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Integrals with a large parameter: Legendre functions of large degree and fixed order

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Abstract

Suppose that a function $f(z, n)$ depends on a large parameter n. A proposed expression $g(z, n)$ is an asymptotic approximation for $f(z, n)$ if it can be shown that the error (i.e. the difference between $f(z, n)$ and $g(z, n)$) is small of an appropriate order when $n \to \infty$. Effective error bounds are particularly useful in numerical work with asymptotic expansions. Most of the existing derivations of error bounds involve complicated calculations.

In the present paper it will be shown how comparatively simple arguments can be used to derive efficient error bounds. The uniformly asymptotic expansions (1·1) and (1·2) for the Legendre functions $Q_n^{-m}(\cosh z)$ and $P_n^{-m}(\cosh z)$ in terms of Bessel functions are used as an illustration of the method which is based on two observations: that the same coefficient functions appear in (1·1) and (1·2), and that these coefficient functions are analytic at $z = 0$ although the Legendre functions and Bessel functions have branch points at $z = 0$. Efficient uniform error bounds for the coefficient functions (and hence for the Legendre functions) are found from simpler non-uniform bounds by applying the maximum-modulus theorem.

1. *Introduction*

The higher transcendental functions of mathematical physics either satisfy second-order differential equations with analytic coefficients, or they have integral representations, or both. Starting either from the differential equation or from an integral we may use well-known techniques (described, e.g. by Olver [**3**]) to obtain asymptotic expansions, valid when some of the variables or parameters are large. The validity of an expansion is established when a suitably small bound is established for the error term (i.e. the difference between the function and the first p terms of its asymptotic expansion). At present, error bounds are derived most frequently from a differential equation, and it has been authoritatively stated ([**3**], ch. 10, § 7·1) that these are on the whole more efficient than bounds obtained from integral representations. Efficient error bounds are particularly important in numerical work when the values of a function are to be calculated from an asymptotic expression (cf. [**3**], ch. 3).

In the present work a method will be described for finding error bounds in a comparatively simple manner. For the sake of clarity we shall confine our attention to the associated Legendre functions $P_n^{-m}(\cosh z)$ and $Q_n^{-m}(\cosh z)$ of large degree n and fixed order m but it is evident that the ideas are applicable to a wide class of functions.

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The expansions for the Legendre functions involve Bessel functions and have the form ([3], ch. 12)

$$e^{mi\pi}u^m Q_n^{-m}(\cosh z) = K_m(uz)\sum_{s=0}^{p-1}\frac{A_s(z^2)}{u^{2s}} - \frac{z}{u}K_{m+1}(uz)\sum_{s=0}^{p-1}\frac{B_s(z^2)}{u^{2s}} + R_p(z,u,Q), \quad (1{\cdot}1)$$

$$u^m P_n^{-m}(\cosh z) = I_m(uz)\sum_{s=0}^{p-1}\frac{A_s(z^2)}{u^{2s}} + \frac{z}{u}I_{m+1}(uz)\sum_{s=0}^{p-1}\frac{B_s(z^2)}{u^{2s}} + R_p(z,u,P), \quad (1{\cdot}2)$$

where $u = n+\frac{1}{2}$, where the functions $A_s(z^2)$ and $B_s(z^2)$ are analytic near $z=0$, and where in the domain of validity each error term R_p is small compared with all the other terms on the right hand side of the corresponding equation when n is large enough. To derive error bounds, Olver used integral equations derived from the associated Legendre equation. An alternative method was used earlier by Szegö [4] who (for $m=0$) derived asymptotic expansions of a different but equivalent form from integral representations of the Legendre functions. In both these treatments the difficulties arise when z is small, of order u^{-1}, when the Legendre functions and Bessel functions vary rapidly.

The method of the present paper is based on the observation that the same two asymptotic series $\Sigma u^{-2s}A_s(z^2)$ and $\Sigma u^{-2s}B_s(z^2)$ occur in both the expansions (1·1) and (1·2), and that they are formally analytic in a domain containing $z=0$. In particular, unlike the Legendre and Bessel functions, they do not have a branch point at $z=0$. Let the functions $\mathfrak{A}^{-m}(z,u)$ and $\mathfrak{B}^{-m}(z,u)$ be defined as the solution of the pair of equations

$$e^{mi\pi}u^m Q_n^{-m}(\cosh z) = K_m(uz)\,\mathfrak{A}^{-m}(z,u) - \frac{z}{u}K_{m+1}(uz)\,\mathfrak{B}^{-m}(z,u), \quad (1{\cdot}3)$$

$$u^m P_n^{-m}(\cosh z) = I_m(uz)\,\mathfrak{A}^{-m}(z,u) + \frac{z}{u}I_{m+1}(uz)\,\mathfrak{B}^{-m}(z,u), \quad (1{\cdot}4)$$

then it will be shown that $\mathfrak{A}^{-m}(z,u)$ and $\mathfrak{B}^{-m}(z,u)$ are indeed analytic in a domain containing $z=0$, and the maximum-modulus theorem for analytic functions (Titchmarsh [5], ch. 5) will then be used to show that they have expansions of the simple form

$$\mathfrak{A}^{-m}(z,u) \sim \sum_{s=0}^{\infty}\frac{A_s(z^2)}{u^{2s}}, \quad \mathfrak{B}^{-m}(z,u) \sim \sum_{s=0}^{\infty}\frac{B_s(z^2)}{u^{2s}}, \quad (1{\cdot}5)$$

and to find efficient error bounds for these expansions. Some of the variables z, u, $-m$ will often be omitted from the function symbols in the subsequent work when this can be done without ambiguity.

2. *The form of the expansion*

In the present work it will be assumed that the parameters m and n are real, and that the complex variable z lies in a region $|z| \leqslant \rho < \pi$ containing the origin. (See however § 5 below). As was stated in the introduction, we shall be concerned with the asymptotic expansions (1·1) and (1·2) of the associated Legendre functions $Q_n^{-m}(\cosh z)$ and $P_n^{-m}(\cosh z)$ when $u = n+\frac{1}{2} \to \infty$ while m is kept fixed.

Let us suppose first that $m > -\frac{1}{2}$; this restriction will be removed in § 5. Then we have the integral representations (Erdélyi [1], vol. 1, § 3·7)

$$e^{mi\pi}Q_n^{-m}(\cosh z) = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{(\sinh z)^{-m}}{\Gamma(\frac{1}{2}+m)}\int_z^{\infty} e^{-ut}(\cosh t - \cosh z)^{m-\frac{1}{2}}\,dt, \quad (2{\cdot}1)$$

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and

$$P_n^{-m}(\cosh z) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \frac{(\sinh z)^{-m}}{\Gamma(\frac{1}{2}+m)} \int_{-z}^{z} e^{-ut}(\cosh z - \cosh t)^{m-\frac{1}{2}}\, dt. \tag{2·2}$$

We shall write

$$\psi_1(z, u, H, m-\tfrac{1}{2}) = \frac{1}{\pi}\int_{-z}^{z} e^{-ut} H(t^2, z^2)\,(z^2-t^2)^{m-\frac{1}{2}}\, dt, \tag{2·3}$$

and

$$\psi_2(z, u, H, m-\tfrac{1}{2}) = \int_{z}^{\infty} e^{-ut} H(t^2, z^2)\,(t^2-z^2)^{m-\frac{1}{2}}\, dt, \tag{2·4}$$

where $H(t^2, z^2)$ denotes any function of t^2 and z^2 which is analytic and satisfies an inequality of the form $|H(t^2, z^2)| < Ae^{k|t|}$ for all t such that $|\operatorname{Im} t| \leqslant \rho < \pi$, and for all z such that $|z| \leqslant \rho < \pi$.

We shall also write

$$G_0(t^2, z^2) = \left(\frac{2\cosh z - 2\cosh t}{z^2 - t^2}\right)^{m-\frac{1}{2}}. \tag{2·5}$$

Then we have

$$P_n^{-m}(\cosh z) = \pi^{\frac{1}{2}} 2^{m-1} \frac{(\sinh z)^{-m}}{\Gamma(\frac{1}{2}+m)} \psi_1(z, u, G_0, m-\tfrac{1}{2}), \tag{2·6}$$

and

$$e^{mi\pi} Q_n^{-m}(\cosh z) = \pi^{\frac{1}{2}} 2^{m-1} \frac{(\sinh z)^{-m}}{\Gamma(\frac{1}{2}+m)} \psi_2(z, u, G_0, m-\tfrac{1}{2}). \tag{2·7}$$

We also have the corresponding representations for Bessel functions (Erdélyi [1], vol. 2, § 7·2)

$$I_s(uz) = \frac{2^{-s} u^s z^{-s}}{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}+s)} \int_{-z}^{z} e^{-ut}(z^2-t^2)^{s-\frac{1}{2}}\, dt, \tag{2·8}$$

and

$$K_s(uz) = \frac{2^{-s} u^s z^{-s}}{\Gamma(\frac{1}{2}+s)} \pi^{\frac{1}{2}} \int_{z}^{\infty} e^{-ut}(t^2-z^2)^{s-\frac{1}{2}}\, dt. \tag{2·9}$$

If we define the symbols

$$\phi_1(z, u, s-\tfrac{1}{2}) = \frac{1}{\pi}\int_{-z}^{z} e^{-ut}(z^2-t^2)^{s-\frac{1}{2}}\, dt, \tag{2·10}$$

and

$$\phi_2(z, u, s-\tfrac{1}{2}) = \int_{z}^{\infty} e^{-ut}(t^2-z^2)^{s-\frac{1}{2}}\, dt, \tag{2·11}$$

then we can write

$$I_s(uz) = \frac{2^{-s} u^s z^{-s}}{\Gamma(\frac{1}{2}+s)} \pi^{\frac{1}{2}} \phi_1(z, u, s-\tfrac{1}{2}), \tag{2·12}$$

and

$$K_s(uz) = \frac{2^{-s} u^s z^{-s}}{\Gamma(\frac{1}{2}+s)} \pi^{\frac{1}{2}} \phi_2(z, u, s-\tfrac{1}{2}). \tag{2·13}$$

To derive the form of the expansion we shall define the sequences of functions $G_p(t^2, z^2)$, $G_p^*(t^2, z^2)$, $G_p^{**}(t^2, z^2)$, $a_p(z^2)$, $b_p(z^2)$, for $p = 0, 1, 2, \ldots$ in the following manner. Suppose that $G_p(t^2, z^2)$ is given, then we write

$$G_p^*(t^2, z^2) = \frac{G_p(t^2, z^2) - G_p(z^2, z^2)}{t^2 - z^2},$$

$$G_p^{**}(t^2, z^2) = \frac{G_p^*(t^2, z^2) - G_p^*(z^2, z^2)}{t^2 - z^2}, \tag{2·14}$$

$$a_p(z^2) = G_p(z^2, z^2), \quad b_p(z^2) = -G_p^*(z^2, z^2). \tag{2·15}$$

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Evidently we have the identity

$$G_p(t^2,z^2) = a_p(z^2) - (t^2-z^2)\,b_p(z^2) + (t^2-z^2)^2\,G_p^{**}(t^2,z^2). \tag{2·16}$$

We now define $G_{p+1}(t^2,z^2)$ in terms of $G_p(t^2,z^2)$ by the relation

$$G_{p+1}(t^2,z^2)\,(z^2-t^2)^{m-\frac{1}{2}} = \frac{d^2}{dt^2}\{G_p^{**}(t^2,z^2)\,(z^2-t^2)^{m+\frac{3}{2}}\} \quad (p=0,1,2,3,\ldots) \tag{2·17}$$

whence

$$G_{p+1}(t^2,z^2) = (z^2-t^2)^2\frac{d^2G_p^{**}}{dt^2} - (z^2-t^2)\,(2m+3)\left\{2t\frac{dG_p^{**}}{dt} + G_p^{**}\right\} + t^2(2m+1)\,(2m+3)\,G_p^{**}. \tag{2·18}$$

(We note incidentally that $G_1(0,0) = G_2(0,0) = \ldots = G_p(0,0) = 0$.) Since $G_0(t^2,z^2)$ is defined by (2·5) we have now defined the sequences $G_p(t^2,z^2)$, $G_p^*(t^2,z^2)$, $G_p^{**}(t^2,z^2)$, $a_p(z^2)$ and $b_p(z^2)$ for $p = 0,1,2,\ldots$. From (2·16) we see that

$$\begin{aligned}
&\psi_1(z,u,G_0,m-\tfrac{1}{2}) - a_0(z^2)\,\phi_1(z,u,m-\tfrac{1}{2}) - b_0(z^2)\,\phi_1(z,u,m+\tfrac{1}{2}) \\
&= \frac{1}{\pi}\int_{-z}^{z} e^{-ut}G_0^{**}(t^2,z^2)\,(z^2-t^2)^{m+\frac{1}{2}}\,dt \\
&= \frac{1}{\pi u^2}\int_{-z}^{z} e^{-ut}\frac{d^2}{dt^2}\{G_0^{**}(t^2,z^2)\,(z^2-t^2)^{m+\frac{3}{2}}\}\,dt, \quad \text{by two integrations by parts,} \\
&= \frac{1}{u^2}\psi_1(z,u,G_1,m-\tfrac{1}{2}), \quad \text{from (2·17),}
\end{aligned} \tag{2·19}$$

and similarly that

$$\psi_1(z,u,G_s,m-\tfrac{1}{2}) = a_s(z^2)\,\phi_1(z,u,m-\tfrac{1}{2}) + b_s(z^2)\,\phi_1(z,u,m+\tfrac{1}{2}) + \frac{1}{u^2}\psi_1(z,u,G_{s+1},m-\tfrac{1}{2}), \quad s=0,1,2,\ldots . \tag{2·20}$$

We thus obtain the relation

$$\begin{aligned}
\psi_1(z,u,G_0,m-\tfrac{1}{2}) &= \phi_1(z,u,m-\tfrac{1}{2})\left(a_0(z^2) + \frac{a_1(z^2)}{u^2} + \ldots + \frac{a_{p-1}(z^2)}{u^{2p-2}}\right) \\
&\quad + \phi_1(z,u,m+\tfrac{1}{2})\left(b_0(z^2) + \frac{b_1(z^2)}{u^2} + \ldots + \frac{b_{p-1}(z^2)}{u^{2p-2}}\right) \\
&\quad + \frac{1}{\pi u^{2p}}\int_{-z}^{z} e^{-ut}G_p(t^2,z^2)\,(z^2-t^2)^{m-\frac{1}{2}}\,dt,
\end{aligned} \tag{2·21}$$

and similarly

$$\begin{aligned}
\psi_2(z,u,G_0,m-\tfrac{1}{2}) &= \phi_2(z,u,m-\tfrac{1}{2})\left(a_0(z^2) + \frac{a_1(z^2)}{u^2} + \ldots + \frac{a_{p-1}(z^2)}{u^{2p-2}}\right) \\
&\quad - \phi_2(z,u,m+\tfrac{1}{2})\left(b_0(z^2) + \frac{b_1(z^2)}{u^2} + \ldots + \frac{b_{p-1}(z^2)}{u^{2p-2}}\right) \\
&\quad + \frac{1}{u^{2p}}\int_{z}^{\infty} e^{-ut}G_p(t^2,z^2)\,(t^2-z^2)^{m-\frac{1}{2}}\,dt.
\end{aligned} \tag{2·22}$$

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It is readily seen that equations (2·21) and (2·22) are equivalent to the asymptotic expansions (1·2) and (1·1), and that the error terms have the formally explicit expressions

$$R_p(z,u,P) = \pi^{\frac{1}{2}}\,2^{m-1}\frac{(\sinh z)^{-m}}{\Gamma(\frac{1}{2}+m)}\cdot\frac{1}{\pi u^{2p}}\int_{-z}^{z} e^{-ut}G_p(t^2,z^2)\,(z^2-t^2)^{m-\frac{1}{2}}\,dt, \tag{2·24}$$

and

$$R_p(z,u,Q) = \pi^{\frac{1}{2}}\,2^{m-1}\frac{(\sinh z)^{-m}}{\Gamma(\frac{1}{2}+m)}\cdot\frac{1}{u^{2p}}\int_{z}^{\infty} e^{-ut}G_p(t^2,z^2)\,(t^2-z^2)^{m-\frac{1}{2}}\,dt. \tag{2·25}$$

The integrals vary rapidly when z is small, of order u^{-1}, (thus $R_p(z,u,Q)$ is exponentially small when $\operatorname{ph} z = 0$ but exponentially large when $\operatorname{ph} z = \pi$), and therefore they cannot easily be estimated efficiently in this neighbourhood.

3. *The functions* $\mathfrak{A}$ *and* $\mathfrak{B}$

Let $\mathfrak{A}^{-m}(z,u)$ and $\mathfrak{B}^{-m}(z,u)$ satisfy equation (1·3) and (1·4). (Henceforth we shall usually write $\mathfrak{A}(z,u)$ instead of $\mathfrak{A}^{-m}(z,u)$, and similarly for $\mathfrak{B}$.) Then it follows that

$$\mathfrak{A}(z,u) = uz\begin{vmatrix} u^m P_n^{-m}(\cosh z) & -I_{m+1}(uz) \\ u^m e^{m i\pi} Q_n^{-m}(\cosh z) & K_{m+1}(uz)\end{vmatrix}, \tag{3·1}$$

where the relation (Erdélyi[1], vol. 2, § 7·11)

$$I_m(uz)\,K_{m+1}(uz) + I_{m+1}(uz)\,K_m(uz) = \frac{1}{uz}, \tag{3·2}$$

has been used. Using the representations (2·3), (2·4), (2·10) and (2·11) we can write this in the form

$$\mathfrak{A}(z,u) = \frac{\pi u^{2m+2}\,2^{-2m-1}\,z^{-m}(\sinh z)^{-m}}{\Gamma(\frac{1}{2}+m)\,\Gamma(\frac{3}{2}+m)}\begin{vmatrix} \psi_1(z,u,G_0,m-\frac{1}{2}) & -\phi_1(z,u,m+\frac{1}{2}) \\ \psi_2(z,u,G_0,m-\frac{1}{2}) & \phi_2(z,u,m+\frac{1}{2})\end{vmatrix} \tag{3·3}$$

$$= \frac{\pi u^{2m+2}\,2^{-2m-1}\,z^{-m}(\sinh z)^{-m}}{\Gamma(\frac{1}{2}+m)\,\Gamma(\frac{3}{2}+m)}\{\psi_1(z,u,G_0,m-\tfrac{1}{2})\,\phi_2(z,u,m+\tfrac{1}{2}) + \psi_2(z,u,G_0,m-\tfrac{1}{2})\,\phi_1(z,u,m+\tfrac{1}{2})\}. \tag{3·4}$$

By analogy, let us define

$$\mathfrak{A}(z,u,H) = \frac{\pi u^{2m+2}\,2^{-2m-1}\,z^{-m}(\sinh z)^{-m}}{\Gamma(\frac{1}{2}+m)\,\Gamma(\frac{3}{2}+m)}\{\psi_1(z,u,H,m-\tfrac{1}{2})\,\phi_2(z,u,m+\tfrac{1}{2}) + \psi_2(z,u,H,m-\tfrac{1}{2})\,\phi_1(z,u,m+\tfrac{1}{2})\}; \tag{3·5}$$

cf. equations (2·3) and (2·4).

Similarly we find that

$$\mathfrak{B}(z,u) = \frac{\pi u^{2m+2}\,2^{-2m}\,z^{-m}(\sinh z)^{-m}}{\{\Gamma(\frac{1}{2}+m)\}^2}\{\psi_1(z,u,G_0,m-\tfrac{1}{2})\,\phi_2(z,u,m-\tfrac{1}{2}) - \psi_2(z,u,G_0,m-\tfrac{1}{2})\,\phi_1(z,u,m-\tfrac{1}{2})\}, \tag{3·6}$$

and we define

$$\mathfrak{B}(z,u,H) = \frac{\pi u^{2m+2}\,2^{-2m}z^{-m}(\sinh z)^{-m}}{\{\Gamma(\frac{1}{2}+m)\}^2}\{\psi_1(z,u,H,m-\tfrac{1}{2})\,\phi_2(z,u,m-\tfrac{1}{2}) - \psi_2(z,u,H,m-\tfrac{1}{2})\,\phi_1(z,u,m-\tfrac{1}{2})\}. \tag{3·7}$$

Thus $$\mathfrak{A}(z,u) = \mathfrak{A}(z,u,G_0) \quad \text{and} \quad \mathfrak{B}(z,u) = \mathfrak{B}(z,u,G_0). \tag{3·8}$$

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Also, by substituting (2·22) and (2·23) in equations (3·4) and (3·6) respectively and using (3·2), we obtain

$$\mathfrak{A}(z,u) = \left(\frac{z}{\sinh z}\right)^m \sum_0^{p-1} \frac{a_s(z^2)}{u^{2s}} + \frac{\mathfrak{A}(z,u,G_p)}{u^{2p}}, \tag{3·9}$$

and

$$\mathfrak{B}(z,u) = (2m+1)\left(\frac{z}{\sinh z}\right)^m \sum_0^{p-1} \frac{b_s(z^2)}{u^{2s}} + \frac{\mathfrak{B}(z,u,G_p)}{u^{2p}}, \tag{3·10}$$

where $a_s(z^2)$ and $b_s(z^2)$ are defined in § 2 above.

Our principal results are Theorem 1 and Theorem 2 below.

THEOREM 1. *The functions $\mathfrak{A}(z,u,G_p)$ and $\mathfrak{B}(z,u,G_p)$, which appear in the error terms of* (3·9) *and* (3·10) *and also depend on m, are analytic functions of z^2 when $|z| \leqslant \rho$ and satisfy the bounds*

$$u^{-2p}|\mathfrak{A}(z,u,G_p)| \leqslant u^{-2p}\left|\left(\frac{z}{\sinh z}\right)^m a_p(z^2)\right| + u^{-2p-2}\left|\frac{z}{\sinh z}\right|^m M_p|z|^2, \tag{3·12}$$

and

$$u^{-2p}|\mathfrak{B}(z,u,G_p)| \leqslant u^{-2p}(2m+1)\left|\left(\frac{z}{\sinh z}\right)^m b_p(z^2)\right| + u^{-2p-2}\left|\frac{z}{\sinh z}\right|^m M_p, \tag{3·13}$$

where the constants M_p are independent of u.

THEOREM 2. *When $|z| \leqslant \rho$, the error term*

$$R_p(z,u,P) = I_m(uz)\frac{\mathfrak{A}(z,u,G_p)}{u^{2p}} + \frac{z}{u} I_{m+1}(uz)\frac{\mathfrak{B}(z,u,G_p)}{u^{2p}} \tag{3·14}$$

in (2·24) *satisfies the inequality*

$$|R_p(z,u,P)| \leqslant u^{-2p}\left|\left(\frac{z}{\sinh z}\right)^m\right| \cdot \left[|a_p(z)\, I_m(uz)| + (2m+1)\left|\frac{zb_p(z^2)}{u} I_{m+1}(uz)\right|\right] + u^{-2p-2}M_p\left[|z^2 I_m(uz)| + \left|\frac{z}{u} I_{m+1}(uz)\right|\right]\left|\frac{z}{\sinh z}\right|^m, \tag{3·15}$$

with a similar result for $R_p(z,u,Q)$, the K-functions everywhere replacing the I-functions. In the notation of § 1 we have

$$|R_p(z,u,P)| \leqslant u^{-2p}\left[|A_p(z^2)\, I_m(uz)| + \left|\frac{zB_p(z^2)}{u} I_{m+1}(uz)\right|\right] + u^{-2p-2}M_p\left[|z^2 I_m(uz)| + \left|\frac{z}{u} I_{m+1}(uz)\right|\right]\left|\frac{z}{\sinh z}\right|^m, \tag{3·16}$$

with a similar result for $R_p(z,u,Q)$.

4. *Proof of the Theorems*

The proof of Theorem 1 relies on three lemmas concerning the functions $\mathfrak{A}(z,u,H)$ and $\mathfrak{B}(z,u,H)$ given by the expressions (3·5) and (3·7). We shall show that these expressions are single-valued and (in a certain sense) bounded although the products $\phi_1\psi_2$ and $\phi_2\psi_1$ have neither of these properties. These properties are established by means of suitable row operations on determinants.

LEMMA A. *Let $H(t^2,z^2)$ denote a function of t^2 and z^2 which is analytic and satisfies an*

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inequality of the form $|H(t^2, z^2)| < Ae^{k|t|}$ *for all* t *such that* $|\operatorname{Im} t| \leqslant \rho \leqslant \pi$ *and all* z *such that* $|z| \leqslant \rho$, k *and* A *being positive constants. Then the functions*

$$\mathfrak{F}_1(z, u, H) = \left(\frac{\sinh z}{z}\right)^m \frac{\Gamma(\frac{1}{2}+m)\,\Gamma(\frac{3}{2}+m)}{\pi u^{2m+2}\,.\,2^{-2m-1}}\,\mathfrak{U}(z, u, H) \tag{4·1}$$

$$= z^{-2m}\{\psi_1(z, u, H, m-\tfrac{1}{2})\,\phi_2(z, u, m+\tfrac{1}{2}) + \psi_2(z, u, H, m-\tfrac{1}{2})\,\phi_1(z, u, m+\tfrac{1}{2})\} \tag{4·2}$$

and

$$\mathfrak{F}_2(z, u, H) \equiv \left(\frac{\sinh z}{z}\right)^m \frac{\{\Gamma(\frac{1}{2}+m)\}^2}{\pi u^{2m+2}\,.\,2^{-2m}}\,\mathfrak{B}(z, u, H) \tag{4·3}$$

$$= z^{-2m}\{\psi_1(z, u, H, m-\tfrac{1}{2})\,\phi_2(z, u, m-\tfrac{1}{2}) - \psi_2(z, u, H, m-\tfrac{1}{2})\,\phi_1(z, u, m-\tfrac{1}{2})\} \tag{4·4}$$

are single-valued functions of z^2 *when* $|z| \leqslant \rho$, *i.e.*

$$\mathfrak{F}_1(ze^{i\pi}) = \mathfrak{F}_1(z), \quad \textit{and} \quad \mathfrak{F}_2(ze^{i\pi}) = \mathfrak{F}_2(z). \tag{4·5}$$

Proof. Consider in turn the functions $\psi_1(z)$, $\psi_2(z)$, $\phi_1(z)$, $\phi_2(z)$ and their analytic continuations when z moves from a value z_0 (near the positive real axis) to $z_0 e^{i\pi}$.

Consider first

$$\psi_1(z, m-\tfrac{1}{2}) = \frac{1}{\pi}\int_{-z}^{z} e^{-ut} H(t^2, z^2)\,(z^2-t^2)^{m-\frac{1}{2}}\,dt = \frac{1}{\pi}\int_{-z}^{z} \cosh ut\,.\,H(t^2, z^2)\,(z^2-t^2)^{m-\frac{1}{2}}\,dt.$$

The final line of integration is the same as the initial line of integration (but in the opposite direction), and the value of the integral is thus unchanged except for a phase factor which can be found from the special case $H(t^2, z^2) = 1$, i.e. the Bessel integral (2·8).

We thus find that

$$\phi_1(z_0 e^{i\pi}, m-\tfrac{1}{2}) = e^{2mi\pi}\phi_1(z_0, m-\tfrac{1}{2}),$$

and

$$\psi_1(z_0 e^{i\pi}, m-\tfrac{1}{2}) = e^{2mi\pi}\psi_1(z_0, m-\tfrac{1}{2}).$$

Thus also

$$\phi_1(z_0 e^{i\pi}, m+\tfrac{1}{2}) = e^{(2m+2)i\pi}\phi_1(z_0, m+\tfrac{1}{2}) = e^{2mi\pi}\phi_1(z_0, m+\tfrac{1}{2}).$$

Next consider

$$\psi_2(z, m-\tfrac{1}{2}) = \int_{z}^{\infty} e^{-ut} H(t^2, z^2)\,(t^2-z^2)^{m-\frac{1}{2}}\,dt.$$

It is evident that $\psi_2(z_0 e^{i\pi}, m-\frac{1}{2}) = \psi_2(z_0, m-\frac{1}{2}) + \text{const.}\,\psi_1(z_0, m-\frac{1}{2})$, where again the constant (which is actually not needed) can be found from the Bessel integrals. It is thus found that

$$\phi_2(z_0 e^{i\pi}, m-\tfrac{1}{2}) = \phi_2(z_0, m-\tfrac{1}{2}) - i\pi^2 e^{mi\pi}\phi_1(z_0, m-\tfrac{1}{2}), \tag{4·6}$$

and

$$\psi_2(z_0 e^{i\pi}, m-\tfrac{1}{2}) = \psi_2(z_0, m-\tfrac{1}{2}) - i\pi^2 e^{mi\pi}\psi_1(z_0, m-\tfrac{1}{2}). \tag{4·7}$$

Thus

$$\begin{aligned}(ze^{i\pi})^{2m}\,\mathfrak{F}_1(ze^{i\pi}) &= \psi_1(ze^{i\pi}, m-\tfrac{1}{2})\,\phi_2(ze^{i\pi}, m+\tfrac{1}{2}) + \psi_2(ze^{i\pi}, m-\tfrac{1}{2})\,\phi_1(ze^{i\pi}, m+\tfrac{1}{2}),\\ &= e^{2mi\pi}\psi_1(z, m-\tfrac{1}{2})\{\phi_2(z, m+\tfrac{1}{2}) + i\pi^2 e^{mi\pi}\phi_1(z, m+\tfrac{1}{2})\}\\ &\quad + e^{2mi\pi}\phi_1(z, m+\tfrac{1}{2})\{\psi_2(z, m-\tfrac{1}{2}) - i\pi^2 e^{mi\pi}\psi_1(z, m-\tfrac{1}{2})\}\\ &= e^{2mi\pi}\{\psi_1(z, m-\tfrac{1}{2})\,\phi_2(z, m+\tfrac{1}{2}) + \psi_2(z, m-\tfrac{1}{2})\,\phi_1(z, m+\tfrac{1}{2})\}\\ &= e^{2mi\pi}z^{2m}\,\mathfrak{F}_1(z), \quad \text{i.e.} \quad \mathfrak{F}_1(ze^{i\pi}) = \mathfrak{F}_1(z).\end{aligned} \tag{4·8}$$

Similarly it can be shown that $\mathfrak{F}_2(ze^{i\pi}) = \mathfrak{F}_2(z)$. (4·9)

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This concludes the proof of Lemma A. It follows that $\mathfrak{F}_1(ze^{2i\pi}) = \mathfrak{F}_1(ze^{i\pi}) = \mathfrak{F}_1(z)$, and that $\mathfrak{F}_1(z)$ can be expanded in a Laurent series involving positive and negative powers of z^2. In fact only positive powers are present as will now be shown.

LEMMA B. *Suppose that* $H(0,0) = 0$. *Then the functions* $\mathfrak{F}_1(z,u,H)$ *and* $\mathfrak{F}_2(z,u,H)$ *defined in* (4·1) *and* (4·3) *are analytic at* $z = 0$ (*and therefore for all* z *in* $|z| \leqslant \rho$) *and satisfy*

$$|\mathfrak{F}_1(z,u,H)| \leqslant M(u,H)\,|z|^2, \quad |\mathfrak{F}_2(z,u,H)| \leqslant M(u,H).$$

Proof. It is sufficient to show that $z^{-2}\mathfrak{F}_1$ and $\mathfrak{F}_2$ are bounded near $z = 0$, for this condition is sufficient for negative powers to be absent in the Laurent series.

Consider first

$$\mathfrak{F}_1(z) = z^{-2m}\begin{vmatrix} \frac{1}{\pi}\int_{-z}^{z} e^{-ut}H(t^2,z^2)\,(z^2-t^2)^{m-\frac{1}{2}}\,dt & -\frac{1}{\pi}\int_{-z}^{z} e^{-ut}(z^2-t^2)^{m+\frac{1}{2}}\,dt \\ \int_{z}^{\infty} e^{-ut}H(t^2,z^2)\,(t^2-z^2)^{m-\frac{1}{2}}\,dt & \int_{z}^{\infty} e^{-ut}(t^2-z^2)^{m+\frac{1}{2}}\,dt \end{vmatrix} \tag{4·10}$$

$$= z^{-2m}H(z^2,z^2)\begin{vmatrix} \phi_1(z,m-\frac{1}{2}) & -\phi_1(z,m+\frac{1}{2}) \\ \phi_2(z,m-\frac{1}{2}) & \phi_2(z,m+\frac{1}{2}) \end{vmatrix} \tag{4·11}$$

$$+ z^{-2m}\begin{vmatrix} -\psi_1(z,H^*,m+\frac{1}{2}), & -\phi_1(z,m+\frac{1}{2}) \\ \psi_2(z,H^*,m+\frac{1}{2}), & \phi_2(z,m+\frac{1}{2}) \end{vmatrix}, \tag{4·12}$$

where $$H^*(t^2,z^2) = \frac{H(t^2,z^2)-H(z^2,z^2)}{t^2-z^2}; \quad \text{cf. equation (2.14)}, \tag{4·13}$$

The determinant in (4·11) can be found by expressing the elements in terms of Bessel functions. By using the relation (3·2) it is thus found that the expression (4·11) is equal to

$$H(z^2,z^2)\,\Gamma(\tfrac{1}{2}+m)\,\Gamma(\tfrac{3}{2}+m)\,\frac{2^{2m+1}}{\pi}\,u^{-2m-2}, \tag{4·14}$$

and since $H(z^2,z^2)$ is $O(z^2)$, this is $O(z^2)$.

Consider next the elements of the determinant in (4·12). (It is assumed throughout that $m+\frac{1}{2} > 0$.) Evidently

$$\left|\frac{1}{\pi}\int_{-z}^{z} e^{-ut}H^*(t^2,z^2)\,(z^2-t^2)^{m+\frac{1}{2}}\,dt\right| < M_1(u,H^*)\int_{-z}^{z}|z^2-t^2|^{m+\frac{1}{2}}\,|dt|$$

$$= M_1(u,H^*)\,.\,|z|^{2m+2}\int_{-1}^{1}(1-v^2)^{m+\frac{1}{2}}\,dv$$

$$= M_2(u,H^*)\,|z|^{2m+2}. \tag{4·15}$$

And

$$\left|\int_{z}^{\infty} e^{-ut}H^*(t^2,z^2)\,(t^2-z^2)^{m+\frac{1}{2}}\,dt\right|$$

is clearly bounded when $|z| \leqslant \rho$, since

$$|H^*(t^2,z^2)| < M_3 e^{k|t|}; \tag{4·16}$$

this inequality can be derived from (4·13). The other elements of the determinant in (4·12) evidently satisfy similar bounds. Thus the term (4·12) is bounded by

$$|z|^{-2m}M_4(u,H^*)\,|z|^{2m+2} = M_4(u,H^*)\,|z|^2,$$

and it now follows from (4·8) that $\mathfrak{F}_1(z,H)$ is analytic at $z = 0$ and $O(z^2)$ near $z = 0$.

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Consider now $\mathfrak{F}_2(z)$. The determinant is unchanged when $H(z^2, z^2)$ times the second column is subtracted from the first column. Thus

$$\mathfrak{F}_2(z, H) = -z^{-2m}\{\psi_1(z, H^*, m+\tfrac{1}{2})\,\phi_2(z, m-\tfrac{1}{2}) + \psi_2(z, H^*, m+\tfrac{1}{2})\,\phi_1(z, m-\tfrac{1}{2})\}. \quad (4{\cdot}17)$$

Estimates for the elements in the first column have just been obtained, while the elements in the second column are related to Bessel functions with known bounds, thus, from (2·8),

$$\left|\frac{1}{\pi}\int_{-z}^{z} e^{-ut}(z^2-t^2)^{m-\frac{1}{2}}\,dt\right| = |\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}+m)\,I_m(uz)\,.\,2^m u^{-m} z^m| \leqslant M_5(u)\,|z|^{2m}, \quad (4{\cdot}18)$$

and

$$\left|\int_z^\infty e^{-ut}(t^2-z^2)^{m-\frac{1}{2}}\,dt\right| = |\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}+m)\,K_m(uz)\,.\,2^m u^{-m} z^m|,$$

$$= M_6(u)\,|z|^m\left|\frac{I_m(uz)-I_{-m}(uz)}{\sin m\pi}\right|,$$

$$= M_7(u)\,(1+|z|^{2m})\log\left(\frac{1}{|z|}\right), \quad (4{\cdot}19)$$

where the logarithmic term is needed only when m is an integer. From these bounds it follows that $\mathfrak{F}_2(z, H)$ is bounded near $z = 0$, and hence analytic. This concludes the proof of Lemma B.

We may observe that alternative proofs of Lemmas A and B can be obtained from the expansions of $P_n^{-m}(\cosh z)$, $Q_n^{-m}(\cosh z)$, $I_m(uz)$ and $K_m(uz)$ near $z = 0$. Thus, to show that $\mathfrak{A}(z, u)$ and $\mathfrak{B}(z, u)$ are analytic when m is not an integer, we may rewrite them in the form

$$\mathfrak{A}(z, u) = \frac{\pi u^{m+1}}{2\sin m\pi}\left\{\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} z P_n^m(\cosh z)\,I_{m+1}(uz) - z P_n^{-m}(\cosh z)\,I_{-m-1}(uz)\right\}, \quad (4{\cdot}20)$$

$$\mathfrak{B}(z, u) = \frac{\pi u^{m+1}}{2\sin m\pi}\left\{P_n^{-m}(\cosh z)\,I_{-m}(uz) - \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} P_n^m(\cosh z)\,I_m(uz)\right\}, \quad (4{\cdot}21)$$

(cf. also (5·16) and (5·17) below), and note that each product in these expressions is evidently analytic at $z = 0$. By a limiting operation it can be shown that this conclusion is still valid when m is an integer. From (4·33) and (4·34) below it is seen that $\mathfrak{A}(z, u, G_p)$ and $\mathfrak{B}(z, u, G_p)$ are also analytic. The earlier method is however considered more appropriate in the present context.

LEMMA C. *Suppose that $H(t^2, z^2)$ satisfies the same conditions as before, and that $H(0, 0) = 0$. Then, for all z in the circle $|z| \leqslant \rho$, we have the bounds*

$$|\mathfrak{A}(z, u, H)| \leqslant \left|\left(\frac{z}{\sinh z}\right)^m H(z^2, z^2)\right| + M(H)\left|\frac{z}{u}\right|^2\left|\frac{z}{\sinh z}\right|^m,$$

and

$$|\mathfrak{B}(z, u, H)| \leqslant (1+2m)\left|\left(\frac{z}{\sinh z}\right)^m H^*(z^2, z^2)\right| + \frac{M(H)}{u^2}\left|\frac{z}{\sinh z}\right|^m,$$

where the bound $M(H)$ is independent of u for all sufficiently large real u.

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Proof. We note that (by Lemma B) the functions $z^{-2}\mathfrak{A}(z,u,H)$ and $\mathfrak{B}(z,u,H)$ are analytic when $|z| \leqslant \rho$; it will thus be sufficient (on account of the maximum-modulus theorem (Titchmarsh[5], ch. 5)) to estimate their magnitude on the boundary $|z| = \rho$. Consider first the expression (4·10) for $\mathfrak{F}_1(z,u,H)$. The term (4·11) is known explicitly, see (4·14). Suppose first that z lies in the fourth quadrant. To estimate the determinant in (4·12), add a suitable multiple of the second row to the first row, in such a way that the elements of the transformed first row are now $(1/\pi)\psi_2(-z,H)$ and $\phi_2(-z)$; this row transformation does not change the value of the determinant. The line of integration in each of the terms $\psi_2(z,H)$, $\psi_2(-z,H)$, $\phi_2(z)$ and $\phi_2(-z)$ is to be parallel to the real z-axis. A transformation of this type exists for each value of z but different multiples are required in the other quadrants. (This is seen most readily from the loop-integral forms given in § 5 below.) We now have the determinant

$$\begin{vmatrix} -\frac{1}{\pi}\int_{-z}^{\infty} e^{-ut}H^*(t^2,z^2)\,(z^2-t^2)^{m+\frac{1}{2}}\,dt & -\frac{1}{\pi}\int_{-z}^{\infty} e^{-ut}(z^2-t^2)^{m+\frac{1}{2}}\,dt \\ \int_{z}^{\infty} e^{-ut}H^*(t^2,z^2)\,(t^2-z^2)^{m+\frac{1}{2}}\,dt & \int_{z}^{\infty} e^{-ut}(t^2-z^2)^{m+\frac{1}{2}}\,dt \end{vmatrix}, \tag{4·22}$$

and, when $|z| = \rho$, the two branch points z and $-z$ are well separated. Thus each term can be estimated for large u, by Watson's Lemma; we observe that the terms in the first row involve an exponential factor e^{uz} while the terms in the second row involve an exponential factor e^{-uz}, and that therefore the product terms in the determinant do not involve any exponential factors. To find the leading term (for large u) it is convenient (in order to avoid cancellations) to subtract $H^*(z^2,z^2)$ times the second column of (4·22) from the first column. The transformation leaves the value of the determinant unchanged but the new determinant is now

$$\begin{vmatrix} \frac{1}{\pi}\int_{-z}^{\infty} e^{-ut}H^{**}(t^2,z^2)\,(z^2-t^2)^{m+\frac{3}{2}}\,dt & -\frac{1}{\pi}\int_{-z}^{\infty} e^{-ut}(z^2-t^2)^{m+\frac{1}{2}}\,dt \\ \int_{z}^{\infty} e^{-ut}H^{**}(t^2,z^2)\,(t^2-z^2)^{m+\frac{3}{2}}\,dt & \int_{z}^{\infty} e^{-ut}(t^2-z^2)^{m+\frac{1}{2}}\,dt \end{vmatrix}, \tag{4·23}$$

where

$$H^{**}(t^2,z^2) = \frac{H^*(t^2,z^2)-H^*(z^2,z^2)}{t^2-z^2}. \tag{4·24}$$

We can now see without much difficulty that

$$\begin{aligned} \mathfrak{F}_1(z,u,H) = {} & H(z^2,z^2)\,\Gamma(\tfrac{1}{2}+m)\,\Gamma(\tfrac{3}{2}+m)\frac{2^{2m+1}}{\pi}u^{-2m-2} \\ & + z^2H^{**}(z^2,z^2)\,\Gamma(\tfrac{3}{2}+m)\,\Gamma(\tfrac{5}{2}+m)\frac{2^{2m+3}}{\pi}u^{-2m-4} \\ & + \text{terms of order } z^2u^{-2m-6}, \end{aligned}$$

whence it follows for $|z| = \rho$ and $-\frac{1}{2}\pi \leqslant \operatorname{ph} z \leqslant 0$, that

$$\left\{\left(\frac{\sinh z}{z}\right)^m \mathfrak{A}(z,u,H) - H(z^2,z^2)\right\} z^{-2} = u^{-2}(1+2m)\,(3+2m)\,H^{**}(z^2,z^2) + \text{terms of order } u^{-4}. \tag{4·25}$$

We note that the product $z^{-2}H(z^2,z^2)$ is analytic since $H(0,0) = 0$.

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Similarly we can find a bound for $\mathfrak{B}(z, u, H)$. The transformations are similar to those for $\mathfrak{A}(z, u, H)$ but simpler, and it is found that

$$\mathfrak{F}_2(z, u, H) = -H^*(z^2, z^2)\,\Gamma(\tfrac{1}{2}+m)\,\Gamma(\tfrac{3}{2}+m)\frac{2^{2m+1}}{\pi}u^{-2m-2} + \text{terms of order } u^{-2m-4}, \tag{4·27}$$

whence it follows that, for $|z| = \rho$ and $-\frac{1}{2}\pi \leqslant \operatorname{ph} z \leqslant 0$,

$$\left(\frac{\sinh z}{z}\right)^m \mathfrak{B}(z, u, H) = -(1+2m)\,H^*(z^2, z^2) + \text{terms of order } u^{-2}. \tag{4·28}$$

Similar calculations can be made for the other quadrants and lead to the same result, as would be expected since $\mathfrak{A}(z, u, H)$ and $\mathfrak{B}(z, u, H)$ are analytic in z. Now apply the maximum-modulus theorem to the expression (4·25). We thus see that, when $|z| \leqslant \rho$, for all values of $\operatorname{ph} z$,

$$\left|\left(\frac{\sinh z}{z}\right)^m \mathfrak{A}(z, u, H) - H(z^2, z^2)\right| |z|^{-2}$$
$$\leqslant \max_{|z|=\rho}\left|\left(\frac{\sinh z}{z}\right)^m \mathfrak{A}(z, u, H) - H(z^2, z^2)\right| \rho^{-2} \leqslant Mu^{-2}, \tag{4·29}$$

i.e.
$$|\mathfrak{A}(z, u, H)| \leqslant \left|\left(\frac{z}{\sinh z}\right)^m H(z^2, z^2)\right| + M(H)\left|\frac{z}{u}\right|^2 \left|\frac{z}{\sinh z}\right|^m \tag{4·30}$$

when $|z| \leqslant \rho$.

Similarly, but more simply, from (4·28) we have

$$\left|\left(\frac{\sinh z}{z}\right)^m \mathfrak{B}(z, u, H) + (1+2m)\,H^*(z^2, z^2)\right| \leqslant Mu^{-2}, \tag{4·31}$$

when $|z| = \rho$, and therefore when $|z| \leqslant \rho$. It follows that

$$|\mathfrak{B}(z, u, H)| \leqslant (1+2m)\left|\left(\frac{z}{\sinh z}\right)^m H^*(z^2, z^2)\right| + \frac{M(H)}{u^2}\left|\frac{z}{\sinh z}\right|^m \tag{4·32}$$

when $|z| \leqslant \rho$. The bounds (4·30) and (4·32) establish Lemma C.

Proof of Theorem 1. It was seen in (3·9) that

$$\mathfrak{A}(z, u) = \left(\frac{z}{\sinh z}\right)^m \sum_0^{p-1} \frac{a_s(z^2)}{u^{2s}} + \frac{\mathfrak{A}(z, u, G_p)}{u^{2p}} \tag{4·33}$$

(where we recall that $a_s(0) = 0$ when $s \geqslant 1$, and that $G_p(0, 0) = 0$ when $p \geqslant 1$). Similarly it was seen in (3·10) that

$$\mathfrak{B}(z, u) = (2m+1)\left(\frac{z}{\sinh z}\right)^m \sum_0^{p-1} \frac{b_s(z^2)}{u^{2s}} + \frac{\mathfrak{B}(z, u, G_p)}{u^{2p}}. \tag{4·34}$$

Theorem 1 now follows at once from the bounds (4·31) and (4·32) given in Lemma C.

Proof of Theorem 2. Theorem 2 is an immediate consequence of (3·12) and (3·13), since $R_p(z, u, P)$ is given by (3·14).

5. *Extension of validity*

The theorems have been proved only for values of z such that $|z| \leqslant \rho$, and for values of m greater than $-\frac{1}{2}$ for which the integral representations (2·1), (2·2), (2·8) and (2·9) are convergent (when $s = m$ or $m+1$). The restrictions on z and m can both be removed.

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As for the restriction on z, it is obvious that the application of the maximum-modulus theorem is not restricted to circular regions; other methods for extending the validity of uniformly asymptotic expansions have been described by Ursell [6], Martin [2] and Olver ([3], ch. 9, § 13), and there can be little doubt that these are applicable to the present problem. As for the restriction on m, this can be removed by using the relations (Erdélyi [1], vol. 1, § 3·3·1)

$$P_n^m(\cosh z) = \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}\left\{P_n^{-m}(\cosh z) + \frac{2}{\pi} e^{mi\pi} \sin m\pi Q_n^{-m}(\cosh z)\right\} \tag{5·1}$$

and
$$e^{-mi\pi}\Gamma(n-m+1)\, Q_n^m(\cosh z) = e^{mi\pi}\Gamma(n+m+1)\, Q_n^{-m}(\cosh z). \tag{5·2}$$

Thus we see that

$$u^m \frac{\Gamma(u-m+\frac{1}{2})}{\Gamma(u+m+\frac{1}{2})} P_n^m(\cosh z) = \left\{I_m(uz) + \frac{2}{\pi}\sin m\pi\, K_m(uz)\right\} \mathfrak{A}^{-m}(z,u)$$
$$+ \frac{z}{u}\left\{I_{m+1}(uz) - \frac{2}{\pi}\sin m\pi\, K_{m+1}(uz)\right\} \mathfrak{B}^{-m}(z,u), \tag{5·3}$$

i.e.,
$$u^m \frac{\Gamma(u-m-\frac{1}{2})}{\Gamma(u+m+\frac{1}{2})} P_n^m(\cosh z) = I_{-m}(uz)\, \mathfrak{A}^{-m}(z,u) + \frac{z}{u} I_{-m-1}(uz)\, \mathfrak{B}^{-m}(z,u). \tag{5·4}$$

On using the relation

$$I_{-m-1}(uz) = I_{-m+1}(uz) - \frac{2m}{uz} I_{-m}(uz), \tag{5·5}$$

we see that

$$u^{-m} P_n^m(\cosh z) = I_{-m}(uz)\, \mathfrak{A}^m(z,u) + \frac{z}{u} I_{-m+1}(uz)\, \mathfrak{B}^m(z,u), \tag{5·6}$$

where
$$\mathfrak{A}^m(z,u) = \left\{\mathfrak{A}^{-m}(z,u) - \frac{2m}{u^2}\mathfrak{B}^{-m}(z,u)\right\} \frac{\Gamma(u+m+\frac{1}{2})}{\Gamma(u-m+\frac{1}{2})\, u^{2m}} \tag{5·7}$$

and
$$\mathfrak{B}^m(z,u) = \mathfrak{B}^{-m}(z,u) \frac{\Gamma(u+m+\frac{1}{2})}{\Gamma(u-m+\frac{1}{2})\, u^{2m}}, \tag{5·8}$$

and similarly that

$$e^{-mi\pi} u^{-m} Q_n^m(\cosh z) = K_{-m}(uz)\, \mathfrak{A}^m(z,u) + \frac{z}{u} K_{-m+1}(uz)\, \mathfrak{B}^m(z,u). \tag{5·9}$$

Alternatively we may extend the validity to negative m by replacing the line-integral representations for the Legendre and Bessel functions (valid for $m+\frac{1}{2} > 0$) by loop integrals valid for all real m. These are derived by standard methods from the line integrals, see Watson ([8], ch. 6), and have the following form:

$$P_n^{-m}(\cosh z) = \frac{\Gamma(\frac{1}{2}-m)}{2^{\frac{1}{2}} i\pi \Gamma(\frac{1}{2})} (\sinh z)^{-m} \int_a^{(-z-,+z+)} e^{-ut} (\cosh t - \cosh z)^{m-\frac{1}{2}}\, dt, \tag{5·10}$$

where $a > |z|$ is real, and where the phase of the integrand vanishes at $t = a$ when z is real;

$$e^{mi\pi} Q_n^{-m}(\cosh z) = -\frac{e^{-mi\pi}\Gamma(\frac{1}{2}-m)}{2^{\frac{1}{2}}\Gamma(\frac{1}{2})} (\sinh z)^{-m} \int_\infty^{(+z+)} e^{-ut} (\cosh t - \cosh z)^{m-\frac{1}{2}}\, dt, \tag{5·11}$$

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where the phase of the integrand vanishes on the upper side of the real t-axis when z is real and $t > |z|$;

$$I_s(uz) = \frac{\Gamma(\frac{1}{2}-s)}{2i\pi\Gamma(\frac{1}{2})} 2^{-s}u^s z^{-s} \int_a^{(-z-,+z+)} e^{-ut}(t^2-z^2)^{s-\frac{1}{2}}\,dt, \tag{5·12}$$

where the phase is chosen as in (5·10);

$$K_s(uz) = -\frac{e^{-si\pi}\Gamma(\frac{1}{2}-s)}{2\Gamma(\frac{1}{2})} 2^{-s}u^s z^{-s} \int_\infty^{(+z+)} e^{-ut}(t^2-z^2)^{s-\frac{1}{2}}\,dt, \tag{5·13}$$

where the phase is chosen as in (5·11).

It is found that (5·11) and (5·13) are not the most convenient forms for the proof of Lemma B; it is in fact more convenient to transform the determinants so as to obtain contours encircling both branch points. The appropriate integrals are

$$P_n^m(\cosh z) = \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \frac{e^{-2mi\pi}\Gamma(\frac{1}{2}-m)}{2^{\frac{1}{2}}i\pi\Gamma(\frac{1}{2})} (\sinh z)^{-m} \times \int_\infty^{(+z+,-z+)} e^{-ut}(\cosh t - \cosh z)^{m-\frac{1}{2}}\,dt, \tag{5·14}$$

and

$$I_{-s}(uz) = \frac{e^{-2si\pi}\Gamma(\frac{1}{2}-s)}{2i\pi\Gamma(\frac{1}{2})} 2^{-s}u^s z^{-s} \int_\infty^{(+z+,-z+)} e^{-ut}(t^2-z^2)^{s-\frac{1}{2}}\,dt, \tag{5·15}$$

where the phases are chosen as in (5·11) above. (For the derivation of (5·15) see Watson([8], § 6·15); the derivation of (5.14) is analogous.)

In effect, in this process we are solving the equations for $\mathfrak{A}^{-m}(z,u)$ and $\mathfrak{B}^{-m}(z,u)$ in the modified form

$$u^m \frac{\Gamma(u-m+\frac{1}{2})}{\Gamma(u+m+\frac{1}{2})} P_n^m(\cosh z) = I_{-m}(uz)\,\mathfrak{A}^{-m}(z,u) + \frac{z}{u} I_{-m-1}(uz)\,\mathfrak{B}^{-m}(z,u), \tag{5·16}$$

$$u^m P_n^{-m}(\cosh z) = I_m(uz)\,\mathfrak{A}^{-m}(z,u) + \frac{z}{u} I_{m+1}(uz)\,\mathfrak{B}^{-m}(z,u), \tag{5·17}$$

where (5·16) is readily deduced from (1·3), (1·4) and (5·1), where (5·17) is identical with (1·4), and where we note that the determinant

$$I_{-m}(uz)\,I_{m+1}(uz) - I_m(uz)\,I_{-m-1}(uz) = \frac{2}{\pi uz}\sin m\pi$$

(see Erdélyi[1], vol. 2, § 7·12) vanishes when m is an integer (cf. (4·20) and (4·21) above). As regards Lemma C, we recall that Watson's Lemma remains applicable to loop integrals ([3], ch. 4, § 5·1) but we shall not pursue this method further here.

6. *Discussion*

In the present work we have found uniform new error bounds for the well-known asymptotic expansions of Legendre functions of large degree n and fixed order m in terms of Bessel functions, and have thus incidentally established the validity of the expansion. Our method was based on the observation that the coefficient functions $\mathfrak{A}(z,u)$ and $\mathfrak{B}(z,u)$ are analytic near $z = 0$, whereas the Legendre functions and Bessel functions have branch points at $z = 0$. This analyticity could be shown either by using integral representations or by using power series expansions valid near

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$z = 0$. The uniform bounds for $\mathfrak{A}(z, u, G_p)$ and $\mathfrak{B}(z, u, G_p)$ for all values of z in the neighbourhood $|z| \leqslant \rho$ were then found by finding bounds on the circle $|z| = \rho$ (on which the simpler non-uniform asymptotic expansions in terms of elementary functions ([3], ch. 10) are valid). By the maximum-modulus theorem the same bounds are then also valid in the whole neighbourhood $|z| \leqslant \rho$. In this part of the argument (Lemma C) it was more convenient to use integral representations rather than the differential equation. Our argument is not only simpler in principle than the usual argument from the differential equation ([3], ch. 12) but the form of the expansions (1·1) and (1·2) suggests that our bounds are also more efficient and more uniform, and perhaps close to best-possible. It should however be noted that our error bounds in their present form are applicable to a finite neighbourhood $|z| \leqslant \rho$ of the origin whereas the usual error bounds are applicable to unbounded intervals; compare [3], ch. 12, § 13·2. It would also be interesting to enquire whether the usual integral equations for the error bounds ([3], ch. 12, § 3) can be transformed into equations of simpler form for the functions $\mathfrak{A}$ and $\mathfrak{B}$.

Ideas related to those of the present work had been used previously in the author's paper [7] on integrals with several nearly coincident saddle points. Coefficient functions were there defined as multilinear functions (analogous to the present $\mathfrak{A}$ and $\mathfrak{B}$) and were obviously analytic functions of the parameters. Thus the only difficulty in that work was with the analogue of Lemma C. Asymptotics in terms of elementary functions were obtained by introducing an additional splitting parameter τ and applying the maximum-modulus theorem to τ. The present work on Legendre functions does not involve this additional complication.

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