n \geq 1$.*

Technique used to prove the result: We first note that the function $t_I(M, n)$ is a function on $\underline{\mathrm{CM}}(A)$ the stable category of MCM A-modules. We also note that $\underline{\mathrm{CM}}(A)$ is a triangulated category [3, 4.4.1]. Let $\underline{\mathrm{CM}}_0(A)$ be the thick subcategory of MCM A-modules which are free on the punctured spectrum $\mathrm{Spec}^0(A)$ of A. The crucial ingredient

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in our proofs is that $\underline{\mathrm{CM}}_0(A)$ has *no* proper thick subcategories, see [11, 6.6]. We first prove Theorem 1.1 for non-free MCM modules in $\underline{\mathrm{CM}}_0(A)$ and then prove for all non-free MCM A-modules by using an induction on $\dim \underline{\mathrm{Hom}}_A(M, M)$. The techniques to prove Theorems 1.2 and 1.3 are similar.

Here is an overview of the contents of this paper. In § 2, we discuss a few preliminaries that we need. In § 3, we prove Theorems 1.1–1.3 when M is free on the punctured spectrum of A. In § 4, we prove Theorems 1.1 and 1.2. In § 5, we prove Theorem 1.4. Finally in § 6, we prove Theorem 1.5.

2. Preliminaries

In this section, we discuss a few preliminary results that we need. We use [7] for notation on triangulated categories. However, we will assume that if $\mathcal{C}$ is a triangulated category then $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set for any objects X, Y of $\mathcal{C}$.

2.1.

Let $\mathcal{C}$ be a skeletally small triangulated category with shift operator $\sum$ and let $\mathbb{I}(\mathcal{C})$ be the set of isomorphism classes of objects in $\mathcal{C}$. By a *weak triangle function* on $\mathcal{C}$, we mean a function $\xi\colon \mathbb{I}(\mathcal{C}) \to \mathbb{Z}$ such that

(1) $\xi(X) \geq 0$ for all $X \in \mathcal{C}$.
(2) $\xi(0) = 0$.
(3) $\xi(X \oplus Y) = \xi(X) + \xi(Y)$ for all $X, Y \in \mathcal{C}$.
(4) $\xi(\Sigma X) = \xi(X)$ for all $X \in \mathcal{C}$.
(5) If $X \to Y \to Z \to \Sigma X$ is a triangle in $\mathcal{C}$ then $\xi(Z) \leq \xi(X) + \xi(Y)$.

2.2.

Set

$$\ker \xi = \{X \mid \xi(X) = 0\}.$$

The following result (essentially an observation) is a crucial ingredient in our proof of Theorem 1.1.

Lemma 2.1. (see [*10*], 2.3) (with hypotheses as above) $\ker \xi$ *is a thick subcategory of* $\mathcal{C}$.

2.3.

Let $(A, \mathfrak{m})$ be a hypersurface ring and let I be an $\mathfrak{m}$-primary ideal in A. Let M be an MCM A-module. Set for $n \geq 0$

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$$t_I(M,n) = \ell(\mathrm{Tor}_1^A(M, A/I^{n+1}))$$
$$s_I(M,n) = \ell(\mathrm{Ext}_A^1(M, A/I^{n+1}))$$
$$e_I(M,n) = \ell(\mathrm{Ext}_A^{d+1}(A/I^{n+1}, M)).$$

Let $\Omega_A^i(M)$ denote the ith-syzygy of M. We prove

Lemma 2.2. *(with hypotheses as above)*

(1) *For all $n \geq 0$, the functions $t_I(-,n), s_I(-,n)$ and $e_I(-,n)$ are functions on $\underline{\mathrm{CM}}(A)$.*
(2) *For all $n \geq 0$, we have $t_I(M,n) = t_I(\Omega_A^1(M),n), s_I(M,n) = s_I(\Omega_A^1(M),n)$ and $e_I(M,n) = e_I(\Omega_A^1(M),n)$.*

Proof. (1) Let $E = M \oplus F = N \oplus G$ where F, G are free A-modules. Then by definition $t_I(E,n) = t_I(M,n) = t_I(N,n)$. Thus $t_I(-,n)$ is a function on $\underline{\mathrm{CM}}(A)$.

The proof for assertions on $s_I(-,n)$ and $e_I(-,n)$ are similar.

(2) We may assume that M has no free summands. Set $N = \Omega_A^1(M)$. Let $0 \to N \to F \to M \to 0$ be the minimal presentation of M with $F = A^r$. Then note as A is a hypersurface ring and M is MCM without free summands we get that a minimal presentation of N is as follows $0 \to M \to G \to N \to 0$ where $G = A^r$, see [4, 6.1(ii)]. By using the first exact sequence, we get

$$0 \to \mathrm{Tor}_1^A(M, A/I^{n+1}) \to N/I^{n+1}N \to F/I^{n+1}F \to M/I^{n+1} \to 0.$$

So we have

$$t_I(M,n) = \ell(N/I^{n+1}N) + \ell(M/I^{n+1}M) - r\ell(A/I^{n+1}A).$$

Using the second exact sequence, we find that $t_I(M,n) = t_I(N,n)$. The result follows.

The proof for assertions on $s_I(-,n)$ and $e_I(-,n)$ is similar. □

2.4.

The only fact we used in the above lemma was that $\Omega_A^2(M) \cong M$ if M is an MCM A-module with no free summands and that A is Gorenstein. Thus, the assertions of the lemma hold if A is a local complete intersection and M is a 2-periodic MCM A-module.

3. $\underline{\mathrm{CM}}_0(A)$

In this section, we give proofs of Theorems 1.1–1.3 when M is free on $\mathrm{Spec}^0(A)$.

Theorem 3.1. *Let $(A, \mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. Then there is an integer $r_I \geq -1$ (depending only on I) such that if $M \in \underline{\mathrm{CM}}_0(A)$ is non-zero then $\deg t_I^M(z) = r_I$.*

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Proof. We first note that for any MCM M we have $\deg t_I^M(z) \leq d-1$, see [12, Corollary 4]. Recall the degree of the zero polynomial to be -1. Set

$$r = \max\{\deg t_I^M(z) \mid M \in \underline{\mathrm{CM}}_0(A)\}.$$

If $r = -1$ then we have nothing to prove. So assume $r \geq 0$. For $M \in \underline{\mathrm{CM}}_0(A)$ define

$$\xi_I(M) = \lim_{n\to\infty} \frac{r!}{n^r} t_I(M,n).$$

We note that $\xi_I(M) \geq 0$ and is zero precisely when $\deg t_I(M,z) < r$.

Claim: $\xi_I(-)$ is a weak triangle function on $\underline{\mathrm{CM}}_0(A)$, see § 2.1.

Assume the claim for the time being. Then $\ker \xi$ is a thick subcategory of $\underline{\mathrm{CM}}_0(A)$. Also if $\deg t_I^L(z) = r$ then $L \notin \ker \xi$. So $\ker \xi \neq \underline{\mathrm{CM}}_0(A)$. As $\underline{\mathrm{CM}}_0(A)$ has no proper thick subcategories, see [11, 6.6], it follows that $\ker \xi = 0$. Therefore $\deg t_I^M(z) = r$ for all $M \neq 0$ in $\underline{\mathrm{CM}}_0(A)$.

It remains to show ξ_I is a weak triangle function on $\underline{\mathrm{CM}}_0(A)$. The first three conditions are trivial to satisfy. By Lemma 2.2(2), it follows that $\xi_I(\Omega_A^{-1}(M)) = \xi_I(M)$. Let $L \to M \to N \to \Omega^{-1}(L)$ is a triangle in $\underline{\mathrm{CM}}_0(A)$ then note that we have a short exact sequence of A-modules

$$0 \to M \to N \oplus F \to \Omega^{-1}(L) \to 0, \quad \text{where } F \text{ is free.}$$

Therefore, we have an inequality

$$t_I(N,n) \leq t_I(M,n) + t_I(\Omega^{-1}(L),n).$$

The result follows. □

The following two results can be proved similarly as in Theorem 3.1. We have to use that $\deg s_I^M(z) \leq d-1$ (see [12, Corollary 4]) and that $\deg e_I^M(z) \leq d$ (see [12, Corollary 7]).

Theorem 3.2. *Let $(A,\mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. Then there is an integer $s_I \geq -1$ (depending only on I) such that if $M \in \underline{\mathrm{CM}}_0(A)$ is non-zero then $\deg s_I^M(z) = s_I$.*

Theorem 3.3. *(= Theorem 1.3) Let $(A,\mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. Then there is an integer $e_I \geq -1$ (depending only on I) such that if $M \in \underline{\mathrm{CM}}_0(A)$ is non-zero then $\deg e_I^M(z) = e_I$.*

4. Proofs of Theorems 1.1 and 1.2

In this section, we give proofs of Theorems 1.1 and 1.2. We need a few preliminaries.

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4.1.

Let M be any finitely generated A-module. Set $L_i(M) = \bigoplus_{n\geq 0} \operatorname{Tor}_i^A(M, A/I^{n+1})$ for $i \geq 0$. Let $\mathcal{R} = A[It]$ be the Rees algebra of I. We have an exact sequence of $\mathcal{R}$-modules

$$0 \to \mathcal{R}(1) \to A[t](1) \to L_0(A) \to 0.$$

Tensoring with M yields an inclusion $0 \to L_1(M) \subseteq \mathcal{R}(1)\otimes M$ and isomorphisms $L_i(M) \cong \operatorname{Tor}_{i-1}^A(\mathcal{R}(1), M)$ for $i \geq 2$. It follows that $L_i(M)$ are finitely generated $\mathcal{R}$-module for all $i \geq 1$. We note that if $\Omega_2^A(M) \cong M$ then we have $L_i(M) \cong L_{i+2}(M)$ for all $i \geq 1$.

4.2.

We also need the following notion. Let $M \in \underline{\mathrm{CM}}(A)$. Let

$$\underline{\operatorname{Supp}}(M) = \{P \mid M_P \text{ is not free } A_P - \text{module}\}.$$

If I is an ideal in A then set $V(I) = \{P \in \operatorname{Spec}(A) \mid P \supseteq I\}$. It is readily verified that $\underline{\operatorname{Supp}}(M) = V(\operatorname{ann} \underline{\operatorname{Hom}}(M, M))$.

Proof of Theorem 1.1. By Theorem 3.1, we have that there exists r_I such that for any non-free MCM module $E \in \underline{\mathrm{CM}}_0(A)$ we have $\deg t_I^E(z) = r_I$.

Claim: For any non-free MCM A-module M, we have $\deg t_I^M(z) = r_I$.

We prove this assertion by induction on $\dim \underline{\operatorname{Supp}}(M)$. If $\dim \underline{\operatorname{Supp}}(M) = 0$ then M is free on $\operatorname{Spec}^0(A)$. In this case, we have nothing to show.

Now assume $\dim \underline{\operatorname{Supp}}(M) > 0$. As $L_1(M)_n, L_2(M)_n$ have finite length for all n and as $L_1(M), L_2(M)$ are finitely generated $\mathcal{R}$-modules it follows that there exists l such that $\mathfrak{m}^l L_i(M)_n = 0$ for all n and for $i = 1, 2$. As M has period two it follows that $\mathfrak{m}^l L_i(M)_n = 0$ for all $i \geq 1$ and all $n \geq 0$.

Let

$$x \in \mathfrak{m}^l \setminus \bigcup_{\substack{P \supseteq \operatorname{ann} \underline{\operatorname{Hom}}(M,M) \\ P \text{ minimal}}} P.$$

Let $M \xrightarrow{x} M \to N \to \Omega^{-1}(M)$ be a triangle in $\underline{\mathrm{CM}}(A)$. It is readily verified that support of $\underline{\operatorname{Hom}}(N, N)$ is contained in the intersection of support of $\underline{\operatorname{Hom}}(M, M)$ and M/xM. So $\dim \underline{\operatorname{Supp}}(N) \leq \dim \underline{\operatorname{Supp}}(M) - 1$. It is also not difficult to prove that N is not free A-module. By induction hypotheses $\deg t_I^N(z) = r_I$. By the structure of triangles in $\underline{\mathrm{CM}}(A)$, see [3, 4.4.1], we have an exact sequence $0 \to G \to N \to M/xM \to 0$ with G-free. It follows that $L_3(N) = L_3(M/xM)$. We also have an exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$. As $x \in \operatorname{ann} L_i(M)$, it follows that we have an exact sequence

$$0 \to L_3(M) \to L_3(M/xM) \to L_2(M) \to 0.$$

As the Hilbert function of $L_3(M)$ and $L_2(M)$ is identical, see Lemma 2.2(2), we get that $2t_I^M(z) = t_I^N(z)$. It follows that $\deg t_I^M(z) = r_I$. By induction, the result follows. □

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4.3.

To prove Theorem 1.2, we need a few preliminaries. Let M be a finitely generated Cohen–Macaulay A-module of dimension r. Let
$E^i(M) = \bigoplus_{n\geq 0} \operatorname{Ext}^i_A(M, A/I^{n+1})$. The exact sequence of $\mathcal{R}$-modules

$$0 \to \mathcal{R}(1) \to A[t](1) \to L_0(A) \to 0,$$

induces an isomorphism $E^i(M) \cong \operatorname{Ext}^{i+1}_A(M, \mathcal{R}(1))$ for all $i > d-r$. In particular, $E^i(M)$ are finitely generated $\mathcal{R}$-modules for all $i > d-r$. We note that if $\Omega^A_2(M) \cong M$ then we have $E^i(M) \cong E^{i+2}(M)$ for all $i \geq 1$. The proof of Theorem 1.2 is mostly similar to the proof of Theorem 1.1. So we mostly sketch the proof.

Sketch of a proof of Theorem 1.2. By Theorem 3.2, we have that there exists s_I such that for any non-free MCM module $L \in \underline{\mathrm{CM}}_0(A)$ we have $\deg s^L_I(z) = s_I$.

Claim: For any non-free MCM A-module M, we have $\deg s^M_I(z) = s_I$.

We prove this assertion by induction on $\dim \underline{\operatorname{Supp}}(M)$. If $\dim \underline{\operatorname{Supp}}(M) = 0$ then M is free on $\operatorname{Spec}^0(A)$. In this case, we have nothing to show.

Now assume $\dim \underline{\operatorname{Supp}}(M) > 0$. As $E^1(M)_n, E^2(M)_n$ have finite length for all n and as $E^1(M), E^2(M)$ are finitely generated $\mathcal{R}$-modules it follows that there exists l such that $\mathfrak{m}^l E^i(M)_n = 0$ for all n and for $i = 1, 2$. As M has period two it follows that $\mathfrak{m}^l E^i(M)_n = 0$ for all $i \geq 1$ and all $n \geq 0$. Let

$$x \in \mathfrak{m}^l \setminus \bigcup_{\substack{P \supseteq \operatorname{ann} \underline{\operatorname{Hom}}(M,M) \\ P \text{ minimal}}} P.$$

Let $M \xrightarrow{x} M \to N \to \Omega^{-1}(M)$ be a triangle in $\underline{\mathrm{CM}}(A)$. As before we have $\dim \underline{\operatorname{Supp}}(N) \leq \dim \underline{\operatorname{Supp}}(M) - 1$ and N is not free. By induction hypotheses $\deg s^N_I(z) = s_I$. By the structure of triangles in $\underline{\mathrm{CM}}(A)$, see [3, 4.4.1], we have an exact sequence $0 \to G \to N \to M/xM \to 0$ with G-free. It follows that $E^3(N) = E^3(M/xM)$. We also have an exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$. As $x \in \operatorname{ann} E^i(M)$ it follows that we have an exact sequence

$$0 \to E^2(M) \to E^3(M/xM) \to E^3(M) \to 0.$$

As the Hilbert function of $E^3(M)$ and $E^2(M)$ is identical, see Lemma 2.2(2), we get that $2s^M_I(z) = s^N_I(z)$. It follows that $\deg s^M_I(z) = s_I$. By induction, the result follows. □

Remark 4.1. Consider $U^i(M) = \bigoplus_{n\geq 0} \operatorname{Ext}^i_A(A/I^{n+1}, M)$. Then for $i \geq d+1$ it is possible to give a natural $\mathcal{R}$-module structure on $U^i(M)$. However, with this structure, $U^i(M)$ is NOT finitely generated (note if $xt \in \mathcal{R}_1$ then $x_1 t U^i(M)_n \subseteq U^i(M)_{n-1}$). Thus, it is not possible to extend the result in Theorem 3.3 to all MCM modules.

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5. Proof of Theorem 1.4

In this section, we give a proof of Theorem 1.4. To prove the result note that we may assume the residue field k of A is infinite (otherwise take the extension $A[t]_{\mathfrak{m}A[t]}$ of A).

5.1.

Recall $x \in I$ is M-superficial with respect to I if there exists c such that $(I^{n+1}M : x) \cap I^c M = I^n M$ for all $n \gg 0$. If $\operatorname{depth} M > 0$ then (as I is $\mathfrak{m}$-primary) it follows that x is M-regular and $(I^{n+1}M : x) = I^n M$ for $n \gg 0$. If k is infinite then I-superficial elements with respect to M exists. In fact in this case, there exists a non-empty open set U_M in the Zariski topology of $I/\mathfrak{m}I$ such that if the image of x is in U_M then x is I-superficial with respect to M.

5.2.

Let $E = \bigoplus_{n\geq 0} E_n$ be a finitely generated graded module over the Rees algebra $\mathcal{R} = A[It]$. Assume E_n has finite length for all n. There exists $xt \in \mathcal{R}_1$ such that xt is E-filter regular, i.e. $(0 :_E xt)_n = 0$ for $n \gg 0$. In fact in this case, there exists a non-empty open set U_E in the Zariski topology of $I/\mathfrak{m}I$ such that if the image of x is in U_E then xt is E-filter regular.

5.3.

Assume $\dim A \geq 2$ and assume M is an MCM A-module. Also assume A is Gorenstein. Let $x \in I$ be such that it is $M \oplus A$-superficial and xt is $L_1(M)$, $E^i(M)$-filter regular for $i = 1, 2$. We note that $(I^{n+1} : x) = I^n$ and $(I^{n+1}M : x) = I^n M$ for $n \gg 0$. Set $B = A/(x)$ and $J = I/(x)$. We have an exact sequence for $n \geq 1$

$$0 \to \ker \alpha_n \to A/I^n \xrightarrow{\alpha_n} A/I^{n+1} \to B/J^{n+1} \to 0,$$

where $\alpha_n(a + I^n) = xa + I^{n+1}$. We note that $\ker \alpha_n = (I^{n+1} : x)/I^n = 0$ for $n \gg 0$. Set $N = M/xM$. Thus for $n \gg 0$ we have an exact sequence

$$\operatorname{Tor}_1^A(M, A/I^n) \xrightarrow{\alpha_n^1} \operatorname{Tor}_1^A(M, A/I^{n+1}) \to \operatorname{Tor}_1^B(N, B/J^{n+1}) \to$$
$$M/I^n M \xrightarrow{\alpha_n^0} M/I^{n+1}M \to N/J^{n+1}N \to 0.$$

We note that $\ker \alpha_n^0 = (I^{n+1}M : x)/I^n M = 0$ for $n \gg 0$. Furthermore, the map α_n^1 is the nth-component of the multiplication map by $xt \in \mathcal{R}_1$ on $L_1(M)$. As xt is $L_1(M)$-filter regular, it follows that $\ker \alpha_n^1 = 0$ for $n \gg 0$. Thus for $n \gg 0$, we have an exact sequence.

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5.4.

$$0 \to \operatorname{Tor}_1^A(M, A/I^n) \xrightarrow{\alpha_n^1} \operatorname{Tor}_1^A(M, A/I^{n+1}) \to \operatorname{Tor}_1^B(N, B/J^{n+1}) \to 0.$$

Similarly, as xt is also $E_1(M)$ and $E_2(M)$-filter regular, we have an exact sequence.

5.5.

$$0 \to \operatorname{Ext}_A^1(M, A/I^n) \xrightarrow{\beta_n^1} \operatorname{Ext}_A^1(M, A/I^{n+1}) \to \operatorname{Ext}_B^1(N, B/J^{n+1}) \to 0.$$

5.6.

We note that if A is a hypersurface and $I \subseteq \mathfrak{m}^2$ is an $\mathfrak{m}$-primary ideal then if we go mod an I-superficial element x then the resulting ring $B = A/(x)$ is no longer a hypersurface. It is, however, a complete intersection. Thus we prove the following result:

Theorem 5.1. *Let $(A, \mathfrak{m})$ be a complete intersection of dimension $d \geq 1$. Let I be an $\mathfrak{m}$-primary ideal. Let M be a non-free MCM A-module with bounded Betti numbers. Then* $\deg t_I^M(z) = \deg s_I^M(z)$.

Proof. We prove the result by induction on dimension d of A. We may assume M has no free summands. As M has bounded Betti numbers, it follows that M is 2-periodic, i.e. $\Omega_A^2(M) \cong M$, (this is essentially contained in [1, 9.2.1]; use $M \cong \Omega^1(\Omega^{-1}(M))$).

We first consider the case when $d = 1$. We note that in this case $\deg t_I^M(z) = 0$ or $= -1$. Similarly, we have $\deg s_I^M(z) = 0$ or $= -1$. We show that $\deg t_I^M(z) = -1$ if and only if $\deg s_I^M(z) = -1$. Suppose $\deg t_I^M(z) = -1$. So $t_I^M(z) = 0$ and therefore $t_I(M, n) = 0$ for $n \gg 0$ (say from $n \geq m$). As $t_I(M, n) = t_I(\Omega_A^1(M), n)$ for all n and M is 2-periodic, we have $\operatorname{Tor}_i^A(M, A/I^{n+1}) = 0$ for all $i \geq 1$ and for $n \geq m$. It follows from [2, Theorem III] that $\operatorname{Ext}^i(M, A/I^{n+1}) = 0$ for all $i \gg 0$ and for all $n \geq m$. As $s_I(M, n) = s_I(\Omega_A^1(M), n)$ for all n and M is 2-periodic, we have $s_I(M, n) = 0$ for all $n \geq m$. It follows that $s_I^M(z) = 0$. Thus $\deg s_I^M(z) = -1$. The proof of when $\deg s_I^M(z) = -1$ then $\deg t_I^M(z) = -1$ is similar and left to the reader.

We now assume that $d \geq 2$ and the result is proved when $\dim A = d - 1$. A proof similar to above shows that $\deg t_I^M(z) = -1$ if and only if $\deg s_I^M(z) = -1$. So assume $\deg t_I^M(z) \geq 0$ (and so $\deg s_I^M(z) \geq 0$). We may assume that the residue field of A is infinite. Let $x \in I$ be $M \oplus A$-superficial and also assume that $xt \in \mathcal{R}_1$ is $L_1(M) \oplus E_1(M) \oplus E_2(M)$-filter regular. We note that $B = A/(x)$ is a complete intersection and $N = M/xM$ is an MCM B-module with bounded Betti numbers. Set $J = I/(x)$. So it follows from our induction hypotheses that $\deg t_J^N(z) = \deg s_J^N(z)$. As $\deg t_I^M(z) \geq 0$, it follows from § 5.4 that $\deg t_J^N(z) = \deg t_I^M(z) - 1$. Similarly, it follows from § 5.5 that $\deg s_J^N(z) = \deg s_I^M(z) - 1$. The result follows. □

We now give

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Proof of Theorem 1.4. Let M be non-free MCM A-module. Then $t_I = \deg t_I^M(z)$ and $s_I = \deg s_I^M(z)$. Note M is 2-periodic. By Theorem 5.1, the result follows. □

6. Proof of Theorem 1.5

In this section, we give

Proof of Theorem 1.5. Let $\mathcal{R} = A[It]$. Recall $L_0(M) = \bigoplus_{n\geq 0} M/I^{n+1}M$. By [9, 5.6], we have

$$\operatorname{Ass}_{\mathcal{R}} G_I(M) = \operatorname{Ass}_{\mathcal{R}} L_0(M).$$

As A is a hypersurface ring, in particular, it is a Gorenstein ring. So there exists an MCM A-module N and an exact sequence $0 \to M \to F \to N \to 0$ with F free. Notice $L_1(N)$ is a $\mathcal{R}$-submodule of $L_0(M)$. In particular, if $L_1(N) \neq 0$ then it is necessarily of dimension d by our assumption on M. So $r_I = d - 1$ if $L_1(N) \neq 0$. If $L_1(N) = 0$ then notice for all $n \geq 0$ we have $t_I(N, n) = t_I(\Omega_1^A(N), n) = 0$. So for all $n \geq 0$ we have $\operatorname{Tor}_i^A(N, A/I^{n+1}) = 0$ for all $i \geq 1$. As N does not have finite projective dimension it follows from [5, 1.9] that $\operatorname{projdim} A/I^{n+1}$ is finite for all $n \geq 0$. □

Remark 6.1. The assumption that $G_I(M)$ is equi-dimensional is a mild one. It is not difficult to show that if A is also universally catenary then $G_I(M)$ is equi-dimensional for any MCM A-module M (so is automatic in our case as A is a hypersurface and so universally catenary). The assumption that $G_I(M)$ is un-mixed is not mild. There are plenty of examples when $\operatorname{depth} G_I(M) = 0$.

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