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Entropy of general diffeomorphisms on line

BAOLIN HE

Shanghai Normal University, Mathematics, Shanghai, 200234, PR China
(e-mail: hebaolin@shnu.edu.cn)

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Abstract. We study the continuity of topological entropy of general diffeomorphisms on line. First, we prove that the entropy map is continuous with respect to the strong C^0-topology on the union of uniformly topologically hyperbolic diffeomorphisms contained in $\mathrm{Diff}_0^r(\mathbb{R})$ (whose first derivative is uniformly away from zero), which is a C^0-open and C^r-dense subset of $\mathrm{Diff}_0^r(\mathbb{R})$, $r = 1, 2, \ldots, \infty$, and ω (real analytic). Second, we give some examples where entropy map is not continuous. Finally, we prove some results on the continuity of entropy of general diffeomorphisms on the (real) line.

Key words: dimension theory, smooth dynamics, topological dynamics
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1. *Introduction*

In recent years, there have been many remarkable works on the continuity of topological entropy of partially hyperbolic diffeomorphisms, for example [**2**–**4**, **7**]. In [**5**], Saghin and Yang posed the following conjecture which is known to many entropy experts.

Conjecture. The topological entropy is continuous on the space of C^1 partially hyperbolic diffeomorphisms of compact manifolds with the dimension of the center equal to one.

Motivated by this conjecture, we study the *topological* entropy of diffeomorphisms on line. The topological entropy of diffeomorphisms on line induces the natural map from $\mathrm{Diff}^r(\mathbb{R})$ to $\mathbb{R}$, which is said to be the entropy map h, $r \geq 1$. In [**1**], we mainly study diffeomorphisms on line with bounded first derivative. The union of C^r diffeomorphisms on line with bounded first derivative is denoted by $\mathrm{Diff}_b^r(\mathbb{R})$, $r \geq 1$.

THEOREM 1. [**1**, Theorem 1.3] *There is a C^0-open and C^r-dense subset $\mathcal{U}$ of $\mathrm{Diff}_b^r(\mathbb{R})$, $r = 1, 2, \ldots, +\infty$, such that the entropy map with respect to the strong C^0-topology is continuous on $\mathcal{U}$; moreover, for any $f \in \mathcal{U}$ such that it is uniformly expanding or $h(f) = 0$, the entropy map is locally constant at f.*

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In this paper we first study diffeomorphisms whose first derivative is uniformly away from zero:

$$\left\{ f \in \mathrm{Diff}^r(\mathbb{R}) : \inf_{x\in\mathbb{R}} |f'(x)| > 0 \right\},$$

which is denoted by $\mathrm{Diff}^r_0(\mathbb{R})$, $r \geq 1$. And we prove a similar result on $\mathrm{Diff}^r_0(\mathbb{R})$ with the above theorem.

THEOREM 2. *The entropy map is continuous (with respect to the strong C^0-topology) on the union of uniformly topologically hyperbolic diffeomorphisms (see the Definition 2.5) contained in $\mathrm{Diff}^r_0(\mathbb{R})$, which is a C^0-open and C^r-dense subset of $\mathrm{Diff}^r_0(\mathbb{R})$, $r = 1, 2, \ldots, \infty$, and ω (real analytic). Moreover, for any uniformly topologically hyperbolic diffeomorphism $f \in \mathrm{Diff}^1_0(\mathbb{R})$ such that it is uniformly expanding or $h(f) = 0$ or ∞, the entropy map is locally constant at f.*

In [**1**, Theorem 1.2], we give an example, $f \in \mathrm{Diff}^\infty_0(\mathbb{R}) \cap \mathrm{Diff}^\infty_b(\mathbb{R})$, whose entropy map is neither lower semi-continuous nor upper semi-continuous at f. The above two theorems show that uniformly topologically hyperbolicity is a sufficient condition for the continuity of entropy map. But, it is not a sufficient condition for general diffeomorphisms on line, by the following two examples. In many places in this paper, diffeomorphisms are understood as *diffeomorphisms on line*.

Example 1. There exists a uniformly topologically hyperbolic orientation-preserving diffeomorphism f whose entropy is 1 such that, for any g satisfying $0 < \inf_{x\in\mathbb{R}}(g - f) \leq \sup_{x\in\mathbb{R}}(g - f) < \infty$, we have $h(g) = 2h(f)$.

Example 2. There exists a uniformly topologically hyperbolic orientation-preserving diffeomorphism f whose entropy is 2 such that, for any g satisfying $0 < \inf_{x\in\mathbb{R}}(f_1 - g) \leq \sup_{x\in\mathbb{R}}(f_1 - g) < \infty$, we have $h(g) = h(f_1)/2$.

On the other hand, we give the range of entropy of a uniformly topologically hyperbolic orientation-preserving diffeomorphism after small perturbation.

PROPOSITION 1.1. *For any uniformly topologically hyperbolic orientation-preserving diffeomorphism f, there exists $\varepsilon > 0$ such that, for any diffeomorphism g satisfying $d_0(f, g) < \varepsilon$, we have that*

$$\tfrac{1}{2}h(f) \leq h(g) \leq 2h(f).$$

By the above two examples, we give the best estimation in the above proposition. Since uniformly topological hyperbolicity is not an open condition for general orientation-reversing diffeomorphisms, we can construct an orientation-reversing diffeomorphism which does not satisfy the above proposition.

Example 3. There exists a uniformly topologically hyperbolic orientation-*reversing* C^r-diffeomorphism f with $h(f) = \infty$ such that, for any $\varepsilon > 0$, there exists C^r-diffeomorphism g satisfying $d^r(f, g) < \varepsilon$ and $h(g) = 0$, $r \geq 1$.

It is not difficult to show that $\mathrm{Diff}^r_0(\mathbb{R}) \cup \mathrm{Diff}^r_b(\mathbb{R})$ is not dense in $\mathrm{Diff}^r(\mathbb{R})$ (see Remark 6.1 in §6). So it is natural to pose the following problem.

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Problem. Does there exist an open and dense subset of $\mathrm{Diff}^r(\mathbb{R})$ such that the entropy map is continuous on it, for $r \geq 1$?

By Theorem 1, one may ask whether the boundedness of the second derivative could ensure the continuity of entropy. But there exist such counterexamples (see Remark 6.3). Here we give two partial results on the problem.

PROPOSITION 1.2. *If f is a uniformly topologically hyperbolic orientation-preserving diffeomorphism and satisfies $h(f) = 0$ or ∞, then the entropy map is locally constant at f.*

This is a direct corollary of Proposition 1.1. Also, we give a generic result on the continuity of entropy map.

PROPOSITION 1.3. *There exists a generic subset of* $\mathrm{Diff}^r(\mathbb{R})$ *such that the entropy map is continuous on it, for $r = 1, 2, \ldots, \infty$, and ω.*

This paper is organized as follows. In §§2 and 3 we mainly recall the definitions of entropy and the length growth rate of diffeomorphisms, and identities between them. In §§4 and 5 we prove Theorem 2. In §6 we construct several diffeomorphisms where the entropy map is not continuous. In §§7 and 8 we mainly prove the three propositions on the continuity of entropy of general diffeomorphisms.

2. *Definitions and notation*

The topological entropy of diffeomorphisms on line is defined *à la* Bowen [**6**]. In [**1**], we introduce the length growth rate of diffeomorphisms on line to simplify the calculation of entropy. First, we review their definitions.

Let f be a continuous endomorphism on the metric space X, and take the compact subset $K \subset X$. For any $\varepsilon > 0$ and integer $n > 0$, a set A is said to be (n, ε)-*separated* if, for any $x, y \in A$, there exists $i \in [0, n)$ such that $d(f^i x, f^i y) \geq \varepsilon$. Let $s(n, \varepsilon, K)$ be the (n, ε)-separated set contained in K with maximal number. Define

$$h(f, \varepsilon, K) = \limsup_{n \to +\infty} \frac{\ln \# s(n, \varepsilon, K)}{n}$$

and

$$h(f, K) = \lim_{\varepsilon \to 0} h(f, \varepsilon, K).$$

The *topological entropy* of f is defined as

$$h(f) = \sup_{K \subset X} h(f, K).$$

For any bounded interval I, $\ell(I)$ is defined as the length of I, and the length growth rate of f on I is defined as

$$l(f, I) = \limsup_{n \to \infty} \frac{\ln \ell(f^n(I))}{n}.$$

Taken over bounded interval $I \subset \mathbb{R}$, the *length growth rate* of f is defined as

$$l(f) = \sup_{I \subset \mathbb{R}} l(f, I).$$

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In [**1**, Proposition 1], we show that the entropy map is not continuous with respect to weak topology. Though the paper, we only consider the *strong topology*. The strong C^r-topology on $\mathrm{Diff}^r(\mathbb{R})$, $r = 0, 1, 2, \ldots$, is induced by

$$d_r(f, g) = \sup\{|f^{(i)}(x) - g^{(i)}(x)|, 0 \le i \le r, x \in \mathbb{R}\}.$$

Here, $\mathrm{Diff}^0(\mathbb{R})$ denotes the homeomorphism space of $\mathbb{R}$. The strong topology on $\mathrm{Diff}^\infty(\mathbb{R})$ is the union of the topologies induced by $\mathrm{Diff}^\infty(\mathbb{R}) \to \mathrm{Diff}^r(\mathbb{R})$ for all r finite. And we give $\mathrm{Diff}^\omega(\mathbb{R})$ the strong topology induced from $\mathrm{Diff}^\infty(\mathbb{R})$.

In our paper we mainly consider uniformly topologically expanding or contracting diffeomorphisms in positive or negative orientation, which were first introduced in [**1**].

Definition 2.1. An orientation-preserving diffeomorphism f is said to be uniformly topologically expanding in positive orientation if there exist $K > 0$ and p such that, for any $x \ge p$, $f(x) - x > K$.

Definition 2.2. An orientation-preserving diffeomorphism f is said to be uniformly topologically expanding in negative orientation if there exist $K > 0$ and p such that, for any $x \le p$, $x - f(x) > K$.

Definition 2.3. An orientation-preserving diffeomorphism f is said to be uniformly topologically contracting in positive orientation if there exist $K > 0$ and a sequence $x_n \to +\infty$ such that $x_n - f(x_n) > K$.

Definition 2.4. An orientation-preserving diffeomorphism f is said to be uniformly topologically contracting in negative orientation if there exist $K > 0$ and a sequence $x_n \to -\infty$ such that $f(x_n) - x_n > K$.

For simplicity, let $\mathcal{C}^+$ be the set of uniformly topologically contracting diffeomorphisms in positive orientation contained in $\mathrm{Diff}^1(\mathbb{R})$, $\mathcal{E}^+$ be the set of uniformly topologically expanding diffeomorphisms in positive orientation contained in $\mathrm{Diff}^1(\mathbb{R})$, $\mathcal{C}^-$ be the set of uniformly topologically contracting diffeomorphisms in negative orientation contained in $\mathrm{Diff}^1(\mathbb{R})$, and $\mathcal{E}^-$ be the set of uniformly topologically expanding diffeomorphisms in negative orientation contained in $\mathrm{Diff}^1(\mathbb{R})$. Then these four sets are open subsets of $\mathrm{Diff}^1(\mathbb{R})$. These subsets are different from those defined in [**1**], where we only consider diffeomorphisms with bounded first derivative.

Let $r = 1, 2, \ldots, \infty$, and ω. Let $\mathrm{Diff}^r_+(\mathbb{R})$ be the union of orientation-preserving diffeomorphisms on $\mathbb{R}$, and $\mathrm{Diff}^r_-(\mathbb{R})$ be the union of orientation-reversing diffeomorphisms on $\mathbb{R}$. Let $\mathcal{H}_+ = (\mathcal{E}^+ \cup \mathcal{C}^+) \cap (\mathcal{E}^- \cup \mathcal{C}^-)$, $\mathcal{H}_- = \{f \in \mathrm{Diff}^1_-(\mathbb{R}) : f^2 \in (\mathcal{E}^+ \cap \mathcal{E}^-) \cup (\mathcal{C}^+ \cap \mathcal{C}^-)\}$, and $\mathcal{H} = \mathcal{H}_+ \cup \mathcal{H}_-$.

Definition 2.5. A diffeomorphism f on line is said to be uniformly topologically hyperbolic, if $f \in \mathcal{H}$.

So $\mathcal{H}_+$ denotes the union of uniformly topologically hyperbolic orientation-preserving diffeomorphisms on $\mathbb{R}$, and $\mathcal{H}_-$ denotes the union of uniformly topologically hyperbolic orientation-reversing diffeomorphisms on $\mathbb{R}$. We conclude this section by recalling the definition of uniformly expanding diffeomorphisms on $\mathbb{R}$.

Definition 2.6. A diffeomorphism f on $\mathbb{R}$ is said to be uniformly expanding if there exist a positive integer n and $\lambda > 1$ such that, for any $x \in \mathbb{R}$, $|(f^n)'(x)| > \lambda^n$.

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3. *Identities on entropy and length growth rate*

In this section we mainly generalize the identities between entropy and length growth rate in [**1**, Propositions 2,3] to every diffeomorphism on line.

Let $a \in \mathbb{R}$ and f be an *orientation-preserving diffeomorphism*. First, we define the length growth rate at every point:

$$l(f, a) = \limsup_{n\to+\infty} \frac{\ln(|f^n(a)| + 1)}{n}.$$

LEMMA 3.1. *If* $\lim_{n\to+\infty} f^n(a) = +\infty$, *then for any* $x \geq a$,

$$l(f, x) = l(f, a).$$

If $\lim_{n\to+\infty} f^n(a) = -\infty$, *then for any* $x \leq a$,

$$l(f, x) = l(f, a).$$

The proof is exactly same as the proof of [**1**, Propositions 2,3], so we omit it. Then we let

$$l^+(f) = \max\left\{0, \sup\left\{l(f, a) : \lim_{n\to+\infty} f^n(a) = +\infty\right\}\right\}$$

and

$$l^-(f) = \max\left\{0, \sup\left\{l(f, a) : \lim_{n\to+\infty} f^n(a) = -\infty\right\}\right\}.$$

By the above lemma, we can see that if $f \in \mathcal{E}^+$ ($\mathcal{E}^-$), then $l^+(f)$ ($l^-(f)$) is as defined in [**1**]. If f is orientation-reversing, we let

$$l^{\pm}(f) = \frac{l^{\pm}(f^2)}{2}.$$

The following is a fundamental identity between entropy and length growth rate.

THEOREM 3. [**1**, Theorem 2.1] *Let* f *be diffeomorphism on* $\mathbb{R}$ *and* J *be an interval.*

$$h(f, J) = \max\{l(f, J), 0\}$$

The following lemma is a basic property on entropy and the length growth rate.

LEMMA 3.2. *For any diffeomorphism* f *on line, and any integer* $k > 0$, *we have that*

$$l^+(f^k) = kl^+(f), \quad l^-(f^k) = kl^-(f), \quad l(f^k) = kl(f), \quad h(f^k) = kh(f).$$

Proof. By the definition of $l^+(\cdot)$, we only need to prove the case where f is orientation-preserving. We suppose that, for any $a \in \mathbb{R}$, $\lim_{n\to+\infty} f^n(a) \neq +\infty$. Then $l^+(f^k) = 0 = kl^+(f)$. Otherwise, there exists $a \in \mathbb{R}$ such that $\lim_{n\to+\infty} f^n(a) = +\infty$. Then, by Lemma 3.1, we have that $l^+(f) = l(f, a)$. Then

$$\begin{aligned} k\, l^+(f) &= k \limsup_{n\to+\infty} \frac{\ln(|f^n(a)|)}{n} \\ &= \max\left\{\limsup_{n\to+\infty} \frac{\ln(|f^{nk+i}(a)|)}{n} : i = 0, 1, 2, \ldots, k-1\right\} \\ &= \max\{l(f^k, f^i(a)) : i = 0, 1, 2, \ldots, k-1\} = l^+(f^k). \end{aligned}$$

Similarly, we can deduce that $l^-(f^k) = k\, l^-(f)$, and $l(f^k) = k\, l(f)$. By Theorem 3, we get that $h(f^k) = k\, h(f)$. □

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We now give our main proposition on the length growth rate and entropy which will be used in the following sections.

PROPOSITION 3.3. *Let f be an orientation-preserving diffeomorphism on line. If $\lim_{n\to+\infty} f^n(p) = +\infty$, then for any $a \geq p$,*

$$l^+(f) = l(f, a) = \limsup_{n\to+\infty} \frac{\ln(|f^n(a)|)}{n}.$$

If $\lim_{n\to+\infty} f^n(p) = -\infty$, then for any $a \leq p$,

$$l^-(f) = l(f, a) = \limsup_{n\to+\infty} \frac{\ln(|f^n(a)|)}{n}.$$

For any diffeomorphism f on line, we have that

$$h(f) = \max\{l^-(f), l^+(f)\}.$$

Proof. The first part is Lemma 3.1. By the definition of $l^+(\cdot)$ and Lemma 3.2, we only need to prove the case where f is orientation-preserving. For any finite interval $[a, b]$, by the definition of the length growth rate, we deduce that

$$l(f, [a, b]) \leq \max\{l(f, a), l(f, b)\}.$$

Recalling the definition of $l^+(\cdot)$ and $l^-(\cdot)$, we have that $\min\{l^-(f), l^+(f)\} \geq 0$. By Theorem 3 and the above inequality, we have that

$$h(f) = \max\{l(f), 0)\} = \max\left\{0, \sup_{[a,b]\subset\mathbb{R}} l(f, [a, b])\right\} \leq \max\{l^-(f), l^+(f)\}.$$

We suppose that, for any $a \in \mathbb{R}$, $\lim_{n\to+\infty} f^n(a) \neq +\infty$. Then $l^+(f) = 0 \leq h(f)$. Otherwise, there exists $a \in \mathbb{R}$ such that $\lim_{n\to+\infty} f^n(a) = +\infty$. Then, by the definition of the length growth rate in $[a, f(a)]$, we have, that for any $\varepsilon > 0$, there exists N such that, for any $n \geq N$,

$$\frac{\ln(f^{n+1}(a) - f^n(a))}{n} < l(f, [a, f(a)]) + \varepsilon.$$

Note that $l(f, [a, f(a)]) \geq 0$. Then, for any $n > N$,

$$f^n(a) \leq f^N(a) + e^{(n+1)(l(f,[a,f(a)])+\varepsilon)}/(e^{l(f,[a,f(a)]))+\varepsilon} - 1).$$

Then, $l(f, a) \leq l(f, [a, f(a)]) + \varepsilon$. So by Theorem 3 again, we have that $l^+(f) = l(f, a) \leq h(f)$. Similarly, we have that $l^-(f) \leq h(f)$. So

$$h(f) = \max\{l^-(f), l^+(f)\}. \qquad \square$$

4. *The openness and density of $\mathcal{H} \cap \mathrm{Diff}_0^r(\mathbb{R})$ in $\mathrm{Diff}_0^r(\mathbb{R})$*

In this section we prove the part of Theorem 2 on the openness and density of $\mathcal{H}$ in $\mathrm{Diff}_0^r(\mathbb{R})$. This is the following proposition.

PROPOSITION 4.1. *$\mathcal{H} \cap \mathrm{Diff}_0^r(\mathbb{R})$ is a C^0-open and C^r-dense subset of $\mathrm{Diff}_0^r(\mathbb{R})$, $r = 1, 2, \ldots, \infty$, and ω.*

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We first we show openness. Before giving the proof, we present the following basic lemma.

LEMMA 4.2. *Let* $f \in \mathrm{Diff}^1_-(\mathbb{R}) \cap \mathrm{Diff}^1_0(\mathbb{R})$. *If* $f^2 \in \mathcal{E}^+$ $(\mathcal{E}^-)$, *then* $f^2 \in \mathcal{E}^-$ $(\mathcal{E}^+)$. *Similarly, if* $f^2 \in \mathcal{C}^+$ $(\mathcal{C}^-)$, *then* $f^2 \in \mathcal{C}^-$ $(\mathcal{C}^+)$.

Proof. For any orientation-reversing diffeomorphism $f \in \mathrm{Diff}^1_0(\mathbb{R})$, if $f^2 \in \mathcal{E}^+$, then there exist $K > 0$ and p such that, for any $x > p$, $f^2(x) - x > K$. By the definition of $\mathrm{Diff}^1_0(\mathbb{R})$, there exists $\lambda > 0$ such that $|f'(x)| > \lambda$ for any $x \in \mathbb{R}$. Then $x - f^2(x) > \lambda K$ for any $x < f(p)$. So $f^2 \in \mathcal{E}^-$. We can deduce the other three cases similarly. □

LEMMA 4.3. $\mathcal{H}_+$ *is a* C^0*-open subset of* $\mathrm{Diff}^1(\mathbb{R})$, *and* $\mathcal{H}_- \cap \mathrm{Diff}^1_0(\mathbb{R})$ *is a* C^0*-open subset of* $\mathrm{Diff}^1_-(\mathbb{R}) \cap \mathrm{Diff}^1_0(\mathbb{R})$.

Proof. By the definition of $\mathcal{H}_+$, we have that $\mathcal{H}_+$ is a C^0-open subset of $\mathrm{Diff}^1(\mathbb{R})$. Take $f \in \mathcal{H}_- \cap \mathrm{Diff}^1_0(\mathbb{R})$. Then there exists $\lambda > 0$ such that, for any $x \in \mathbb{R}$, $|f'(x)| > \lambda$. If $f^2 \in \mathcal{C}^+$, then there exist $K > 0$ and a sequence $\{x_n : n > 0\}$ such that $\lim_{n\to\infty} x_n = +\infty$, and, for any $n > 0$, $x_n - f^2(x_n) > K$. For any $K/2 > \varepsilon > 0$ and any $g \in B(f, 1/2\lambda^2\varepsilon)$, we let $y_n = x_n - \varepsilon$. Then

$$\begin{aligned} g(y_n) - f(x_n) &> f(y_n) - f(x_n) - \tfrac{1}{2}\lambda^2\varepsilon > (\lambda - \tfrac{1}{2}\lambda^2)\varepsilon, \\ f^2(x_n) - g^2(y_n) &= f(f(x_n)) - f(g(y_n)) + f(g(y_n)) \\ &\quad -g^2(y_n) > \lambda((\lambda - \tfrac{1}{2}\lambda^2)\varepsilon) - \tfrac{1}{2}\lambda^2\varepsilon > 0. \end{aligned}$$

Then, for any $n > 0$,

$$y_n - g^2(y_n) = y_n - f^2(x_n) + f^2(x_n) - g^2(y_n) > K - \varepsilon > \frac{K}{2}.$$

By Lemma 4.2, we have that g is contained in $\mathcal{C}^+ \cap \mathcal{C}^-$.

If $f^2 \notin \mathcal{C}^+$, then $f^2 \in \mathcal{E}^+$ by the fact that $f^2 \in \mathcal{H}_+$. Then, there exist $K > 0$ and p such that, for any $x > p$, $f^2(x) - x > K$. For any $K/2 > \varepsilon > 0$, and any $x > p$, and any $g \in B(f, \lambda^2\varepsilon/2)$, we let $y = x + \varepsilon$ and we have that

$$f(x) - g(y) > f(x) - f(y) - \tfrac{1}{2}\lambda^2\varepsilon > (\lambda - \tfrac{1}{2}\lambda^2)\varepsilon,$$

$$g^2(y) - f^2(x) = -f(f(x)) + f(g(y)) - f(g(y)) + g^2(y) > \lambda((\lambda - \tfrac{1}{2}\lambda^2)\varepsilon) - \tfrac{1}{2}\lambda^2\varepsilon > 0.$$

Then, for any $y > p + K/2$,

$$g^2(y) - y = g^2(y) - f^2(x) + f^2(x) - y > K - \varepsilon > \frac{K}{2}.$$

By Lemma 4.2 again, we have that g is contained in $\mathcal{E}^+ \cap \mathcal{E}^-$.
So, $\mathcal{H}_- \cap \mathrm{Diff}^1_0(\mathbb{R})$ is a C^0 open subset of $\mathrm{Diff}^1_0(\mathbb{R}) \cap \mathrm{Diff}^1_-(\mathbb{R})$. □

For the case $r = 1, 2, \ldots, \infty$ of the above proposition, we can deduce it by the same proof of the former part of [**1**, Theorem 1.3]. But when f is orientation-reversing, the function $\alpha(x)$ in the proof of [**1**, Theorem 1.3] is not C^ω. To deal with this case, we introduce the following two technical lemmas.

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LEMMA 4.4. *Let* $\alpha(x) = \sin(x)$ *or* $\cos(x)$. *For any sequence* $\{a_n \in \mathbb{R} : n \geq 1\}$ *satisfying* $\lim_{n\to\infty} a_n = \infty$, *and any interval* $[b_1, b_2] \subset (-1, 1)$, *there exist* $0 < \lambda < 1$ *and a subsequence* $\{a_{n_k} : k \geq 1\}$ *such that* $\alpha(\lambda a_{n_k})$ *is contained in* $[b_1, b_2]$ *for any* $k \geq 1$.

Proof. Toward a contradiction, we suppose there exists a sequence $\{a_n : n \geq 1\}$ satisfying $\lim_{n\to\infty} a_n = \infty$, and interval $[b_1, b_2] \subset (-1, 1)$ such that, for any $0 < \lambda < 1$, there exists N_λ such that, for any $n > N_\lambda$,

$$\alpha(\lambda a_n) \notin [b_1, b_2].$$

For any $n > 0$, let $A_n = \{\lambda : N_\lambda = n\}$. Then

$$\bigcup_{n>0} A_n = (0, 1).$$

By $(0, 1)$ is a Baire space, there exists n_0 such that the closure $\overline{A_{n_0}}$ contains some interval $[c_1, c_2]$. Take $m > n_0$ such that $|a_m(c_2 - c_1)| > 2\pi$. Then $\{\alpha(\lambda a_m) : \lambda \in A_{n_0}\}$ is dense in $[-1, 1]$. But, by the definition of A_{n_0}, we have that, for any $n > n_0$ and any $\lambda \in A_{n_0}$,

$$\alpha(\lambda a_n) \notin [b_1, b_2],$$

a contradiction! This finishes the proof of this lemma. □

LEMMA 4.5. *Let* $\{x_n \in \mathbb{R} : n \geq 1\}$ *satisfying* $\lim_{n\to\infty} x_n = +\infty$, *and* $\{y_n \in \mathbb{R} : n \geq 1\}$ *satisfying* $\lim_{n\to\infty} y_n = -\infty$. *Then, for any* $0 < b < \sqrt{2} - 1$, *there exist a subsequence* $\{n_k\}$ *and* $\lambda \in (0, 1)$ *such that* $\alpha(x) = \pm\sin(\lambda x)$ *or* $\pm\cos(\lambda x)$, *and*

$$\alpha(x_{n_k}) > b, \quad \alpha(y_{n_k}) < -b.$$

Proof. For any positive integer n, let $a_n = x_n - y_n$. Take $1 > \varepsilon > 0$ satisfying $\sqrt{2-\varepsilon} - 1 = b$. By the above lemma, there exist $\lambda \in (0, 1)$ and a subsequence of positive integers $\{n_k\}$ such that

$$\cos(\lambda(x_{n_k} - y_{n_k})) < -1 + \varepsilon.$$

Then

$$(\cos(\lambda x_{n_k}) - \cos(\lambda y_{n_k}))^2 + (\sin(\lambda x_{n_k}) - \sin(\lambda y_{n_k}))^2 > 4 - 2\varepsilon.$$

Then there a exists subsequence of $\{n_k\}$, still denoted by $\{n_k\}$ for simplicity, such that $|\cos(\lambda x_{n_k}) - \cos(\lambda y_{n_k})| > \sqrt{2-\varepsilon}$ or $|\sin(\lambda x_{n_k}) - \sin(\lambda y_{n_k})| > \sqrt{2-\varepsilon}$. Without loss of generality, we suppose that, for any $k \geq 1$, $\sin(\lambda x_{n_k}) - \sin(\lambda y_{n_k}) > \sqrt{2-\varepsilon}$. Letting $\alpha(x) = \sin(\lambda x)$, we have that $\alpha(x_{n_k}) > \sqrt{2-\varepsilon} - 1 = b$ and $\alpha(y_{n_k}) < -b$. □

Proof of Proposition 4.1. Fix r. In the following, we will deal with both orientation-preserving and orientation-reversing diffeomorphisms. We suppose that f is contained in $\mathrm{Diff}^r_+(\mathbb{R})$. If $f \notin \mathcal{E}^+$, then

$$\liminf_{x\to+\infty} f(x) - x \leq 0.$$

For any $\varepsilon > 0$, let $g = f - \varepsilon$ be the perturbation of f. Then

$$\liminf_{x\to+\infty} g(x) - x \leq -\varepsilon.$$

Then g is contained in $\mathcal{C}^+$. This deduces the C^r-density of $(\mathcal{E}^+ \cup \mathcal{C}^+)$ in $\mathrm{Diff}^r_+(\mathbb{R})$. Similarly, we have the C^r-density of $(\mathcal{E}^- \cup \mathcal{C}^-)$ in $\mathrm{Diff}^r_+(\mathbb{R})$. Then, by Lemma 4.3, we

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have that both $(\mathcal{E}^+ \cup \mathcal{C}^+)$ and $(\mathcal{E}^- \cup \mathcal{C}^-)$ in $\mathrm{Diff}^r_+(\mathbb{R})$ are C^r open and dense subsets of $\mathrm{Diff}^r_+(\mathbb{R})$. As we know, $\mathrm{Diff}^r(\mathbb{R})$ in the strong C^r-topology is a Baire space. So $\mathcal{H}_+ = (\mathcal{E}^+ \cup \mathcal{C}^+) \cap (\mathcal{E}^- \cup \mathcal{C}^-)$ is a C^0-open and C^r-dense subset of $\mathrm{Diff}^r_+(\mathbb{R})$.

We suppose that f is contained in $\mathrm{Diff}^r_-(\mathbb{R})$. If $f^2 \notin \mathcal{E}^+$, then

$$\liminf_{x\to+\infty} f^2(x) - x \le 0.$$

Then, for any $(\sqrt{2}-1)/2 > \varepsilon > 0$, there exists a sequence of real numbers $\{x_n : n \ge 1\}$ such that $\lim_{n\to\infty} x_n = +\infty$ and $f^2(x_n) - x_n < (\sqrt{2}-1)\varepsilon/4$. By Lemma 4.5, there exist a subsequence of positive integers $\{n_k\}$ and a C^ω function $\alpha(x)$ such that $||\alpha||_\omega = \sup\{|\alpha^{(i)}(x)|, 0 \le i < \infty, x \in \mathbb{R}\} < 1$, and

$$\alpha(x_{n_k}) > 0, \quad \alpha(f(x_{n_k})) < 1 - \sqrt{2}.$$

Let $g = f + \varepsilon\alpha$ be the C^ω-perturbation of f. Then

$$\begin{aligned} g^2(x) - x &= f(f(x) + \varepsilon\alpha(x)) + \varepsilon\alpha(f(x) + \varepsilon\alpha(x)) - x \\ &= f^2(x) - x + \varepsilon f'(\xi)\alpha(x) + \varepsilon\alpha(f(x) + \varepsilon\alpha(x)). \end{aligned}$$

By $||\alpha||_\omega < 1$, $\varepsilon < (\sqrt{2}-1)/2$, and $\alpha(f(x_{n_k})) < 1 - \sqrt{2}$ for any positive integer k, we have that

$$\alpha(f(x_{n_k}) + \varepsilon\alpha(x_{n_k})) < \alpha(f(x_{n_k})) + \varepsilon < \frac{-\sqrt{2}+1}{2}.$$

Then

$$g^2(x_{n_k}) - x_{n_k} < \frac{(1-\sqrt{2})\varepsilon}{4}.$$

So g^2 is contained in $\mathcal{C}^+$, and, by Lemma 4.2, we also have that g^2 is contained in $\mathcal{C}^-$. This shows that $\mathcal{H}_- = \{f \in \mathrm{Diff}^1_-(\mathbb{R}) : f^2 \in (\mathcal{E}^+ \cap \mathcal{E}^-) \cup (\mathcal{C}^+ \cap \mathcal{C}^-)\}$ is C^r-dense in $\mathrm{Diff}^r_0(\mathbb{R}) \cap \mathrm{Diff}^r_-(\mathbb{R})$. By Lemma 4.3, we deduce that $\mathcal{H} \cap \mathrm{Diff}^r_0(\mathbb{R}) = (\mathcal{H}_+ \cup \mathcal{H}_-) \cap \mathrm{Diff}^r_0(\mathbb{R})$ is a C^0-open and C^r-dense subset of $\mathrm{Diff}^r_0(\mathbb{R})$.

Remark 4.6. By Example 3 in §6, $\mathcal{H}_-$ is not an open subset of $\mathrm{Diff}^1_-(\mathbb{R})$. But we can prove that $\mathcal{H}$ is C^r-dense subset of $\mathrm{Diff}^r(\mathbb{R})$, $r = 1, 2, \ldots, \infty$, by slightly improving the proof of [**1**, Theorem 1.3].

Remark 4.7. The proof of the above proposition is also valid for the case of $\mathrm{Diff}^r_b(\mathbb{R})$, $r \ge 1$. So Theorem 1 is also valid for the C^ω-case.

5. *Continuity of entropy map on* $\mathrm{Diff}^r_0(\mathbb{R})$

In this section we finish the proof of Theorem 2. First, we prove the continuity of the entropy map. By the identity between the length growth rate and entropy, we only need to check the continuity of the length growth rate.

For simplicity, we let

$$\mathcal{H}_0^+ = (\mathcal{E}^+ \cup \mathcal{C}^+ \cup \{f \in \mathrm{Diff}^1_-(\mathbb{R}) : f^2 \in \mathcal{E}^+ \cup \mathcal{C}^+\}) \cap \mathrm{Diff}^1_0(\mathbb{R}),$$

$$\mathcal{H}_0^- = (\mathcal{E}^- \cup \mathcal{C}^- \cup \{f \in \mathrm{Diff}^1_-(\mathbb{R}) : f^2 \in \mathcal{E}^- \cup \mathcal{C}^-\}) \cap \mathrm{Diff}^1_0(\mathbb{R}).$$

The following proposition shows the continuity of the length growth rate.

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PROPOSITION 5.1. *$l^+(\cdot)$ with respect to the strong C^0-topology is continuous on $\mathcal{H}_0^+$. Moreover, for every $f \in \mathcal{H}_0^+$ such that it is uniformly expanding or $h(f) = 0$ (∞), $l^+(\cdot)$ is locally constant around f. It has the same properties on $\mathcal{H}_0^-$ and $l^-(\cdot)$.*

Proof. If $f \in (\mathcal{C}^+ \cup \{f \in \mathrm{Diff}_-^1(\mathbb{R}) : f^2 \in \mathcal{C}^+\}) \cap \mathrm{Diff}_0^1(\mathbb{R})$, then $l^+(\cdot)$ is locally zero at f. Otherwise, f is contained in $(\mathcal{E}^+ \cup \{f \in \mathrm{Diff}_-^1(\mathbb{R}) : f^2 \in \mathcal{E}^+\}) \cap \mathrm{Diff}_0^1(\mathbb{R})$. Then there exists $\lambda > 0$ such that

$$\inf\{|f'(x)| : x \in \mathbb{R}\} > \lambda,$$

and there exist $p \in \mathbb{R}$ and $K > 0$ such that, for any $x \geq p$, we have that

$$f^2(x) - x > K.$$

Fix $\varepsilon > 0$. Take a diffeomorphism g satisfying $d_0(f, g) < \varepsilon$. By the monotonicity of f^2, for any positive integer n, there exists m_n such that $f^{2m_n}(p) \leq g^{2n}(p) < f^{2m_n+2}(p)$. Then $g^{2n}(p) < f^{2m_n+4}(p) - K$. By $d_0(f, g) < \varepsilon$ and $\inf\{|f'(x)| : x \in \mathbb{R}\} > \lambda$, it is not difficult to calculate that

$$g^{2n+2}(p) < f^{2m_n+6}(p) - (\lambda^2 K - (\varepsilon + \lambda\varepsilon)).$$

Take $t_\varepsilon = [|\ln_{\lambda^2}(\varepsilon/(K(1-\lambda)+\varepsilon))|]$. By induction, we can deduce that, for any $1 \leq i \leq t_\varepsilon - 1$, we have that

$$g^{2n+2i}(p) < f^{2m_n+4+2i}(p) - \lambda^{2i} K + \sum_{j=0}^{2i-1} \lambda^j \varepsilon < f^{2m_n+4+2i}(p).$$

Then $m_{n+i} \leq m_n + 1 + i$ for any $0 \leq i \leq t_\varepsilon - 1$. Then for any $n > t_\varepsilon$,

$$\frac{m_n}{n} < \frac{t_\varepsilon + 3}{t_\varepsilon + 1}.$$

And then

$$l^+(g) = \limsup_{n\to+\infty} \frac{\ln(|g^{2n}(p)|)}{2n} \leq \limsup_{n\to\infty} \frac{\ln(|f^{2m_n+2}(p)|)}{2m_n+2} \frac{2m_n+2}{2n} \leq l^+(f) \frac{t_\varepsilon + 3}{t_\varepsilon + 1}.$$

On the other hand, $g^{2n}(p) > f^{2m_n-2}(p) + K$ and

$$g^{2n+2}(p) > f^{2m_n+2i-2}(p) + \lambda^2 K - (\varepsilon + \lambda\varepsilon).$$

By induction again, we can show that, for any $1 \leq i \leq t_\varepsilon - 1$,

$$g^{2n+2i}(p) > f^{2m_n+2i-2}(p) + \lambda^{2i} K - \sum_{j=0}^{2i-1} \lambda^j \varepsilon > f^{2m_n+2i-2}(p).$$

By similar estimation, we have that, for any $n > t_\varepsilon$,

$$l^+(g) \geq l^+(f) \frac{t_\varepsilon + 1}{t_\varepsilon + 3}.$$

By $\lim_{\varepsilon\to 0} t_\varepsilon = +\infty$, we can get the continuity of $l^+(\cdot)$ at f. Moreover, if $l^+(f) = 0$ or ∞, then $l^+(\cdot)$ is locally constant at f.

Particularly if f is uniformly expanding, let $p' = (p + f^2(p))/2$. For any small perturbation g of f, we have that $f^{2n}(p) < g^{2n}(p') < f^{2n+2}(p)$ for any $n > 0$. By Proposition 3.3 again, we have that $l^+(\cdot)$ is locally constant at f.

Let $\varphi(x) = -x$. For any $f \in \mathcal{H}^-$, $\varphi^{-1} f \varphi \in \mathcal{H}^+$, and $l^-(f) = l^+(\varphi^{-1} f \varphi)$. Therefore, we can deduce the same property on $\mathcal{H}_0^-$ and $l^-(\cdot)$ as on $\mathcal{H}_0^+$ and $l^+(\cdot)$. □

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Proof of Theorem 2. By Proposition 4.1, we have that $\mathcal{H} \cap \mathrm{Diff}_0^r(\mathbb{R})$ is a C^0-open and C^r-dense subset of $\mathrm{Diff}_0^r(\mathbb{R})$, $r = 1, 2, \ldots, \infty$, and ω. Recall Proposition 3.3, we have that for any diffeomorphism f on line, $h(f) = \max\{l^-(f), l^+(f)\}$. Then we get the same properties of the entropy map on $\mathcal{H} \cap \mathrm{Diff}_0^r(\mathbb{R})$ as on $l^+(\cdot)$ and $l^-(\cdot)$ in Proposition 5.1. This finishes the proof of Theorem 2.

6. *Examples where the entropy map is not continuous*

In this section, we will construct several examples where the entropy map is not continuous. We begin by proving that the entropy in Example 1 could suddenly become bigger as a result of an arbitrarily small perturbation.

Proof of Example 1. Let f satisfy $f(0) = 2$ and $f'(x) = 0.5$ for any $x \le -1$. By induction, we can make f such that, for any non-negative integer n,

$$f\left(f^n(0) + \frac{1}{n+1}\right) = f^{n+2}(0) - \frac{1}{n+1}, \quad f^{n+1}(0) = 2e^n.$$

Then $f \in \mathcal{C}^- \cap \mathcal{E}^+$. By Proposition 3.3, we have that

$$h(f) = \limsup_{n\to\infty} \frac{\ell(f^n(0))}{n} = 1.$$

Fix any $\varepsilon > 0$, and let $N = \max\{[1/\varepsilon] + 1, [\ln \varepsilon] + 2\}$, and $g = f + 2\varepsilon$. Then, for any $n \ge N$, and $f^n(0) + \varepsilon < y < f^n(0) + 2\varepsilon$, we have that

$$f(f^n(0) + \varepsilon) > f\left(f^n(0) + \frac{1}{n+1}\right) > f^2(f^n(0)) - \varepsilon,$$

$$g(y) > g(f^n(0) + \varepsilon) = f(f^n(0) + \varepsilon) + 2\varepsilon > f^{n+2}(0) + \varepsilon,$$

and

$$f^{n+1}(0) - f^n(0) = 2(e^n - e^{n-1}) > 2\varepsilon,$$

$$g(y) < g(f^n(0) + 2\varepsilon) = f(f^n(0) + 2\varepsilon) + 2\varepsilon < f^2(f^n(0)) + 2\varepsilon.$$

Then $f^{n+2}(0) + \varepsilon < g(y) < f^{n+2}(0) + 2\varepsilon$. By induction, for any $n \ge N$, $f^n(0) + \varepsilon < y < f^n(0) + 2\varepsilon$, and positive integer k, we have that

$$f^{n+2k}(0) + \varepsilon < g^k(y) < f^{n+2k}(0) + 2\varepsilon.$$

Then $g \in \mathcal{C}^- \cap \mathcal{E}^+$. By Proposition 3.3, we have that

$$h(g) = \limsup_{n\to\infty} \frac{g^n(f^N(0) + 1.5\varepsilon)}{n} = 2h(f).$$

Then it is not difficult to deduce that for any g satisfying $0 < \inf_{x\in\mathbb{R}}(g - f) \le \sup_{x\in\mathbb{R}}(g - f) < \infty$, we have that

$$h(g) = 2h(f). \qquad \square$$

Remark 6.1. Although entropy is not continuous in the above diffeomorphism, it is not difficult to show that, for any $\varepsilon \ne 0$, there exists an open neighborhood of $f + \varepsilon$ such that the entropy is continuous on it.

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Remark 6.2. In the above example, we have that

$$\lim_{n\to\infty} f^{n+1}(0) - f^n(0) = \infty.$$

For any g contained in the connected component of f, there exist t_1, t_2 such that $f + t_1 < g < f + t_2$. Then

$$\limsup_{x\in\mathbb{R}} g'(x) = \infty.$$

Since f is uniformly topologically contracting in negative orientation, there exists a neighborhood $\mathcal{U}$ of f such that, for any $g \in \mathcal{U}$,

$$\liminf_{x\in\mathbb{R}} g'(x) = 0.$$

So the union of diffeomorphisms whose first derivative is bounded or uniformly away from zero is not dense in $\mathrm{Diff}^r(\mathbb{R})$, $r \geq 1$.

The construction and proof of Example 2, whose entropy suddenly becomes smaller by arbitrarily small perturbation, is quite similar to that of Example 1.

Proof of Example 2. Let f satisfy $f(0) = 2$ and $f'(x) = 0.5$ for any $x \leq -1$. By induction, we can make f such that, for any non-negative integer n,

$$f\left(f^{n+1}(0) - \frac{1}{n+1}\right) = f^{n+1}(0) + \frac{1}{n+1}, \quad f^{n+1}(0) = 2e^n.$$

It is not difficult to show that, for any $x \geq 0$, $f^2(x) - x > 2$. Then $f^2 \in \mathcal{E}^+ \cap \mathcal{C}^-$. Fix any positive number $\varepsilon < 1$, and let $N = \max\{[1/\varepsilon] + 1, [\ln \varepsilon] + 1\}$ and $g = f^2 - 2\varepsilon$. Note that, for any $n \geq N$, $f^{n+1}(0) - f^n(0) > 2\varepsilon$ and $f^2(f^{n+1}(0) - \varepsilon) < f^{n+2}(0) + \varepsilon$. Then, for any $n \geq N$ and $f^{n+1}(0) - 2\varepsilon < y < f^{n+1}(0) - \varepsilon$, we have that

$$f^{n+2}(0) - 2\varepsilon < g(y) < f^{n+2}(0) - \varepsilon.$$

Let $f^2 = f_1$. Together with the Proposition 3.3, we have that, for any $\varepsilon > 0$,

$$h(f_1 - 2\varepsilon) = \frac{h(f_1)}{2} = 1 > 0.$$

And then, for any g satisfying $0 < \inf_{x\in\mathbb{R}}(f_1 - g) \leq \sup_{x\in\mathbb{R}}(f_1 - g) < \infty$,

$$h(g) = \tfrac{1}{2}h(f_1). \qquad \square$$

Remark 6.3. In fact, we can construct a more special example whose entropy map is not continuous. It is also contained in $\mathcal{C}^- \cap \mathcal{E}^+$, and its second derivative is bounded. It has the same flavor as the above example, and is constructed by induction, see Figure 1.

Here, we give some explanation of the graph and the non-continuity of the entropy map at f. There, $a_0 = f^{4n}(1)$, $a_i = f^i(a_0)$, $1 \leq i \leq 4$, $f'(a_0) = e^n$, $b_0 - a_0 = 1/n$, and $b_1 = f(b_0)$. We make $a_1 - a_0 > e^n$ and $a_4 - a_3 > e^n$ such that there is enough space to make f with bounded second derivative. Then, by Proposition 3.3, we have that

$$h(f) \geq \limsup_{n\to\infty} \frac{\ln e^n}{4n} = \frac{1}{4}.$$

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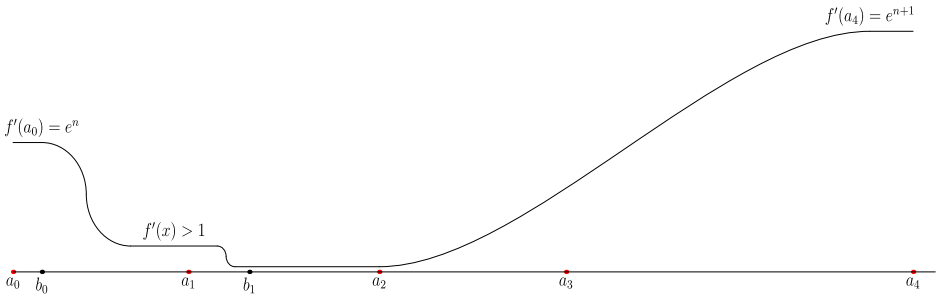


FIGURE 1. The graph of $f(x)$ in its nth block.

Obviously, we can make $h(f) < \infty$. And $f'(x)$ is very, very small in the interval $[b_1, a_2]$ such that $a_3 - f(b_1) < 1/n$. Let $f_n = f + 1/n$. Then

$$f_n^2(a_0) = f\left(f(a_0) + \frac{1}{n}\right) + \frac{1}{n} = f^2(b_0) + \frac{1}{n} > f^3(a_0).$$

By $f_n^3(a_0) > f^4(a_0)$, we have that, for any $\varepsilon > 0$,

$$h(f + \varepsilon) \geq \frac{4h(f)}{3}.$$

Then the entropy map is not continuous at f.

In the following, we prove Example 3, which states that a uniformly topologically hyperbolic orientation-*reversing* C^r-diffeomorphism f with $h(f) = \infty$ satisfying that the entropy suddenly becomes zero by an arbitrarily small perturbation. This is mainly due to the fact that uniformly topological hyperbolicity is not an open condition for general orientation-reversing diffeomorphisms. But this will not occur in the orientation-preserving case by the Proposition 1.2.

Proof of Example 3. By induction, we can construct a diffeomorphism f satisfying

$$f(0) = 0, \quad \int_0^{-1} f'(x)\,dx = 1, \quad \int_0^1 f'(x)\,dx = f(1) = -2,$$

$|f'(x)| > 1$ for any $x \in [f^{199}(1), f^3(1)]$, and for any positive integer n, we have the following properties:

(1) $f^{2n+1}(1) - f^{2n-1}(1) = -1$;

(2) $\int_{f^{2n-1}(1)}^{f^{2n+1}(1)} f'(x)\,dx = f^{2n+2}(1) - f^{2n}(1) = n^n$;

(3) if $n > 200$, then $f(f^{2n+1}(1) + 1/n) = f^{2n}(1) + 1/n$,

$$f^2\left(\frac{f^{2n-1}(1) + f^{2n+1}(1)}{2}\right) = f\left(f^{2n}(1) + \frac{1}{2n}\right) = \frac{f^{2n+3}(1) + f^{2n+1}(1)}{2},$$

and

$$f^2\left(\frac{f^{2n}(1) + f^{2n+2}(1)}{2}\right) = f\left(f^{2n+3}(1) + \frac{1}{2n+2}\right) = \frac{f^{2n+2}(1) + f^{2n+4}(1)}{2}.$$

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By the last two identities of the third property of f, we have that, for any $n > 400$,

$$f^2\left(\frac{f^n(1) + f^{n+1}(1)}{2}\right) = \frac{f^{n+2}(1) + f^{n+3}(1)}{2}.$$

The above identities and the first two properties ensure that there exists $K > 0$ such that, for any $x < f(1) = -2$, $f^2(x) < x - K$, and for any $x > 1$, $f^2(x) - x > K$. Then $f^2 \in \mathcal{E}^+ \cap \mathcal{E}^-$. By Proposition 3.3, we have that

$$l^-(f^2) = \limsup_{n\to+\infty} \frac{\ln(|f^{2n+1}(1)|)}{2n+1} = 0,$$

$$l^+(f^2) = \limsup_{n\to+\infty} \frac{\ln(|f^{2n}(1)|)}{2n} = \infty,$$

and $h(f) = \infty$. Let α be a smooth function satisfying

$$\alpha(x) = 1,\ x \geq -50, \quad \text{and} \quad \alpha(x) = -1,\ x < -99.$$

Let $g = f + \varepsilon\alpha$, $K > \varepsilon > 0$. Note that $[f^{199}(1),\ f^3(1)] = [-101, -3] \supset [-99, 50]$. Then, by $f'(x) > 1$ in $[f^{199}(1),\ f^3(1)]$, we have that $g' \neq 0$ and g is diffeomorphic on line. We take an integer n satisfying $n > \max\{1/\varepsilon, 200\}$. Then

$$f^{2n+1}(1) + \varepsilon \in \left[f^{2n+1}(1) + \frac{1}{n},\ f^{2n-1}(1)\right] = \left[-(n+2) + \frac{1}{n},\ -(n+1)\right].$$

Together with the construction of f and α, we have that

$$g^2(f^{2n}(1)) = g(f^{2n+1}(1) + \varepsilon) = f(f^{2n+1}(1) + \varepsilon) - \varepsilon < f^{2n}(1) + \frac{1}{n} - \varepsilon < f^{2n}(1).$$

This shows that

$$l^+(g^2) = 0.$$

By the uniformly topological expansion of f^2, we have that, for any $x < f(1)$,

$$g(x) \leq f(x) + \varepsilon < f(x) + K < f^3(x),$$

and for any $x > 1$, $g(x) \geq f(x) - \varepsilon > f^3(x)$. Then, for any $x < f(1)$, we have that $f^3(x) > 1$ and

$$g^2(x) > g(f^3(x)) > f^6(x).$$

Then, by Proposition 3.3 again, we deduce that $l^-(g^2) \leq 3l^-(f^2) = 0$, and $h(g) = 0$. □

We conclude this section with two simple examples where zero entropy suddenly becomes infinite entropy, and infinite entropy suddenly becomes zero entropy by an arbitrarily small perturbation. By Proposition 1.2, these two orientation-preserving examples cannot be uniformly topologically hyperbolic.

Example 4. There exists an orientation-preserving diffeomorphism f with $h(f) = \infty$ such that, for any $\varepsilon > 0$, $h(f - \varepsilon) = 0$.

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Proof. The diffeomorphism f is contained in the closure of topologically contracting diffeomorphisms. We construct a diffeomorphism f that is uniformly contracting in negative orientation and, for any positive integer $n > 1$,

$$f^{2n}(1) = n^n \quad \text{and} \quad f^{2n+1}(1) = n^n + \frac{1}{n}.$$

Then, by Proposition 3.3, we have that

$$h(f) = \limsup_{n\to\infty} \frac{\ln(|f^{2n}(1)|)}{2n} = \infty.$$

And for any $n > 1$, $f - 1/n \in \mathcal{C}^- \cup \mathcal{C}^+$. Then $h(f - 1/n) = 0$. □

Example 5. There exists an orientation-preserving diffeomorphism f with $h(f) = 0$ such that, for any $\varepsilon > 0$, $h(f + \varepsilon) = \infty$.

Proof. We construct a diffeomorphism f that is uniformly contracting in negative orientation and, for any $x > 0$, $f(x) \geq x$, and there exists a sequence $\{a_n > 1 : n > 0\}$ such that, for any $n > 0$,

$$a_{n+1} > n^{a_n} + 2, \quad f(a_n) = n^{a_n} \quad \text{and} \quad f(n^{a_n} + 1) = n^{a_n} + 1.$$

By the definition of $l^+(\cdot)$ and $l^-(\cdot)$, we have that $l^+(f) = l^-(f) = 0$. Then $h(f) = \max\{l^+(f), l^-(f)\} = 0$. For any $\varepsilon > 0$, let $g = f + \varepsilon$. Then $g(x) \geq x + \varepsilon$ for any $x > 0$. And then, for any $n > 0$,

$$g^{[a_n/\varepsilon]+1}(1) > a_n \quad \text{and} \quad g^{[a_n/\varepsilon]+2}(1) > g(a_n) > n^{a_n}.$$

By Proposition 3.3, we have that

$$h(g) \geq \limsup_{n\to\infty} \frac{\ln(|g^{[a_n/\varepsilon]+2}(1)|)}{[a_n/\varepsilon] + 2} = \infty.$$ □

7. *General uniformly topologically hyperbolic orientation-preserving diffeomorphisms on line*

In Examples 1 and 2, we construct uniformly topologically hyperbolic diffeomorphisms on line such that entropy map is not continuous. In this section we will prove that for any uniformly topologically hyperbolic orientation-preserving diffeomorphism on line with zero entropy or infinite entropy, entropy map is locally constant. (By Example 3, we show that this is not valid for orientation-reversing diffeomorphisms.) By Proposition 3.3, we can get Propositions 1.1 and 1.2 by the following two propositions on the length growth rate.

PROPOSITION 7.1. *If $f \in \mathcal{E}^+$ satisfies $l^+(f) = 0$ or ∞, then $l^+(\cdot)$ is locally constant at f. If $f \in \mathcal{E}^-$ satisfies $l^-(f) = 0$ or ∞, then $l^-(\cdot)$ is locally constant at f.*

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PROPOSITION 7.2. *For any $f \in \mathcal{E}^+ \cup \mathcal{C}^+$, there exists $\varepsilon > 0$ such that, for any diffeomorphism g satisfying $d_0(f, g) < \varepsilon$, we have that*

$$\tfrac{1}{2} l^+(f) \le l^+(g) \le 2l^+(f).$$

For any $f \in \mathcal{E}^- \cup \mathcal{C}^-$, $l^-(\cdot)$ has the same property.

Proposition 7.1 is directly deduced by the above proposition. We now give its proof.

Proof. Take $f \in \mathcal{E}^+$. Suppose there exist p and $K > 0$ such that, for any $x \ge p$, $f(x) > x + K$. Then, for any $0 < \varepsilon < K/2$, any $x \ge p$, and any diffeomorphism g satisfying that $d_0(g, f) < \varepsilon$, we have that $g(x) > x + K - \varepsilon$, and then

$$g^2(x) > g(x) + K - \varepsilon > f(x).$$

Similarly, we have that $f^2(x) > g(x)$. By Proposition 3.3, which says that, for any $a \ge p$,

$$l^+(f) = \limsup_{n\to\infty} \frac{\ln(|f^n(a)|)}{n},$$

we have that

$$\tfrac{1}{2} l^+(f) \le l^+(g) \le 2l^+(f).$$

Obviously the above inequality is also valid for $f \in \mathcal{C}^+$. For any $f \in \mathcal{E}^- \cup \mathcal{C}^-$, we can deduce the same property on $l^-(\cdot)$ and $\mathcal{E}^- \cup \mathcal{C}^-$. □

8. *The generic continuity of the entropy map on* $\mathrm{Diff}^r(\mathbb{R})$

We now prove Proposition 1.3. Let X be a Baire space. A subset of X is said to be a *generic* (*residual*) *subset* of X if it is the intersection of countably many subsets of X with dense interior.

The following lemma is a very basic property on the continuity points of functions on Baire spaces. For completion, we give its proof.

LEMMA 8.1. *Let X be a Baire space and f be a function from X to $\mathbb{R}$. If f is continuous on a dense subset D of X, then f is continuous on a generic subset of X.*

Proof. For any $x \in D$ and any $\varepsilon > 0$, let $B_f(x, \varepsilon)$ be an open subset of X containing x and such that for any $y \in B_f(x, \varepsilon)$, $|f(y) - f(x)| < \varepsilon$. Let

$$\mathcal{U} = \bigcap_{m>0} \bigcup_{x\in D} B_f\left(x, \frac{1}{m}\right).$$

Then it is not difficult to show that f is continuous on $\mathcal{U}$ and $\mathcal{U}$ is a generic subset of X. □

LEMMA 8.2. *For any $f \in \mathrm{Diff}^1(\mathbb{R})$, let h_f be the map from $\mathbb{R}$ to $\mathbb{R}^+$:*

$$h_f(t) = h(f + t).$$

Then there exists a dense subset $\mathcal{D}$ of $\mathbb{R}$ such that h_f is continuous on it.

By the monotonicity of h_f, we get the above lemma.

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Proof of Proposition 1.3. Fix r. By Lemma 8.2, we have that, for any $f \in \mathrm{Diff}^r(\mathbb{R})$ and $\varepsilon > 0$, there exists ε_0 such that $|\varepsilon_0| < \varepsilon$ and h_f is continuous at ε_0. It is not difficult to show that the entropy map is continuous at $f + \varepsilon_0$. Then there exists a dense subset $\mathcal{D}$ of $\mathrm{Diff}^r(\mathbb{R})$ such that the entropy map is continuous on it. So, by Lemma 8.1, there exists a generic subset of $\mathrm{Diff}^r(\mathbb{R})$ such that the entropy map is continuous on it. This finishes the proof of this proposition.

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REFERENCES

[**1**] B. He. Entropy of diffeomorphisms of line. *Discrete Contin. Dyn. Syst.* **37** (2017), 4753–4766.
[**2**] G. Liao, M. Viana and J. Yang. The entropy conjecture for diffeomorphisms away from tangencies. *J. Eur. Math. Soc. (JEMS)* **15** (2013), 2043–2060.
[**3**] J. Milnor and W. Thurston. On iterated maps of the interval. *Dynamical Systems (Lecture Notes in Mathematics, 1342)*. Springer, Berlin, 1988, pp. 465–563.
[**4**] S. Newhouse. Continuity properties of entropy. *Ann. of Math.* **129** (1989), 215–235.
[**5**] R. Saghin and J. Yang. Continuity of topological entropy for perturbation of time-one maps of hyperbolic flows. *Israel J. Math.* **215** (2016), 857–875.
[**6**] P. Walters. *Ergodic Theory: Introductory Lectures (Lecture Notes in Mathematics, 458)*. Springer, Berlin, 1975.
[**7**] Y. Yomdin. Volume growth and entropy. *Israel J. Math.* **57** (1987), 285–300.

https://doi.org/10.1017/etds.2020.3 Published online by Cambridge University Press