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Normalized solutions for nonlinear Schrödinger systems

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We consider the existence of *normalized* solutions in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ for systems of nonlinear Schrödinger equations, which appear in models for binary mixtures of ultracold quantum gases. Making a solitary wave ansatz, one is led to coupled systems of elliptic equations of the form

$$-\Delta u_1 = \lambda_1 u_1 + f_1(u_1) + \partial_1 F(u_1, u_2),$$
$$-\Delta u_2 = \lambda_2 u_2 + f_2(u_2) + \partial_2 F(u_1, u_2),$$
$$u_1, u_2 \in H^1(\mathbb{R}^N),\ N \geqslant 2,$$

and we are looking for solutions satisfying

$$\int_{\mathbb{R}^N} |u_1|^2 = a_1, \qquad \int_{\mathbb{R}^N} |u_2|^2 = a_2,$$

where $a_1 > 0$ and $a_2 > 0$ are prescribed. In the system, λ_1 and λ_2 are unknown and will appear as Lagrange multipliers. We treat the case of homogeneous nonlinearities, i.e. $f_i(u_i) = \mu_i |u_i|^{p_i - 1} u_i$, $F(u_1, u_2) = \beta |u_1|^{r_1} |u_2|^{r_2}$, with positive constants β, μ_i, p_i, r_i. The exponents are Sobolev subcritical but may be L^2-supercritical. Our main result deals with the case in which $2 < p_1 < 2 + 4/N < p_2$, $r_1 + r_2 < 2^*$ in dimensions $2 \leqslant N \leqslant 4$. We also consider the cases in which all of these numbers are less than $2 + 4/N$ or all are bigger than $2 + 4/N$.

Keywords: nonlinear Schrödinger systems; solitary waves; normalized solutions; variational methods; constrained linking

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1. Introduction

Elliptic systems of the form

$$\left.\begin{aligned} -\Delta u_1 &= \lambda_1 u_1 + f_1(u_1) + \partial_1 F(u_1, u_2), \\ -\Delta u_2 &= \lambda_2 u_2 + f_2(u_2) + \partial_2 F(u_1, u_2), \\ u_1, u_2 &\in H^1(\mathbb{R}^N), \end{aligned}\right\} \tag{1.1}$$

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have been investigated in the last decades by many authors. Surprisingly little is known about the existence of normalized solutions, i.e. solutions that satisfy the constraint

$$\int_{\mathbb{R}^N} |u_1|^2 = a_1 \quad \text{and} \quad \int_{\mathbb{R}^N} |u_2|^2 = a_2 \tag{1.2}$$

with $a_1, a_2 > 0$ prescribed. One motivation driving the search for normalized solutions of (1.1) are coupled systems of nonlinear Schrödinger equations

$$\left.\begin{aligned} -\mathrm{i}\partial_t \Psi_1 &= \Delta \Psi_1 + g_1(|\Psi_1|)\Psi_1 + \partial_1 G(|\Psi_1|^2, |\Psi_2|^2)\Psi_1, \\ -\mathrm{i}\partial_t \Psi_2 &= \Delta \Psi_2 + g_2(|\Psi_2|)\Psi_2 + \partial_2 G(|\Psi_1|^2, |\Psi_2|^2)\Psi_2 \end{aligned}\right\} \quad \text{in } \mathbb{R} \times \mathbb{R}^N. \tag{1.3}$$

Since the *masses*

$$\int_{\mathbb{R}^N} |\Psi_1|^2 \quad \text{and} \quad \int_{\mathbb{R}^N} |\Psi_2|^2$$

are preserved along trajectories of (1.3), it is natural to consider them as prescribed. A solitary wave of (1.3) is a solution having the form

$$\Psi_1(t,x) = \mathrm{e}^{-\mathrm{i}\lambda_1 t} u_1(x) \quad \text{and} \quad \Psi_2(t,x) = \mathrm{e}^{-\mathrm{i}\lambda_2 t} u_2(x)$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$. This ansatz leads to (1.1) for (u_1, u_2) with $f_1(u_1) = g_1(|u_1|)u_1$, $f_2(u_2) = g_2(|u_2|)u_2$ and $F(u_1, u_2) = \frac{1}{2} G(|u_1|^2, |u_2|^2)$.

The question of finding normalized solutions is already interesting for scalar equations and provides features and difficulties that are not present when the normalization condition has been dropped. Since the scalar setting will of course be relevant when treating systems, let us recall a few facts. Solutions $u \in H^1(\mathbb{R}^N)$ of

$$-\Delta u = \lambda u + f(u), \quad \int_{\mathbb{R}^N} |u|^2 = a, \tag{1.4}$$

with $a > 0$ fixed can be obtained as critical points of the functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} F(u) \quad \text{with } F(s) = \int_0^s f(t)\,\mathrm{d}t,$$

constrained to the L^2-sphere $S_a := \{u \in H^1(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} |u|^2 = a\}$, provided that f is subcritical and superlinear. The model nonlinearity is $f(s) = |s|^{p-2}s$ with $2 < p < 2^* = 2N/(N-2)$. The parameter λ in the equation appears then as Lagrange multiplier.

The best studied cases of (1.4) correspond to the situation in which a solution can be found as a global minimizer of J on $S(a)$, which is the case if $2 < p < 2 + 4/N$ for the model nonlinearity. This research mainly started with the work of Stuart [30,31]. A bit later the concentration compactness principle of Lions [23,24] was used in this type of problem. The case in which J is unbounded from below (and from above) on S_a, i.e. $2 + 4/N < p < 2^*$ for the model nonlinearity, is much less studied. For this case, a mountain pass structure was exploited in [20] leading to the existence of one normalized solution. The existence of infinitely many normalized solution was later proved in [7], where a 'fountain'-type structure on the L^2-sphere was discovered that is somewhat reminiscent to the one for the free functional from [6]; see also [33]. More results on normalized solutions for scalar equations can be found

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in [9,11,21]. Technical difficulties in dealing with the constrained functional are that the existence of bounded Palais–Smale sequences requires new arguments, that Lagrange multipliers have to be controlled, and that weak limits of Palais–Smale sequences *a priori* do not necessarily lie on the same L^2-sphere.

The goal of this paper is to find positive radial solutions of systems like (1.1) under various growth conditions on f_1, f_2, F. In order to keep the ideas and the results simple, and in order to avoid technicalities, we only deal with homogeneous nonlinearities $f_1(s) = \mu_1|s|^{p_1-2}s$, $f_2(s) = \mu_2|s|^{p_2-2}s$ and $F(s,t) = \beta|s|^{r_1}|t|^{r_2}$. Thus we look for positive radial solutions $u_1, u_2 \in E := H^1_{\mathrm{rad}}(\mathbb{R}^N) \times H^1_{\mathrm{rad}}(\mathbb{R}^N)$ of the system

$$\left.\begin{aligned} -\Delta u_1 &= \lambda_1 u_1 + \mu_1|u_1|^{p_1-2}u_1 + r_1\beta|u_1|^{r_1-2}|u_2|^{r_2}u_1, \\ -\Delta u_2 &= \lambda_2 u_2 + \mu_2|u_2|^{p_2-2}u_2 + r_2\beta|u_1|^{r_1}|u_2|^{r_2-2}u_2, \end{aligned}\right\} \tag{1.5}$$

which are L^2-normalized in the sense of (1.2). Throughout the paper we require $N \geqslant 2$, $p_1, p_2 \in]2, 2^*[\setminus \{2 + 4/N\}$ and $\beta, \mu_1, \mu_2, r_1, r_2, a_1, a_2 > 0$ with $2 \leqslant r_1 + r_2 < 2^*$. Thus we treat various self-focusing cases and attractive interaction. These constants are prescribed whereas the parameters λ_1 and λ_2 are unknown and will appear as Lagrange multipliers. The system comes from mean-field models for binary mixtures of Bose–Einstein condensates or for binary gases of fermion atoms in degenerate quantum states (Bose–Fermi mixtures, Fermi–Fermi mixtures); see [1,5,14,26] and the references therein. The most famous case is that of coupled Gross–Pitaevskii equations in dimension $N \leqslant 3$ with $p_1 = p_2 = 4$, $r_1 = r_2 = 2$, which models Bose–Einstein condensation. However, models for other ultracold quantum gases use different exponents.

The particular case in which $p_1 = p_2 = 4$, $r_1 = r_2 = 2$ for coupled Gross–Pitaevskii equations in $\mathbb{R}^3$ is being treated in the companion paper [8]. In the present paper we deal with general exponents and distinguish between the cases $p_1, p_2 < 2 + 4/N$, $p_1 < 2 + 4/N < p_2$ and $p_1, p_2 > 2 + 4/N$. The exponent $2 + 4/N$ is critical for the normalized solution problem and will not be treated here. Other results on the existence of prescribed L^2-norm solutions for systems can be found in [2, 3, 17, 19, 27, 28, 32]. In these papers, either the solutions obtained were global minimizers of the associated functional (for example, in the defocusing repelling case $\mu_1, \mu_2, \beta < 0$), or only the case of small masses $a_1, a_2 \sim 0$ was treated, as in [28]. In [28] the system included a trapping potential or was defined on a bounded domain. Requiring the masses to be small is a bifurcation-type result.

To the best of our knowledge the results of this paper and those of its companion paper [8] are the first results in which one obtains normalized solutions for systems when the associated functional, here J, is unbounded from below in the constraint, and when the masses need not be small.

The paper is organized as follows. In § 2 we state and discuss our results, § 3 contains some preliminary results, and §§ 4 and 5 are devoted to the proofs of the theorems from § 2. Throughout the paper we use the notation $|u|_p$ to denote the L^p-norm.

2. Statement of results

We fix $N \geqslant 2$, $p_1, p_2 \in]2, 2^*[$, and $\beta, \mu_1, \mu_2, r_1, r_2, a_1, a_2 > 0$ with $2 \leqslant r_1 + r_2 < 2^*$. We seek solutions in the space $E := H^1_{\mathrm{rad}}(\mathbb{R}^N) \times H^1_{\mathrm{rad}}(\mathbb{R}^N)$ that are pairs of radial

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functions in $H^1(\mathbb{R}^N)$. Our first result on (1.2), (1.5) deals with a case in which it is possible to minimize the functional on the constraint.

THEOREM 2.1. *Problem (1.2), (1.5) has, for some $\lambda_1, \lambda_2 < 0$, a solution $(u_1, u_2) \in E$ satisfying $u_1 > 0$, $u_2 > 0$ in each of the following cases:*

(a) $2 \leqslant N \leqslant 4$ *and* $p_1, p_2, r_1 + r_2 < 2 + 4/N$;

(b) $N \geqslant 5$ *and* $p_1, p_2 < 2 + 2/(N-2)$ *and* $r_1 + r_2 < 2 + 4/N$.

We do not know whether theorem 2.1(a) holds true for all $N \geqslant 2$, i.e. whether the hypothesis $p_1, p_2 < 2 + 2/(N-2)$ in theorem 2.1(b) can be replaced by $p_1, p_2 < 2 + 4/N$. Our proof of theorem 2.1 ultimately relies on the use of a Liouville-type result for an associated scalar equation. More precisely, we require that

$$-\Delta u_1 = \lambda_1 u_1 + \mu_1 u_1^{p_1 - 1} + r_1 \beta u_1^{r_1 - 1} u_2$$

not have a solution $u_1 > 0$ if $\lambda_1 \geqslant 0$ and $u_2 \geqslant 0$. This is responsible for the restriction that $N \leqslant 4$ in (a) and that $p_1, p_2 < 2 + 2/(N-2)$ in (b).

Setting $S(a) = S_a \cap H^1_{\mathrm{rad}}(\mathbb{R}^N) = \{u \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \colon |u|_2^2 = a\}$, the solution in theorem 2.1 will be a minimizer of the functional

$$J(u_1, u_2) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2 \, \mathrm{d}x - \int_{\mathbb{R}^N} \frac{\mu_1}{p_1} |u_1|^{p_1} + \frac{\mu_2}{p_2} |u_2|^{p_2} + \beta |u_1|^{r_1} |u_2|^{r_2} \, \mathrm{d}x$$

constrained to $S(a_1) \times S(a_2) \subset E$.

It is easy to prove that any minimizing sequence $\{(u_1^n, u_2^n)\} \subset S(a_1) \times S(a_2)$ associated with J is bounded. Thus we can assume without restriction that $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2)$ weakly in E for some $(u_1, u_2) \in E$. From the weak convergence in E we deduce that (u_1, u_2) satisfies (1.5) for some associated (λ_1, λ_2). To prove theorem 2.1 one still has to show that $(u_1, u_2) \in S(a_1) \times S(a_2)$. Even if we work in the space of radially symmetric functions this question is, with respect to the scalar case, challenging, as was already observed in [19].

Our second result deals with the case in which p_2 and $r_1 + r_2$ are bigger than $2 + 4/N$ so that J is unbounded from below and minimization does not work. We require the following hypotheses on the coefficients:

(H1) $2 < p_1 < 2 + 4/N < p_2 < 2^*$;

(H2) $2 + 4/N < r_1 + r_2 < 2^*$, $r_2 > 2$.

Consider the functional $I \colon H^1_{\mathrm{rad}}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x - \frac{\mu}{p} \int_{\mathbb{R}^N} |u|^p \, \mathrm{d}x$$

constrained to the L^2-sphere $S(a)$. For $p \in]2, 2^*[\setminus \{2 + 4/N\}$ we denote by $m_p^\mu(a)$ the ground state level, i.e.

$$m_p^\mu(a) = \inf\{I(u) \colon u \in S(a) \text{ such that } (I_{|S(a)})'(u) = 0\}.$$

We discuss the properties of this ground state level in lemma 3.1.

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THEOREM 2.2. *Assume that (H1) and (H2) hold and that* $2 \leqslant N \leqslant 4$. *If*

$$m_{p_1}^{\mu_1}(a_1) + m_{p_2}^{\mu_2}(a_2) < 0, \tag{2.1}$$

then, for some $\lambda_1 < 0$ *and* $\lambda_2 < 0$, *there exists a solution* $(u_1, u_2) \in E$ *of (1.2), (1.5), satisfying* $u_1 > 0$, $u_2 > 0$.

As a corollary of theorem 2.2 we obtain the following.

COROLLARY 2.3. *Assume that (H1) and (H2) hold and that* $2 \leqslant N \leqslant 4$.

(a) *For any* $a_2 > 0$ *there exists* $\bar{a}_1 > 0$ *such that for any* $a_1 \geqslant \bar{a}_1$ *there exists a positive solution* $(u_1, u_2) \in E$ *of (1.2), (1.5), for some* $\lambda_1 < 0$ *and* $\lambda_2 < 0$. *In addition,* $\bar{a}_1 \to 0$ *as* $a_2 \to \infty$.

(b) *For any* $a_1 > 0$ *there exists* $\bar{a}_2 > 0$ *such that for any* $a_2 \geqslant \bar{a}_2$ *there exists a positive solution* $(u_1, u_2) \in E$ *of (1.2), (1.5), for some* $\lambda_1 < 0$ *and* $\lambda_2 < 0$. *In addition,* $\bar{a}_2 \to 0$ *as* $a_1 \to \infty$.

With respect to theorem 2.1, the proof of theorem 2.2 presents new difficulties. First one needs to identify a possible critical level $\gamma(a_1, a_2)$ where one can find Palais–Smale sequences. The construction of this minimax level, which is of mountain pass type, is the heart of the proof and is carried out in lemmas 5.1–5.3. We need the restrictions on N and on the exponents r_1, r_2 in (H2) in order to control the mixed term $\int_{\mathbb{R}^N} |u_1|^{r_1}|u_2|^{r_2}\,\mathrm{d}x$ when proving the mountain pass structure, which is of different type compared to the standard one. By Ekeland's variational principle there exists a Palais–Smale sequence associated with $\gamma(a_1, a_2)$. One then needs to find a bounded Palais–Smale sequence. We manage to find a special Palais–Smale sequence $\{(u_1^n, u_2^n)\} \subset S(a_1) \times S(a_2)$ having the additional property that $Q(u_1^n, u_2^n) \to 0$, where $Q\colon E \to \mathbb{R}$ is given by

$$\begin{aligned} Q(u_1, u_2) = |\nabla u_1|_2^2 + |\nabla u_2|_2^2 - \frac{\mu_1}{p_1} N\left(\frac{p_1}{2} - 1\right)|u_1|_{p_1}^{p_1} - \frac{\mu_2}{p_2} N\left(\frac{p_2}{2} - 1\right)|u_2|_{p_2}^{p_2} \\ - N\beta\left(\frac{r_1 + r_2}{2} - 1\right)\int_{\mathbb{R}^N} |u_1|^{r_1}|u_2|^{r_2}\,\mathrm{d}x. \end{aligned} \tag{2.2}$$

The condition $Q(u_1, u_2) = 0$ corresponds to a natural constraint of Pohozaev type on the solutions of (1.2), (1.5); see remark 5.6. To construct a Palais–Smale sequence having the additional property that $Q(u_1^n, u_2^n) \to 0$, we employ similar arguments as in [7, 11, 20, 25]; see also [4, 18].

From the property that $Q(u_1^n, u_2^n) \to 0$ we deduce that $\{(u_1^n, u_2^n)\} \subset E$ is bounded. Finally, in order to ensure the strong convergence of our Palais–Smale sequence we combine estimate (2.1) with the Liouville argument already used in the proof of theorem 1.1.

In our last result we assume the inequalities $p_1, p_2, r_1 + r_2 > 2 + 4/N$.

THEOREM 2.4. *Assume that* $p_1, p_2, r_1 + r_2 > 2 + 4/N$ *and that* $2 \leqslant N \leqslant 4$.

(a) *There exists* $\beta_1 = \beta_1(a_1, a_2, \mu_1, \mu_2) > 0$ *such that if* $\beta \leqslant \beta_1$, *then (1.2), (1.5) has a positive solution* $(u_1, u_2) \in E$ *for some* $\lambda_1 < 0$ *and* $\lambda_2 < 0$.

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(b) *There exists* $\beta_2 = \beta_2(a_1, a_2, \mu_1, \mu_2) > 0$ *such that if* $\beta \geqslant \beta_2$, *then (1.2), (1.5) has a positive solution* $(u_1, u_2) \in E$ *for some* $\lambda_1 < 0$ *and* $\lambda_2 < 0$.

We would like to emphasize that the proof yields explicit estimates for β_1 from below and β_2 from above in terms of p_1, p_2, r_1, r_2 and a_1, a_2, μ_1, μ_2; in particular, β_1 and β_2 are not obtained by limiting processes.

Theorem 2.4 is a generalization of the result from [8], where the case in which $N = 3$, $p_1 = p_2 = 4$, $r_1 = r_2 = 2$ was considered. The proof of theorem 2.4(a) is based on a two-dimensional linking on the constraint $S = S(a_1) \times S(a_2)$, whereas the proof of theorem 2.4(b) uses a mountain pass argument on S. As in theorem 2.2, one obtains a special Palais–Smale sequence $\{(u_1^n, u_2^n)\} \subset S(a_1) \times S(a_2)$ at the suspected critical level $\theta(a_1, a_2)$ such that $Q(u_1^n, u_2^n) \to 0$. This leads in particular to its boundedness. In order to obtain the strong convergence, an upper bound for β is needed in (a), and a lower bound in (b). Concerning estimates for β_1 or β_2, we just mention that in the setting of [8] one has $\beta_1 \to \infty$ if $\mu_1 = \mu_2 \to \infty$ and a_1, a_2 are fixed. Similarly, $\beta_2 \to 0$ if $\mu_1 = \mu_2 \to 0$ and a_1, a_2 are fixed. Since the proof in [8] for the special case in which $N = 3$, $p_1 = p_2 = 4$, $r_1 = r_2 = 2$ generalizes easily, we simply refer the reader to [8] and do not give any details here.

Remark 2.5. The results presented in this paper for $N \geqslant 2$ can be extended to $N = 1$. The difference between the $N = 1$ and $N \geqslant 2$ cases is that the compact embedding $H^1_{\mathrm{rad}}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ for $q \in]2, 2^*[$ only holds when $N \geqslant 2$. When $N = 1$ the corresponding compactness can, however, be regained by working with Palais–Smale sequences of almost Schwartz-symmetric functions. In order to avoid additional technicalities, we do not deal with the $N = 1$ case in this paper but instead refer the reader to [21], where a similar issue is treated. The results are identical in the $N = 1$ and $N \geqslant 2$ cases except that in the $N = 1$ case one requires in addition that $r_2 > 4$ in (H2) (this restriction originates from the adapted version of lemma 5.1).

3. Preliminary results

Throughout the paper we denote by H the space $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ equipped with the standard norm $|\cdot|$, so $E = H \times H$. We denote by $\mathcal{S}$ the constraint $S(a_1) \times S(a_2)$. We recall (see, for example, [12]) that if $u_n \rightharpoonup u$ weakly in H, then $u_n \to u$ strongly in $L^q(\mathbb{R}^N)$ for $q \in]2, 2^*[$.

Let us first observe that the functional J is well defined. For $2 \leqslant r_1 + r_2 \leqslant 2^*$ there exists $q > 1$ with

$$\max\left\{\frac{2}{r_1}, \frac{2^*}{2^* - r_2}\right\} \leqslant q \leqslant \min\left\{\frac{2^*}{r_1}, \frac{2}{(2 - r_2)^+}\right\}, \tag{3.1}$$

which implies $2 \leqslant r_1 q, r_2 q' \leqslant 2^*$, and hence

$$\int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} \, \mathrm{d}x \leqslant |u_1|^{r_1}_{r_1 q} |u_2|^{r_2}_{r_2 q'} < \infty.$$

The Gagliardo–Nirenberg inequality

$$|u|_p \leqslant C(N, p) |\nabla u|_2^{\alpha} |u|_2^{1-\alpha}, \quad \text{where } \alpha = \frac{N(p-2)}{2p},$$

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which holds for $u \in H^1(\mathbb{R}^N)$ and $2 \leqslant p \leqslant 2^*$, implies for $u_1 \in S(a_1)$, $u_2 \in S(a_2)$ that

$$\left.\begin{aligned} \int_{\mathbb{R}^N} |u_1|^{p_1} &\leqslant C(N, p_1, a_1)|\nabla u_1|_2^{N(p_1-2)/2}, \\ \int_{\mathbb{R}^N} |u_2|^{p_2} &\leqslant C(N, p_2, a_2)|\nabla u_2|_2^{N(p_2-2)/2}, \end{aligned}\right\} \tag{3.2}$$

and

$$\int_{\mathbb{R}^N} |u_1|^{r_1}|u_2|^{r_2}\,\mathrm{d}x \leqslant |u_1|_{r_1 q}^{r_1}|u_2|_{r_2 q'}^{r_2} \leqslant C|\nabla u_1|_2^{N(r_1 q-2)/2q}|\nabla u_2|_2^{N(r_2 q'-2)/2q'} \tag{3.3}$$

with $C = C(N, r_1, r_2, a_1, a_2, q)$.

LEMMA 3.1. *Assume that $p \in]2, 2^*[\setminus \{2 + 4/N\}$ and let $\mu > 0$ be given. For any $a > 0$ there exists a unique couple $(\lambda_a, u_a) \in \mathbb{R}^+ \times H$ solving*

$$-\Delta u + \lambda u = \mu|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N), \tag{3.4}$$

and such that $u_a > 0$ and $|u_a|_2^2 = a$. Moreover, u_a corresponds to the least energy level $m_p^\mu(a)$ of the functional $I\colon H \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p\,\mathrm{d}x$$

constrained to the L^2-sphere $S(a)$. If $p \in]2, 2 + 4/N[$, then $m_p^\mu(a) < 0$ for all $a > 0$, the map $a \mapsto m_p^\mu(a)$ is strictly decreasing, and $m_p^\mu(a) \to -\infty$ as $a \to \infty$. If $p \in]2 + 4/N, 2^*[$, then $m_p^\mu(a) > 0$ for all $a > 0$, the map $a \mapsto m_p^\mu(a)$ is strictly decreasing, and $m_p^\mu(a) \to 0$ as $a \to \infty$.

Proof. It is standard (see [22]) that the equation

$$-\Delta u + \lambda u = \mu|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N), \tag{3.5}$$

has, for any $\lambda > 0$, a unique positive radial solution u_λ. By direct calculations one can show that u_λ is given by

$$u_\lambda(x) = \lambda^{1/(p-2)}w(\sqrt{\lambda}x),$$

where w is the unique positive radial solution of

$$-\Delta u + u = \mu|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N).$$

Since

$$|u_\lambda|_2^2 = \lambda^{2/(p-2)-N/2}|w|_2^2,$$

for any $a > 0$ there exists a unique $\lambda_a > 0$, explicitly given by

$$\lambda_a = \left(\frac{a}{|w|_2^2}\right)^{2(p-2)/(4-N(p-2))},$$

such that $u_{\lambda_a} \in H^1(\mathbb{R}^N)$ satisfies $|u_{\lambda_a}|_2^2 = a$ and is the unique positive radial solution of

$$-\Delta u + \lambda_a u = \mu|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N).$$

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The solution u_{λ_a} corresponds to the least energy solution of the functional $I\colon H \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x$$

constrained to the L^2-sphere $S(a)$. Here $\lambda_a > 0$ appears as the associated Lagrange parameter. To prove this statement two cases have to be distinguished.

Case 1 ($p \in]2, 2+4/N[$). The least energy solution corresponds to the energy level

$$m_p^\mu(a) = \inf_{u\in S(a)} I(u).$$

It is standard [30, 31] (see also [10]) that $m_p^\mu(a) < 0$, that the map $a \mapsto m_p^\mu(a)$ is strictly decreasing, and that $m_p^\mu(a) \to -\infty$ as $a \to \infty$.

Case 2 ($p \in]2+4/N, 2^*[$). The least energy solution corresponds to the energy level

$$m_p^\mu(a) = \inf_{u\in V(a)} I(u).$$

Here

$$V(a) = \left\{ u \in S(a)\colon |\nabla u|_2^2 = \frac{\mu N(p-2)}{2p}|u|_p^p \right\} \tag{3.6}$$

is a natural constraint that contains all the critical points of I constrained to $S(a)$. This was proved in [11, lemma 9.3]; see also [20]. Also, it was shown in [11, lemma 9.3] that the map $a \mapsto m_p^\mu(a)$ is strictly decreasing and that $m_p^\mu(a) \to 0$ as $a \to \infty$. □

Lemma 3.2. *Let $\{(u_1^n, u_2^n)\} \subset \mathcal{S}$ be a bounded Palais–Smale sequence of J restricted to $\mathcal{S}$. Then there exist $(u_1, u_2) \in E$, $(\lambda_1, \lambda_2) \in \mathbb{R}\times\mathbb{R}$ and a sequence $\{(\lambda_1^n, \lambda_2^n)\} \subset \mathbb{R}\times\mathbb{R}$ such that, up to a subsequence, the following hold:*

(a) *for $i = 1, 2$, $u_i^n \rightharpoonup u_i$ weakly in H and in $L^2(\mathbb{R}^N)$, $u_i^n \to u_i$ in $L^q(\mathbb{R}^N)$ for any $q \in]2, 2^*[$;*

(b) *$(\lambda_1^n, \lambda_2^n) \to (\lambda_1, \lambda_2)$ in $\mathbb{R}\times\mathbb{R}$;*

(c) *$J'(u_1^n, u_2^n) - \lambda_1^n(u_1^n, 0) - \lambda_2^n(0, u_2^n) \to 0$ in E^*;*

(d) *(u_1, u_2) is a solution of system (1.5), where (λ_1, λ_2) are given in (b).*

In addition, if $\lambda_1 < 0$, then $u_1^n \to u_1$ strongly in H. Similarly, if $\lambda_2 < 0$, then $u_2^n \to u_2$ strongly in H.

Proof. Point (a) is trivial. Since $\{(u_1^n, u_2^n)\} \subset H \times H$ is bounded, following Berestycki and Lions [12, lemma 3], we know that $(J|_{\mathcal{S}})'(u_1^n, u_2^n) \to 0$ in E^* is equivalent to

$$J'(u_1^n, u_2^n) - \frac{1}{|u_1^n|_2^2}\langle J'(u_1^n, u_2^n), (u_1^n, 0)\rangle(u_1^n, 0) - \frac{1}{|u_2^n|_2^2}\langle J'(u_1^n, u_2^n), (0, u_2^n)\rangle(0, u_2^n) \to 0$$

in E^*. Therefore, we obtain

$$J'(u_1^n, u_2^n) - \lambda_1^n(u_1^n, 0) - \lambda_2^n(0, u_2^n) \to 0 \quad \text{in } E^*$$

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with

$$\lambda_1^n = \frac{1}{|u_1^n|_2^2}\left(|\nabla u_1^n| - \mu_1 |u_1^n|_{p_1}^{p_2} - \beta r_1 \int_{\mathbb{R}^N} |u_1^n|^{r_1} |u_2^n|^{r_2}\,\mathrm{d}x\right) \tag{3.7}$$

and

$$\lambda_2^n = \frac{1}{|u_2^n|_2^2}\left(|\nabla u_2^n| - \mu_1 |u_2^n|_{p_1}^{p_2} - \beta r_2 \int_{\mathbb{R}^N} |u_2^n|^{r_2} |u_1^n|^{r_1}\,\mathrm{d}x\right). \tag{3.8}$$

This proves (c). To prove (b), namely that $\{(\lambda_1^n, \lambda_2^n)\} \subset \mathbb{R}\times\mathbb{R}$ is bounded, it suffices to recall that $\{(u_1^n, u_2^n)\} \subset E$ is bounded and to use the estimates (3.2) and (3.3). From (b) and (c) it is now standard to deduce (d).

It remains to show that if $\lambda_1 < 0$, then $u_1^n \to u_1$ strongly in $H^1(\mathbb{R}^N)$, and in particular in $L^2(\mathbb{R}^N)$. Since

$$|u_1^n|_{p_1}^{p_1} \to |u_1|_{p_1}^{p_1} \quad \text{and} \quad \int_{\mathbb{R}^N} |u_1^n|^{r_1} |u_2^n|^{r_2}\,\mathrm{d}x \to \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2}\,\mathrm{d}x,$$

and using the fact that

$$\langle J'(u_1^n, u_2^n) - \lambda_1^n (u_1^n, 0), (u_1^n, 0)\rangle \to \langle J'(u_1, u_2) - \lambda_1 (u_1, 0), (u_1, 0)\rangle = 0,$$

we deduce that

$$|\nabla u_1^n|_2^2 - \lambda_1^n |u_1^n|_2^2 \to |\nabla u_1|_2^2 - \lambda_1 |u_1|_2^2. \tag{3.9}$$

As a consequence of the weak convergence $u_i^n \rightharpoonup u_i$, we obtain

$$|\nabla u_1|_2^2 \leqslant \liminf |\nabla u_1^n|_2^2 \quad \text{and} \quad |u_1|_2^2 \leqslant \liminf |u_1^n|_2^2.$$

Finally, since $\lambda_1^n \to \lambda_1$ we deduce from (3.9) that

$$|\nabla u_1^n|_2^2 \to |\nabla u_1|_2^2 \quad \text{and} \quad |u_1^n|_2^2 \to |u_1|_2^2,$$

from which the strong convergence follows. The case in which $\lambda_2 < 0$ is treated in the same way. □

Lemma 3.3.

(a) *Suppose that $q \in \,]1, N/(N-2)]$ when $N \geqslant 3$ and $q \in \,]1, \infty[$ when $N = 1, 2$. Let $u \in L^q(\mathbb{R}^N)$ be a smooth non-negative function satisfying $-\Delta u \geqslant 0$ in $\mathbb{R}^N$. Then $u \equiv 0$ holds.*

(b) *For $1 < q \leqslant 1 + 2/(N-2)$ the inequality $-\Delta u \geqslant u^q$ does not have a positive classical solution in $\mathbb{R}^N$.*

Proof. Part (a) can be found in [19, lemma A.2]. Part (b) is due to [16]; a simple proof can be found in [29]. □

Lemma 3.4. *Assume that $N \leqslant 4$, or $p_1 \leqslant 2 + 2/(N-2)$ if $N \geqslant 5$. Then if $(u_1, u_2) \in E$ is a solution of (1.5) with $u_1 \gneqq 0$ and $u_2 \geqslant 0$, it holds that $\lambda_1 < 0$. Similarly, suppose that $N \leqslant 4$, or $p_2 \leqslant 2+2/(N-2)$ if $N \geqslant 5$. Then if $(u_1, u_2) \in E$ is a solution of (1.5) with $u_2 \gneqq 0$ and $u_1 \geqslant 0$, it holds that $\lambda_2 < 0$.*

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Proof. In the first case, since $u_1 \gneqq 0$ satisfies

$$-\Delta u_1 = \lambda_1 u_1 + \mu_1 u_1^{p_1-1} + r_1\beta u_1^{r_1-1}u_2^{r_2}$$

and since all summands on the right-hand side are non negative if $\lambda_1 \geqslant 0$, we conclude by lemma 3.3 that $u_1 = 0$. This contradicts the assumption that $u_1 \gneqq 0$. The proof of the other part is identical. □

4. Proof of theorem 2.1

LEMMA 4.1. *If $p_1, p_2, r_1 + r_2 < 2 + 4/N$, then J is bounded from below and coercive on $\mathcal{S}$ for any $a_1, a_2 > 0$. In addition there exists a bounded Palais–Smale sequence $\{(u_1^n, u_2^n)\} \subset \mathcal{S}$ that satisfies $(u_1^n)^- \to 0$ and $(u_2^n)^- \to 0$ in H. Here $(u_i^n)^- = \max\{0, -u_i^n\}$ for $i = 1, 2$.*

Proof. Observe that $N(p_i - 2)/2 < 2$ because $p_i < 2 + 4/N$, $i = 1, 2$, and that

$$\frac{N(r_1 q - 2)}{2q} + \frac{N(r_2 q' - 2)}{2q'} < 2 \tag{4.1}$$

since $r_1 + r_2 < 2 + 4/N$. It follows easily from (3.2), (3.3) and (4.1) that J is bounded from below and coercive on $\mathcal{S}$.

Now let $\{(v_1^n, v_2^n)\} \subset \mathcal{S}$ be a minimizing sequence for J on $\mathcal{S}$. By the coerciveness of J, this sequence is bounded and also, without restriction, we can assume that $v_1^n \geqslant 0$ and $v_2^n \geqslant 0$. Using Ekeland's variational principle [13, 15], we deduce that there exists a minimizing sequence $\{(u_1^n, u_2^n)\} \subset S$ that is a Palais–Smale sequence for J restricted to $\mathcal{S}$ and that satisfies $(u_1^n, u_2^n) - (v_1^n, v_2^n) \to 0$ in E. In particular, $(u_1^n)^- \to 0$ and $(u_2^n)^- \to 0$ in H. □

Proof of theorem 2.1. From lemmas 3.1 and 4.1 we deduce the existence of a bounded Palais–Smale sequence $\{(u_1^n, u_2^n)\} \subset \mathcal{S}$ such that $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2)$ weakly in E with $u_1 \geqslant 0$ and $u_2 \geqslant 0$. We also obtain a pair $(\lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}$ for which (u_1, u_2) is a solution of system (1.5). To conclude the proof it remains to show that $u_1^n \to u_1$ and $u_2^n \to u_2$ in H. Indeed, if this is the case, then we have both that $u_1 \in S(a_1)$ and $u_2 \in S(a_2)$, and that (u_1, u_2) is a least energy solution. In addition by the strong maximum principle, applied separately to each equation, we obtain that $u_1 > 0$ and $u_2 > 0$. In order to show the strong convergence in H we define

$$m(a_1, a_2) := \inf_{(u_1,u_2)\in\mathcal{S}} J(u_1, u_2).$$

Since $\beta \geqslant 0$, we clearly have

$$m(a_1, a_2) \leqslant m_{p_1}^{\mu_1}(a_1) + m_{p_2}^{\mu_2}(a_2) < 0, \tag{4.2}$$

where the last inequality comes from lemma 3.1. We now distinguish four cases and show that only the last one may occur.

CASE 1 ($u_1 = 0$ and $u_2 = 0$). In this case, $|u_1^n|_{p_1}^{p_1} \to 0$, $|u_2^n|_{p_2}^{p_2} \to 0$ and

$$\int_{\mathbb{R}^N} |u_1^n|^{r_1} |u_2^n|^{r_2} \, \mathrm{d}x \to 0.$$

Thus, $\lim J(u_1^n, u_2^n) \geqslant 0$, which contradicts (4.2).

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CASE 2 ($u_1 = 0$ and $u_2 \neq 0$). Here, setting $\bar{a}_2 := |u_2|_2^2 \leqslant a_2$ we obtain

$$\lim J(u_1^n, u_2^n) \geqslant \tfrac{1}{2}|\nabla u_2|_2^2 - \frac{\mu_2}{p_2}|u_2|_{p_2}^{p_2} \geqslant m(\bar{a}_2).$$

By lemma 3.1 we know that $m(\bar{a}_2) \geqslant m(a_2)$, and since $m(a_1) < 0$ we have a contradiction with (4.2).

CASE 3 ($u_1 \neq 0$ and $u_2 = 0$). Reversing the role of u_1 and u_2 we obtain a contradiction similar to case 2.

CASE 4 ($0 < |u_1|_2^2 = \bar{a}_1 \leqslant a_1$ or $0 < |u_2|_2^2 = \bar{a}_2 \leqslant a_2$). Necessarily this case occurs. Now using lemma 3.4, we deduce that $\lambda_1 < 0$ and $\lambda_2 < 0$. Then lemma 3.2 implies that $u_1^n \to u_1$ and $u_2^n \to u_2$ in H. At this point the proof of the theorem is completed. □

5. Proof of theorem 2.2

For $c > 0$ we define the sets

$$A_c = \{u_2 \in S(a_2) \colon |\nabla u_2|_2^2 \leqslant c\} \quad \text{and} \quad B_c = \{u_2 \in S(a_2) \colon |\nabla u_2|_2^2 = 2c\}.$$

Setting $J_{u_1}(u_2) := J(u_1, u_2)$ for $u_1 \in S(a_1)$ we observe that

$$J_{u_1}(u_2) = J_{u_1}(0) + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_2|^2 \,\mathrm{d}x - \frac{\mu_2}{p_2}\int_{\mathbb{R}^N} |u_2|^{p_2} \,\mathrm{d}x - \beta \int_{\mathbb{R}^N} |u_1|^{r_1}|u_2|^{r_2} \,\mathrm{d}x.$$

LEMMA 5.1. *There exists a continuous function $c\colon S(a_1) \to \mathbb{R}$, $u_1 \to c(u_1)$, such that*

$$\sup_{A_{c(u_1)}} J_{u_1} < \inf_{B_{c(u_1)}} J_{u_1} \quad \textit{for all } u_1 \in S(a_1).$$

The function c is bounded, and it is bounded away from 0 on bounded subsets of $S(a_1)$.

Proof. Fixing $u_1 \in S(a_1)$ we first observe that for $u_2 \in A_c$ it holds that

$$J_{u_1}(u_2) \leqslant J_{u_1}(0) + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_2|^2 \,\mathrm{d}x \leqslant J_{u_1}(0) + \tfrac{1}{2}c.$$

For $u_2 \in B_c$, q as in (3.1), and $\gamma = N(r_2 q' - 2)/2q'$ we have, using the Gagliardo–Nirenberg inequality (see (3.2), (3.3)),

$$\begin{aligned} J_{u_1}(u_2) &\geqslant J_{u_1}(0) + c - \frac{\mu_2}{p_2} c(p_2, N) |\nabla u_2|_2^{N(p_2-2)/2} |u_2|_2^{p_2(1-N/2)+N} - \beta |u_1|_{r_1 q}^{r_1} |u_2|_{r_2 q'}^{r_2} \\ &= J_{u_1}(0) + c - K_1 c^{N(p_2-2)/4} - K_2 |u_1|_{r_1 q}^{r_1} c^{\gamma/2}. \end{aligned}$$

Here $K_1 = K_1(N, \mu_2, p_2, a_2)$ and $K_2 = K_2(N, \beta, r_2, a_2, q)$. Observe that $N(p_2 - 2)/4 > 1$ because $p_2 > 2 + 4/N$, and $\gamma > 2$ provided that $q < 2N/(2N - r_2 N + 4)$. We can choose q satisfying this inequality and (3.1) because

$$\frac{2N}{2N - r_2 N + 4} > \max\left\{\frac{2}{r_1}, \frac{2^*}{2^* - r_2}\right\},$$

which is a consequence of $r_1 + r_2 > 2 + 4/N$ and $r_2 > 2$.

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Observe that $K_1 c^{N(p_2-2)/4} \leqslant c/8$ if $c > 0$ is small because $N(p_2-2)/4 > 1$, and $K_2 |u_1|_{r_1 q}^{r_1} c^{\gamma/2} \leqslant c/8$ if $c > 0$ is small because $\gamma > 2$. More precisely, if $c\colon S(a_1) \to \mathbb{R}^+$ satisfies

$$c(u_1) \leqslant \min\{(8K_1)^{-4/(N(p_2-2)-4)}, (8K_2)^{-2/(\gamma-2)} |u_1|_{r_1 q}^{-2r_1/(\gamma-2)}\}, \tag{5.1}$$

then we have for $u_2 \in B_{c(u_1)}$ that

$$\begin{aligned} J_{u_1}(u_2) &\geqslant J_{u_1}(0) + c(u_1) - \tfrac{1}{8} c(u_1) - \tfrac{1}{8} c(u_1) \\ &> J_{u_1}(0) + \tfrac{1}{2} c(u_1) \\ &\geqslant \sup_{A_{c(u_1)}} J_{u_1}. \end{aligned} \tag{5.2}$$

Clearly, we may define a continuous function $c\colon S(a_1) \to \mathbb{R}^+$ that satisfies (5.1) and is bounded away from 0 on bounded subsets of $S(a_1)$. In fact, the right-hand side of (5.1) may serve as a definition. By (5.1), c is also bounded from above. □

Now we set

$$A(u_1) = A_{c(u_1)}, \qquad B(u_1) = B_{c(u_1)}$$

and

$$B = \{(u_1, u_2)\colon u_1 \in S(a_1),\ u_2 \in B(u_1)\}.$$

Let $\underline{u} \in S(a_1)$ be such that

$$J(\underline{u}, 0) = \min_{u \in S(a_1)} J(u, 0) < 0. \tag{5.3}$$

The existence of $\underline{u}$ is insured by lemma 3.1.

Lemma 5.2. *There exist $\bar{v} \in A(\underline{u})$ and $\bar{w} \in S(a_2) \setminus A_{2c(\underline{u})}$ such that*

$$\max\{J(\underline{u}, \bar{v}), J(\underline{u}, \bar{w})\} < \inf_{(u_1,u_2)\in B} J(u_1, u_2).$$

Proof. Since $J(\underline{u}, u_2) \to J(\underline{u}, 0)$ as $|\nabla u_2|_2 \to 0$, in order to obtain $\bar{v} \in A(\underline{u})$ it is sufficient to prove that $J(\underline{u}, 0) < \inf_B J$. The functional $J(\cdot, 0)\colon S(a_1) \to \mathbb{R}$ is coercive because $2 < p_1 < 2 + 4/N$. Choose $R > 0$ such that $J(u_1, 0) \geqslant J(\underline{u}, 0) + 1$ if $|\nabla u_1|_2 \geqslant R$. Then we have, for $(u_1, u_2) \in B$ with $|\nabla u_1|_2 \geqslant R$ (see (5.2)),

$$J(u_1, u_2) \geqslant J(u_1, 0) + \tfrac{3}{4} c(u_1) > J(\underline{u}, 0) + 1.$$

By lemma 5.1 it holds that

$$\varepsilon := \inf_{|\nabla u_1|_2 \leqslant R} c(u_1) > 0,$$

which implies for $(u_1, u_2) \in B$ with $|\nabla u_1|_2 \leqslant R$ that

$$J(u_1, u_2) \geqslant J(u_1, 0) + \tfrac{3}{4} c(u_1) \geqslant J(\underline{u}, 0) + \tfrac{3}{4}\varepsilon.$$

In order to find $\bar{w} \in S(a_2) \setminus A_{2c(\underline{u})}$ as required, we define for each $u \in S(a_2)$ and $t \in \mathbb{R}$ the scaled function $t * u$ by $(t * u)(x) = \mathrm{e}^{tN/2} u(\mathrm{e}^t x)$. Clearly, $t * u \in S(a_2)$ for every $t > 0$, and $|\nabla(t * u)|_2 \to \infty$ as $t \to \infty$. Now, since $p_2 > 2 + 4/N$, fixing an arbitrary $u \in S(a_2)$ we see that $J(\underline{u}, (t * u)) \to -\infty$ as $t \to \infty$. □

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As a consequence of lemma 5.2, the set

$$\Gamma := \Big\{ g \in \mathcal{C}([0,1],\ \mathcal{S})\colon g(0) = (v_1, v_2),\ g(1) = (w_1, w_2),\ v_2 \in A(v_1),\\ w_2 \notin A_{2c(w_1)},\ \max\{J(v_1, v_2), J(w_1, w_2)\} < \inf_B J \Big\}$$

is non-empty.

LEMMA 5.3. *We have*

$$\gamma(a_1, a_2) := \inf_{g\in\Gamma} \max_{t\in[0,1]} J(g(t)) \geqslant \inf_B J.$$

Proof. We just need to show that for each $g(t) = (g_1(t), g_2(t)) \in \Gamma$ there exists a $t \in [0,1]$ such that $g(t) \in B$. The map $\alpha\colon [0,1] \to \mathbb{R}$ given by $t \to |\nabla g_2(t)|_2^2 - 2c(g_1(t))$ satisfies

$$\alpha(0) = |\nabla v_2|_2^2 - 2c(v_1) \leqslant c(v_1) - 2c(v_1) < 0$$

and

$$\alpha(1) = |\nabla w_2|_2^2 - 2c(w_1) > 0.$$

Thus there exists a $t \in [0,1]$ such that $\alpha(t) = 0$, which means that $g(t) \in B$. □

For future reference we also need the following.

LEMMA 5.4. *Assume that (H1) and (H2) hold. Then for any $a_1 > 0$ and $a_2 > 0$ we have*

$$\gamma(a_1, a_2) \leqslant m_{p_1}^{\mu_1}(a_1) + m_{p_2}^{\mu_2}(a_2). \tag{5.4}$$

Proof. Let $\bar{u} \in S(a_2)$ be such that

$$J(0, \bar{u}) = I(\bar{u}) = \min_{u\in V(a_2)} I(u) = m_{p_2}^{\mu_2}(a_2)$$

whose existence and characterization is recalled in lemma 3.1, with $V(a)$ defined in (3.6). Since $\bar{u} \in V(a_2)$, it is readily seen that

$$\max_{t\in\mathbb{R}} I(t * \bar{u}) = I(0 * \bar{u}) = I(\bar{u}). \tag{5.5}$$

We now consider the path $h\colon [0,1] \to S$ given by $h(t) = (\underline{u}, h_s(t))$, where

$$h_s(t)(x) = \mathrm{e}^{s(2t-1)N/2}\bar{u}(\mathrm{e}^{s(2t-1)}x).$$

Here $s > 0$ is chosen sufficiently large so that

$$h_s(0)(\cdot) = \mathrm{e}^{-sN/2}\bar{u}(\mathrm{e}^{-s}\cdot) \in A(\underline{u}), \qquad h_s(1)(\cdot) = \mathrm{e}^{sN/2}\bar{u}(\mathrm{e}^{s}\cdot) \notin A_{2c(\underline{u})}$$

and

$$\max J(\underline{u}, h_s(1)) < 0.$$

Thus h belongs to Γ. Now using (5.5) and $\beta \geqslant 0$, we obtain

$$\max_{t\in[0,1]} J(h(t)) \leqslant J(\underline{u}, 0) + \max_{t\in[0,1]} J(0, h_s(t)) = m_{p_1}^{\mu_1}(a_1) + m_{p_2}^{\mu_2}(a_2).$$ □

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LEMMA 5.5. *Assume that (H1) and (H2) hold. There exists a Palais–Smale sequence* $\{(u_1^n, u_2^n)\} \subset \mathcal{S}$ *for* J *at the level* $\gamma(a_1, a_2)$ *that satisfies* $(u_1^n)^- \to 0$, $(u_2^n)^- \to 0$ *in* H *and the additional property that* $Q(u_1^n, u_2^n) \to 0$, *where* Q *is given in (2.2).*

REMARK 5.6. It is possible to prove that any solution (u_1, u_2) of (1.2), (1.5) must satisfy $Q(u_1, u_2) = 0$. Thus $Q(u_1, u_2) = 0$ is a natural constraint. This condition is directly related to the Pohozaev identity adapted to the presence of the constraint S. Formally it can be obtained by looking at the function $t \mapsto (t * u_1, t * u_2)$ for $(u_1, u_2) \in \mathcal{S}$. Then $Q(u_1, u_2) = 0$ corresponds to the condition that the derivative of $t \mapsto J(t * u_1, t * u_2)$ is zero when $t = 0$.

Results in the spirit of lemma 5.5 have now been proved in a variety of situations [7,9,18,20,21,25] and we shall offer only a sketch here, referring the reader to these papers for more details. We recall the stretched functional first introduced in [20]:

$$\tilde{J}\colon \mathbb{R} \times E \to \mathbb{R}, \quad (s, (u_1, u_2)) \mapsto J(s * u_1, s * u_2).$$

In what follows we write $s*(u_1, u_2) := (s*u_1, s*u_2)$ and recall that $s*(u_1, u_2) \in \mathcal{S}$ if $(u_1, u_2) \in \mathcal{S}$. Now we define the set of paths

$$\tilde{\Gamma} := \Big\{ \tilde{g} \in C([0,1], \mathbb{R} \times \mathcal{S})\colon \tilde{g}(0) = (0, (v_1, v_2)),\ \tilde{g}(1) = (0, (w_1, w_2)),$$
$$v_2 \in A(v_1),\ w_2 \notin A_{2c(w_1)},\ \max\{J(v_1, v_2), J(w_1, w_2)\} < \inf_B J \Big\}$$

and

$$\tilde{\gamma}(a_1, a_2) := \inf_{\tilde{g} \in \tilde{\Gamma}} \max_{t \in [0,1]} \tilde{J}(\tilde{g}(t)).$$

Observe that $\tilde{\gamma}(a_1, a_2) = \gamma(a_1, a_2)$. Indeed, by the definitions of $\tilde{\gamma}(a_1, a_2)$ and $\gamma(a_1, a_2)$, this identity follows immediately from the fact that the maps

$$\varphi\colon \Gamma \to \tilde{\Gamma}, \quad g \mapsto \varphi(g) := (0, g),$$

and

$$\psi\colon \tilde{\Gamma} \to \Gamma, \quad \tilde{g} = (\sigma, g) \mapsto \psi(\tilde{g}) := \sigma * g, \quad \text{with } (\sigma * g)(t) = \sigma(t) * g(t),$$

satisfy

$$\tilde{J}(\varphi(g)) = J(g) \quad \text{and} \quad J(\psi(\tilde{g})) = \tilde{J}(\tilde{g}).$$

Proof of lemma 5.5. From the observation that $\tilde{\gamma}(a_1, a_2) = \gamma(a_1, a_2)$, we obtain a sequence $\{(v_1^n, v_2^n)\} \subset \Gamma$ such that

$$\max_{t \in [0,1]} \tilde{J}(0, (v_1^n(t), v_2^n(t))) \to \tilde{\gamma}(a_1, a_2).$$

Since $J(u_1, u_2) = J(|u_1|, |u_2|)$ for $(u_1, u_2) \in E$ we can assume that $v_1^n(t) \geqslant 0$ and $v_2^n(t) \geqslant 0$ for $t \in [0, 1]$.

Now, the existence of a Palais–Smale sequence $\{(s_n, (u_1^n, u_2^n))\}$ for $\tilde{J}$ restricted to $\mathbb{R} \times \mathcal{S}$ at the level $\gamma(a_1, a_2)$ such that $s_n \to 0$ and $u_i^n - v_i^n \to 0$ for $i = 1, 2$ is implied by [15, theorem 4.1]. It follows that $(u_1^n)^- \to 0$ and $(u_2^n)^- \to 0$. From $\tilde{J}(s, (u_1, u_2)) = \tilde{J}(0, s * (u_1, u_2))$ we deduce that

$$(\partial_s \tilde{J})(s, (u_1, u_2)) = (\partial_s \tilde{J})(0, s * (u_1, u_2))$$

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and, for $u = (u_1, u_2)$, $\phi = (\phi_1, \phi_2)$,

$$(\partial_u \tilde{J})(s, u)[\phi] = (\partial_u \tilde{J})(0, s * u)[s * \phi].$$

As a consequence, $\{(0, s_n * (u_1^n, u_2^n))\}$ is also a Palais–Smale sequence for $\tilde{J}$ restricted to $\mathbb{R} \times \mathcal{S}$ at the level $\gamma(a_1, a_2)$. Thus we may assume that $s_n = 0$. This implies firstly that $\{(u_1^n, u_2^n)\} \subset S$ is a Palais–Smale sequence for J restricted to S at the level $\gamma(a_1, a_2)$, and secondly, using $\partial_s \tilde{J}(0, (u_1^n, u_2^n)) \to 0$, that $Q(u_1^n, u_2^n) \to 0$ holds. □

LEMMA 5.7. *Assume that (H1) and (H2) hold. Then the sequence $\{(u_1^n, u_2^n)\} \subset \mathcal{S}$ obtained in lemma 5.5 is bounded.*

Proof. This property is directly related to the fact that the functional J restricted to the set $Q(u_1, u_2) = 0$ is coercive. Indeed we can write, for any $\varepsilon > 0$,

$$J(u_1, u_2) = \frac{\varepsilon}{2}|\nabla u_1|_2^2 + \frac{\varepsilon}{2}|\nabla u_2|_2^2 + a(\varepsilon)|u_1|_{p_1}^{p_1} + b(\varepsilon)|u_2|_{p_2}^{p_2} + \beta c(\varepsilon) \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} \, \mathrm{d}x + \frac{1-\varepsilon}{2} Q(u_1, u_2),$$

where

$$a(\varepsilon) = \frac{(1-\varepsilon)\mu_1 N}{2p_1}\left(\frac{p_1}{2} - 1\right) - \frac{\mu_1}{p_1}, \qquad b(\varepsilon) = \frac{(1-\varepsilon)\mu_2 N}{2p_2}\left(\frac{p_2}{2} - 1\right) - \frac{\mu_2}{p_2}$$

and

$$c(\varepsilon) = \frac{(1-\varepsilon)N}{2}\left(\frac{r_1 + r_2}{2} - 1\right) - 1.$$

The coefficient $a(\varepsilon)$ is strictly negative but the corresponding term can be easily controlled by $\varepsilon|\nabla u_1|_2^2$ (using the Gagliardo–Nirenberg inequality once more) because $p_1 < 2 + 4/N$. Next observe that $b(\varepsilon) > 0$ holds for $\varepsilon > 0$ small enough, because $p_2 > 2 + 4/N$. Now, concerning the term $\beta c(\varepsilon) \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} \, \mathrm{d}x$ we immediately obtain that $c(\varepsilon) > 0$ for $\varepsilon > 0$ small. Using $Q(u_1^n, u_2^n) \to 0$ yields the boundedness of our Palais–Smale sequence. □

At this point, using lemma 3.2 we can assume that $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2)$ weakly in E with $u_1 \geqslant 0$ and $u_2 \geqslant 0$. In order to get the strong convergence, according to lemmas 3.2 and 3.4, we just need to show that $u_1 \neq 0$ and $u_2 \neq 0$.

LEMMA 5.8. *Assume that (H1) and (H2) hold, and that $\gamma(a_1, a_2) \neq 0$. Then $u_1 \neq 0$ and $u_2 \neq 0$.*

Proof. Suppose by contradiction that at least one of u_1 or u_2 is zero. Then the strong convergence in $L^q(\mathbb{R}^N)$ for $q \in (2, 2^*)$ implies that

$$\beta \int_{\mathbb{R}^N} |u_1^n|^{r_1} |u_2^n|^{r_2} \, \mathrm{d}x \to 0.$$

Thus, since $\{(u_1^n, u_2^n)\}$ satisfies $Q(u_1^n, u_2^n) \to 0$ it follows that

$$\begin{aligned} J(u_1^n, u_2^n) &= \frac{\mu_1}{p_1}\left[\frac{N}{2}\left(\frac{p_1}{2} - 1\right) - 1\right]|u_1^n|_{p_1}^{p_1} + \frac{\mu_2}{p_2}\left[\frac{N}{2}\left(\frac{p_2}{2} - 1\right) - 1\right]|u_2^n|_{p_2}^{p_2} + o(1) \\ &= -D_1|u_1|_{p_1}^{p_1} + D_2|u_2|_{p_2}^{p_2} + o(1), \end{aligned} \tag{5.6}$$

where $D_1 > 0$ and $D_2 > 0$. We now distinguish three cases.

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Case 1 ($u_1 = 0$ and $u_2 = 0$). From (5.6) we obtain that $J(u_1^n, u_2^n) \to 0$. Thus, since we have assumed that $\gamma(a_1, a_2) \neq 0$, this case cannot occur.

Case 2 ($u_1 = 0$ and $u_2 \neq 0$). First note that by lemma 3.4 we have $\lambda_2 < 0$, and hence $u_2^n \to u_2 \in S(a_2)$ strongly in H as a consequence of lemma 3.2. Using $u_1 = 0$ it follows from (5.6) that

$$J(u_1^n, u_2^n) - I(u_2^n) \to 0 \quad \text{and} \quad J(u_1^n, u_2^n) \to D_2|u_2|_{p_2}^{p_2}. \tag{5.7}$$

Since (u_1, u_2) is a solution of system (1.5) we see that $u_2 \gneqq 0$ satisfies

$$-\Delta u - \lambda_2 u = \mu_2 |u|^{p_2 - 2} u.$$

From lemma 3.1 and (5.7) we deduce that $D_2|u_2|_{p_2}^{p_2} = m_{p_2}^{\mu_2}(a_2) > 0$. Therefore, in order to obtain a contradiction it suffices to show that $\gamma(a_1, a_2) < m_{p_2}^{\mu_2}(a_2)$. But this is immediate from lemma 5.4 because $m_{p_1}^{\mu_1}(a_1) < 0$. Thus case 2 is not possible.

Case 3 ($u_1 \neq 0$ and $u_2 = 0$). As in case 2, we can show that $u_n^1 \to u_1 \in S(a_1)$ strongly in H. Now, since $u_2 = 0$ it follows that

$$J(u_1^n, u_2^n) - I(u_1^n) \to 0 \quad \text{and} \quad J(u_1^n, u_2^n) \to -D_1|u_1|_{p_1}^{p_1}.$$

Arguing as in case 2, we identify $-D_1|u_1|_{p_1}^{p_1}$ with the least energy level of

$$-\Delta u - \lambda_1 u = \mu_1 |u|^{p_1 - 2} u, \tag{5.8}$$

namely

$$-D_1|u_1|_{p_1}^{p_1} = m_{p_1}^{\mu_1}(a_1).$$

Therefore, in order to avoid this case happening it suffices to show that $\gamma(a_1, a_2) > m_{p_1}^{\mu_1}(a_1) = I(\underline{u})$. But this is precisely what we can deduce from lemmas 5.1, 5.3 and the definition of B.

Having proved that cases 1, 2 and 3 are all impossible, this concludes the proof of the lemma. □

Proof of theorem 2.2. In view of lemmas 3.2, 5.5, 5.7 and 5.8, in order to establish the theorem it is enough to prove that $\gamma(a_1, a_2) < 0$. We see from lemma 5.4 that this is the case if $m_{p_1}^{\mu_1}(a_1) + m_{p_2}^{\mu_2}(a_2) < 0$. Note also that $u_1 > 0$ and $u_2 > 0$ follow directly from the strong maximum principle because $u_1 \gneqq 0$ and $u_2 \gneqq 0$. □

Proof of corollary 2.3. From lemma 3.1 we know that $m_{p_1}^{\mu_1}(a_1) < 0$ and $m_{p_1}^{\mu_1}(a_1) \to -\infty$ as $a_1 \to \infty$. Also, $m_{p_2}^{\mu_2}(a_2) > 0$ and $m_{p_2}^{\mu_2}(a_2) \to 0$ as $a_2 \to \infty$. Therefore the corollary follows directly from theorem 2.2. □

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