# POINT-CURVE CORRESPONDENCES

## III. CORRESPONDENCES ON A SINGLE SURFACE

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In this paper we consider the theory of point-curve correspondences on a single surface $F$, i.e. correspondences in which to each point of $F$ corresponds an algebraic curve of $F$. The results previously obtained for induced and extended correspondences between two surfaces require some modification here, as we can consider the self-correspondences induced on a curve $C$ of $F$, and the related theory of extended correspondences, which is now complicated by the existence of the identical correspondence on $C$. We also develop a theory of correspondences with (non-zero) valency, and show that for a surface whose Riemann matrix is pure and without complex multiplication (and hence for surfaces of 'general moduli'), all correspondences are valency correspondences, this result being exactly analogous to the well-known theorem for curves. We also consider generalized valency correspondences which are an extension to surfaces of the concept of correspondences of multiple valency on a curve. The analogy between our theory and the known results for curves does break down in one important respect. It is not true that every surface possesses valency correspondences even of the generalized kind, the existence of such correspondences involving restrictions on the intersection group of the surface.

This work depends on the results of (2) as well as the two preceding papers of this series ((3) and (4)), and we shall also need to refer to the results of (5). §11, like the corresponding section of (4), makes use of Severi's work on the self-correspondences on a curve variable in a linear system on a surface.

### 10. Notation

This paper is to be regarded as a continuation of (3) and (4), and the numbering of the paragraphs‡ and references to the earlier papers will be treated accordingly. The notation will be much the same as in these papers, and we shall retain our dummy suffix convention by which summation takes place over every repeated *lower* suffix. *Positive* upper indices attached to a number or a matrix do not mean powers, but the index $-1$ will be used to denote the inverse.

$F$ is an algebraic surface, and the same symbol is used to denote a suitable Riemannian model of it. On $F$ we take bases $\gamma_1, \ldots, \gamma_{2q}$ and $\Gamma_1, \ldots, \Gamma_{2q}$ for the 1- and 3-cycles of $F$ so that

$$(\gamma_i . \Gamma_j) = \delta_{ij},$$

where $\delta_{ij}$ is the Kronecker delta. The $\gamma_i$ are not necessarily a fundamental base, and the $\Gamma$'s may well be fractional. But the 1-cycles are chosen so that we may take a set

‡ §§1–3 are in (3) and §§4–9 in (4).

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$u_1, \ldots, u_q$ of simple integrals of the first kind on $F$ whose period matrix relative to these cycles takes the canonical form

$$\Omega = \begin{pmatrix} \Omega^1 & & & \\ & \Omega^2 & & \\ & & \ddots & \\ & & & \Omega^s \end{pmatrix},$$

where no two of the $\Omega^i$ are related and

$$\Omega^i = \begin{pmatrix} \omega^i & & \\ & \ddots & \\ & & \omega^i \end{pmatrix}$$

consists of $r^i$ submatrices of genus $q^i$ each equal to $\omega^i$.

With each $\Omega^i$ there is naturally associated a set of integrals and a set of cycles (i.e. the $r^iq^i$ integrals whose periods on the $2r^iq^i$ cycles are the elements of the matrix $\Omega^i$). These sets will be called *primary sets* of integrals and cycles respectively. Each primary set of integrals (cycles) is made up of $r^i$ *secondary sets* of integrals (cycles) associated in the same way with the pure submatrices $\omega^i$. With our usual base for the 1-cycles on a proper curve $C^*$ of $F$ (4·1) we can also talk of the corresponding primary and secondary sets on $C^*$. The matrices $\Omega^i$ and $\omega^i$ will be called respectively primary and secondary submatrices of $\Omega$.

Corresponding to any curve $C$ of $F$ we define the $a$-matrix of the curve by the relation

$$\Gamma_i . C \approx a_{ij}\gamma_j \quad (i, j = 1, \ldots, 2q).$$

We shall take a canonical base (in the sense of (5)) $C_1, \ldots, C_\rho$ for the algebraic curves of $F$. This means that the $a$-matrices of the first $t$ curves of the base are linearly independent, and those of the remaining $\rho - t$ are zero, $t$ being the dimension of the *algebraic intersection group* of $F$ introduced in (2). The $\rho \times \rho$ matrices $\mathbf{x}$, $\mathbf{y}$ are defined by the relations

$$x_{kl} = (C_k . C_l), \quad \mathbf{xy} = \mathbf{1}_\rho.$$

As a base for the invariant module $\mathfrak{Q}$ defined in (5) we take the (possibly fractional) cycles $Q_1, \ldots, Q_t$ such that

$$Q_r \approx y_{rk} C_k \quad (r = 1, \ldots, t;\ k = 1, \ldots, \rho).$$

## 11. Induced and extended correspondences

Instead of correspondences between two surfaces $F$ and $G$ such as we considered in (3) and (4), we now consider the special case of correspondences between $F$ and $F'$, $F'$ being a copy of $F$. The cycles of various dimensions of $F'$ will be denoted by the same symbols as the corresponding cycles for $F$ with a prime attached. The numerical characters, including period matrices and cycle-intersection matrices, will be the same for $F'$ as for $F$, so that the prime will not be needed for them, and no confusion should be caused if we continue to use primes to denote the transpose of a matrix. A self-correspondence $T$ on $F$ can therefore be represented by a variety $W$ on $F \times F'$, and instead of (3·2) we shall have the homology

$$W \approx F \times Y' + \xi_{ij} \Gamma_i \times \Gamma'_j + X \times F'. \tag{11·1}$$

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$X$ and $Y$ are algebraic curves of $F$ such that $T(x)$ is (weakly) homologous to $Y$ and $T^{-1}(x)$ (i.e. the locus of points $y$ such that $T(y)$ passes through $x$) is homologous to $X$. If $T(x)$ coincides with $T^{-1}(x)$ for every $x$ the correspondence is symmetrical, and this will require not only that $X \approx Y$ but also that the matrix $\xi$ is skew (cf. (3), p. 138). These conditions are, of course, not sufficient for $T$ to be symmetrical, but are necessary for there to be a symmetrical correspondence in the class of correspondences defined by primals homologous to $W$.

The theory of induced and extended correspondences for a pair of distinct proper curves $C^*$ and $D^*$, developed in §§ 5 and 7, can be taken over directly by replacing $G$ with $F'$. However, something new arises in this case from the possibility that $D^*$ may coincide with $C^*$. In this case we speak of the correspondence induced *on* $C^*$, and also of the extension of a correspondence on $C^*$. As remarked in § 5 the theory of induced correspondences is not affected by letting $C^*$ and $D^*$ coincide. And it is also easy to see, in the same way as for the similar problem in § 7, that every invariant correspondence has, at least, a fractional extension. We thus have the following analogue of Theorem 7·1:

THEOREM 11·1. *There is a base for the singular invariant correspondences on $C^*$ such that each member of the base has an extension to a point-curve correspondence on $F$. These extended correspondences form a base for the singular point-curve correspondences on $F$.*

Theorem 7·2, however, is not to be taken over so quickly. Indeed, a sufficiently general pair of curves $C^*$ and $D^*$ (in the sense of § 7) are not permitted to be coincident. However, Severi has shown (6) that, if $C^*$ is a sufficiently general proper curve of $F$ (belonging to a system which is at least $\infty^2$), every correspondence on $C^*$ is of valency on the vanishing cycles. Hence every correspondence on $C^*$ is homologous to a sum of an invariant correspondence and a multiple of the identical correspondence on $C^*$. Instead of Theorem 7·2 we thus have

THEOREM 11·2. *There exists a base, including the identical correspondence, for the correspondences on a sufficiently general curve $C^*$ of $F$, such that every member of the base, excepting only the identical correspondence, can be extended to a point-curve correspondence on $F$. These extended correspondences, together with a further $2p-2$ correspondences of valency zero, form a base for the point-curve correspondences on $F$.*

It is to be noted that this theorem, in the form given, would certainly not apply to a curve $C^*$ whose genus is equal to the irregularity of $F$, since in this case all correspondences on $C^*$ are invariant. Such curves (which are not 'sufficiently general' in our sense) can only exist on special types of surfaces (cf. (7), p. 142).

## 12. STRICT VALENCY CORRESPONDENCES

A correspondence of valency zero induces on any curve a correspondence of valency zero, and any correspondence which is an extension of a correspondence of valency zero on a proper curve of $F$ is necessarily of valency zero. It is natural then to attempt to define correspondences of non-zero valency by the type of correspondences they induce. We cannot have correspondences of non-zero valency induced on a proper curve

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whose genus is greater than $q$, since such a correspondence is not invariant. It is thus clear that the simplest type of induced correspondences, after correspondences of valency zero, are correspondences of double valency in which every invariant cycle is transformed into a cycle homologous to a non-zero multiple of itself, and every vanishing cycle into a bounding cycle (on the curve). A point-curve correspondence which induces on *every* proper curve‡ a correspondence of this type will be called a strict (non-zero) valency correspondence. That there are actually surfaces on which such correspondences exist will appear shortly.

It is convenient at this stage to introduce some new nomenclature. We shall say that a pure Riemann matrix is *pseudo-simple* if it has (apart from constant factors) only a single principal matrix, and that it is *simple* if this principal matrix is its only non-zero multiplier. If all the primary submatrices of a matrix are pure (i.e. the primary and secondary elements coincide) and are simple (pseudo-simple) we shall say that the matrix is *semi-simple* (*semi-pseudo-simple*). Further, we shall apply terms such as simple, semi-simple, etc., to the varieties whose Riemann matrices are of the corresponding types; in particular, a surface of general moduli is simple in this sense. It has already been shown that if a surface is pseudo-simple then its algebraic intersection group is a free cyclic group ((2), p. 420). We can now prove the following fundamental results for valency correspondences:

THEOREM 12·1. *A surface possesses point-curve correspondences of non-zero valency if, and only if, its algebraic intersection group is cyclic. In particular, valency correspondences exist on any pseudo-simple surface.*

THEOREM 12·2. *Every correspondence on a surface is a valency correspondence if, and only if, the surface is simple.*

For if $\xi$ is the matrix of a correspondence $T$, and $\mathbf{a}^*$ is the $a$-matrix of a proper curve $C^*$, it follows from the formula (5·11) that the induced correspondence on $C^*$ is of double valency if, and only if, $\xi\mathbf{a}^*$ is a multiple of the unit matrix. Hence if $T$ is a valency correspondence $\mathbf{a}^*$ is a multiple of $\xi^{-1}$ for every proper $C^*$, which shows that the algebraic intersection group $\mathfrak{J}_A$ is cyclic. Conversely, if $\mathfrak{J}_A$ is cyclic, $\mathbf{a}^*$ is a multiple of the generator $\mathbf{a}$ of the group. But $\mathbf{a}$ is a non-singular multiplier of $\Omega$, and hence $\mathbf{a}^{-1}$ is an anti-multiplier. There is thus a correspondence whose matrix $\xi$ is a multiple of $\mathbf{a}^{-1}$, and this will induce a double valency correspondence on every $C^*$, since $\xi\mathbf{a}^*$ will be a scalar multiple of the unit matrix. This establishes Theorem 12·1.

If, further, $F$ is simple and $\mathbf{c}$ the (unique) principal matrix of $\Omega$, then the matrix $\xi$ of every correspondence is a multiple of $\mathbf{c}^{-1}$, and at the same time every $a$-matrix is a multiple of $\mathbf{c}$. It follows that $\xi\mathbf{a}^*$ is always a scalar multiple of the unit matrix, and hence that every correspondence is a valency correspondence. Conversely, if every correspondence is a valency correspondence, and if $\mathbf{a}$ is a generator of $\mathfrak{J}_A$ (which we have already shown to be cyclic if a valency correspondence exists at all), then every permissible $\xi$ is a multiple of $\mathbf{a}^{-1}$, so that $\Omega$ can have only a single multiplier, and is therefore simple.

‡ It does not follow, as it does for correspondences of valency zero, that, if $T$ induces a correspondence of this type on any one proper curve, it will do so on all others.

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It is to be noted that Theorem 12·2 is exactly the same as the corresponding result for correspondences between curves. However, there is no analogue for curves to Theorem 12·1, since valency correspondences exist on every curve.

Let us now consider further a valency correspondence $T$. The group $\mathfrak{J}_A$ is necessarily cyclic, and we shall assume further that $F$ is non-overlapped (cf. (5)). This may be a restrictive assumption, but the only known overlapped surfaces with $\mathfrak{J}_A$ cyclic are ruled surfaces, and it does not seem worth pursuing the overlapped case until other examples are found. (In any case no very great modifications need be made in the argument; the chief difference will be the lack of a simple interpretation of a constant corresponding to $g$.) We may thus ((5), § 2) take our canonical base $C_1, \ldots, C_\rho$ so that $(C_1 . C_r) = 0$ $(r = 2, \ldots, \rho)$, and so that $C_1$ is the generator $Q$ (say) of the module $\mathfrak{Q}$. The grade, $x_{11}$, of $Q$ is a positive integer which will be denoted by $g$, so that $Q_1 = \frac{1}{g} Q$. The $a$-matrix, $\mathbf{a}$, of $Q$ will be the generator of $\mathfrak{J}_A$, and the matrix $\boldsymbol{\xi}$ of $T$ will be a multiple, say $\kappa \mathbf{a}^{-1}$, of the inverse of this matrix. The united curve of $T$ will be given by the homology ((5), formula 3·10)

$$U \approx X + Y + \text{trace}(\boldsymbol{\xi}\mathbf{a})\, Q_1 \quad \text{or} \quad U \approx X + Y + (2q\kappa/g)\, Q. \tag{12·1}$$

This result is strongly reminiscent of the formula $\alpha + \beta + 2vp$ for the number of united points of an $(\alpha, \beta)$-correspondence of valency $v$ on a curve of genus $p$, or of the functional form $u \equiv T(x) + T^{-1}(x) + v(2x + k)$, $u$ being the group of united points of the correspondence and $k$ a canonical group (cf. (1), p. 8). It is natural at this point to inquire whether we can attach a numerical value to the valency of a correspondence. This can be done in more than one reasonable manner, but we cannot see that any one of them is clearly superior to the others, and it seems wisest to leave the final choice of the definition to possible subsequent writers on this topic.

Let us assume that of the two alternative (equal and opposite) generators of $\mathfrak{J}_A$ (to which correspond two possible generators of $\mathfrak{Q}$) we have chosen that one which is a principal matrix of $\boldsymbol{\Omega}$. The $a$-matrix of any proper curve $C^*$ is then a positive multiple $\lambda^* \mathbf{a}$ of $\mathbf{a}$. Further, the matrices $\boldsymbol{\xi}$ of the virtual and effective valency correspondences on $F$ form another cyclic group, and we can pick for one of its two possible generators, that matrix $\boldsymbol{\eta}$ which is a positive multiple $\mu \mathbf{a}^{-1}$ of $\mathbf{a}^{-1}$. (It follows from (12·1) that $2q\mu$ is divisible by $g$, but $\mu$ need not necessarily be an integer.) We can, incidentally, confine ourselves to considering effective correspondences because the existence of a virtual correspondence implies the existence of an effective one with the same matrix $\boldsymbol{\xi}$. For, as we have pointed out in (3) (p. 143), the system

$$|K| \equiv |F \times P' + P \times F'|,$$

where $P$ is a 'prime section' of $F$, forms a system of 'prime sections' of $F \times F'$. Thus, if $W$ is any virtual variety on $F \times F'$, there will exist positive integers $r$ such that the system $|W + rK|$ contains an effective variety with the same matrix as $W$.

An effective correspondence whose matrix is $\nu\boldsymbol{\eta}$ $(= \nu\mu\mathbf{a}^{-1})$ will induce on $C^*$ an effective correspondence whose valency on the invariant cycles is $\nu\mu\lambda^*$, which has the same sign as $\nu$. Thus the sign of the valency of the induced correspondence is the same for

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all proper curves, and we take this sign to be that of the valency of $T$. Thus all that remains is to define the magnitude $v$ of the valency of $T$.

The first suggested definition of $v$ is to take it equal to $\nu$. We then have effective correspondences of every (positive or negative) integral valency, and no correspondences of non-integral valency. In this case (12·1) for a correspondence of valency $v$ takes the form

$$U \approx X + Y + vQ^*, \tag{12·2}$$

where $Q^*$ is a (non-fractional) curve defined by

$$Q^* \approx (2q\mu/g)\,Q. \tag{12·3·1}$$

The second method is to take $v = \nu\mu$. Compared with the first definition it has the disadvantage that we can no longer expect correspondences of arbitrary integral valencies, and if $\mu$ is not an integer we will have fractional values of $v$. On the other hand, $v$ is now defined as soon as the algebraic intersection group $\mathfrak{Z}_A$ is known, and we do not require to know (as we did before) the cyclic group of the matrices of the valency correspondences. For a curve $C^*$ whose $a$-matrix is $\lambda^*\mathbf{a}$ the valency on the invariant cycles is simply $v\lambda^*$. Again, we can reduce (12·1) to the form (12·2), but now we have

$$Q^* \approx (2q/g)\,Q. \tag{12·3·2}$$

In this case, unlike the previous one, it is possible that $Q^*$ may be fractional, but $vQ^*$ is always integral. (If $\mu = 1$ this definition coincides with the previous one. It is possible that this may be the case for all surfaces, for some particular surfaces, or for no surfaces. At present we have no grounds for deciding between these three possibilities.)

A third method of procedure‡ is as follows. Among the proper curves of $F$ is at least one, $C^\dagger$, whose $a$-matrix $\lambda^\dagger\mathbf{a}$ is such that $\lambda^\dagger$ is the least of all possible values of $\lambda^*$. We then define $v = \mu\nu\lambda^\dagger$. This is the valency on the invariant cycles of the correspondence induced by $T$ on $C^\dagger$, and is the least valency for any of the correspondences induced on proper curves. In this case we again have the relation (12·2), but now,

$$Q^* \approx (2q/\lambda^\dagger g)\,Q. \tag{12·3·3}$$

As before $U$ is not fractional so that $2vq$ must be divisible by $\lambda^\dagger g$. Unless therefore $\lambda^\dagger g$ is a factor of $2q$ this imposes serious restrictions (possibly not the only ones) on the permissible values of $v$. Of the three definitions suggested this one has perhaps the greatest geometrical significance; nevertheless, the author's bias is in favour of the first method.

## 13. A first generalization of valency correspondences

The idea of valency correspondences can be generalized to correspondences of multiple valency. Let $\mathscr{S}_1, \ldots, \mathscr{S}_n$ be various sets of 1-cycles on $F$. To each $\mathscr{S}_i$ corresponds a set $\mathscr{S}_i^*$ of invariant cycles on the proper curve $C^*$. We shall say that $T$ is a (*multiple*) *valency correspondence* with non-zero valencies on the sets $\mathscr{S}_1, \ldots, \mathscr{S}_n$ if $T$ induces on every proper $C^*$ a correspondence with valencies on the sets $\mathscr{S}_1^*, \ldots, \mathscr{S}_n^*$ (i.e. each cycle $\gamma^*$ of $\mathscr{S}_i^*$ is transformed by the induced correspondence into a cycle homologous, on $C^*$, to a multiple $k_i^*\gamma^*$ of itself; $k_i^*$ is the same for all cycles of the set $\mathscr{S}_i^*$; its value may

‡ Based on a friendly suggestion by Prof. B. Segre.

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alter between one proper curve and another, but it must not be zero for all such curves), and of zero valency on all cycles of $C^*$ having zero intersection, on $C^*$, with all the cycles of $\mathscr{S}_1^*, \ldots, \mathscr{S}_n^*$. If this latter condition is omitted, and possibly some or all of the $k_i^*$ are permitted to vanish, we speak of a *semi-valency correspondence* with non-zero valencies on the sets $\mathscr{S}_i$ for which the corresponding $k_i^*$ do not all vanish. The existence of a valency, or semi-valency, correspondence with valencies on the sets $\mathscr{S}_1, \ldots, \mathscr{S}_n$ naturally implies possible restrictions on these sets (except in the case of a semi-valency correspondence having no sets with non-zero valencies). It is clear that a correspondence with valencies on the sets $\mathscr{S}_1, \ldots, \mathscr{S}_n$ is not necessarily a correspondence with valency on the set $\mathscr{S}_1 + \mathscr{S}_2 + \ldots + \mathscr{S}_n$, and that the latter is the more restrictive property. In this section, except for Theorem 13·2A, we shall consider only *primary systems* of cycles consisting of one or more complete primary sets. The associated valency correspondences (if such exist) will be called *primary valency correspondences.*

Let $\mathscr{S}$ be a primary system of cycles consisting of a finite number of primary sets $\mathscr{P}_1, \ldots, \mathscr{P}_n$. We introduce the idea of the *sections* $\boldsymbol{\Omega}(\mathscr{S})$ and $\mathfrak{J}_A(\mathscr{S})$ of the period matrix $\boldsymbol{\Omega}$, and of the intersection group $\mathfrak{J}_A$ of $F$. The matrix $\boldsymbol{\Omega}(\mathscr{S})$ is obtained from $\boldsymbol{\Omega}$ by striking out the columns corresponding to cycles not in $\mathscr{S}$, and the rows corresponding to the integrals which have no periods over the cycles of $\mathscr{S}$. $\boldsymbol{\Omega}(\mathscr{S})$ is thus itself a Riemann matrix. $\mathfrak{J}_A(\mathscr{S})$ is the additive group made up of the matrices of $\mathfrak{J}_A$ from which have been struck out the rows and columns corresponding to cycles not in $\mathscr{S}$. Using these notions we can now give the following extensions of the results of the last section.

THEOREM 13·1A. *A surface $F$ possesses primary valency correspondences with non-zero valency on a system $\mathscr{S}$ of cycles if, and only if, the section $\mathfrak{J}_A(\mathscr{S})$ of the intersection group of $F$ is cyclic.*

THEOREM 13·1B. *A surface $F$ possesses primary valency correspondences with non-zero valencies on all the primary sets $\mathscr{P}_1, \ldots, \mathscr{P}_n$ of a primary system $\mathscr{S}$ of cycles if, and only if, each of the sections $\mathfrak{J}_A(\mathscr{P}_i)$ is cyclic. In particular such correspondences exist for any system $\mathscr{S}$ such that the Riemann matrix $\boldsymbol{\Omega}(\mathscr{S})$ is semi-pseudo-simple.*

THEOREM 13·2A. *If every correspondence on $F$ is a semi-valency correspondence with a valency on a system $\mathscr{S}$ of cycles, then $\mathscr{S}$ is primary and $\boldsymbol{\Omega}(\mathscr{S})$ is simple, and hence pure. (This implies that $\mathscr{S}$ has no secondary proper subsets.)*

THEOREM 13·2B. *Every correspondence on $F$ is a semi-valency correspondence with valencies on each of the primary sets $\mathscr{P}_1, \ldots, \mathscr{P}_n$ of a primary system $\mathscr{S}$ of cycles if, and only if, $\boldsymbol{\Omega}(\mathscr{S})$ is semi-simple. (In this case none of the primary sets have secondary proper subsets.)*

The proofs of these theorems involve no new points of any interest, and are accordingly omitted. (It is, perhaps, worth noting that we can always suppose a primary system $\mathscr{S}$ to consist of the leading primary sets, since a renumbering of the primary sets alters nothing essential.)

Just as in the last section, we wish to discuss the united curve, and to inquire if there is a feasible method of defining numerical valencies. To discuss the question in the greatest possible generality would contribute nothing of particular interest to the subject, and would merely obscure the relevant points under a welter of detail. We

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shall therefore consider first the simple case of a surface $F$ which has only two primary sets, $\mathscr{P}_1$ and $\mathscr{P}_2$, of cycles, and possesses correspondences of multiple valency with valencies on each set.

Let $\Omega = \begin{pmatrix} \Omega^1 & 0 \\ 0 & \Omega^2 \end{pmatrix}$ be the period matrix of $F$, $\Omega^1$ and $\Omega^2$ being of genera $q_1$ and $q_2$ respectively. By Theorem 13·1B the intersection group will be generated by two matrices

$$\mathbf{a}^{(1)} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{a}^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha_2 \end{pmatrix},$$

where $\alpha_1$ and $\alpha_2$ are respectively principal matrices of $\Omega^1$ and $\Omega^2$. We shall assume for the moment that $\mathfrak{I}_A$ is not generated by a single linear combination of $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$, but will return to consider this case later. We take a canonical base $C_1, C_2, \ldots, C_\rho$ derived from the given base of $\mathfrak{I}_A$. If $C^* \approx \lambda_i^* C_i$ is any proper curve of $F$, then

$$\mathbf{a}^* = \lambda_1^* \mathbf{a}^{(1)} + \lambda_2^* \mathbf{a}^{(2)}.$$

Any double valency correspondence $T$ will have a matrix $\xi$ of the form

$$\xi = \begin{pmatrix} \kappa_1 \alpha_1^{-1} & 0 \\ 0 & \kappa_2 \alpha_2^{-1} \end{pmatrix}, \tag{13·1}$$

and the united curve $U$ will then be given ((5), formula 3·8) by the homology

$$U \approx X + Y + 2\kappa_1 q_1 Q_1 + 2\kappa_2 q_2 Q_2. \tag{13·2}$$

The correspondence $T$ induces on $C^*$ a correspondence of valency $\kappa_1 \lambda_1^*$ on the first primary set of cycles, of valency $\kappa_2 \lambda_2^*$ on the second primary set, and of valency zero on the vanishing cycles. Just as in the last section there will be a least positive number $\mu_1$ such that (for arbitrary integers $\nu$) every matrix of the form

$$\begin{pmatrix} \nu\mu_1 \alpha_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

can be the matrix of an effective valency correspondence with non-zero valency only on the first primary set, and there will be another number $\mu_2$ similarly defined for the second set.

Let us write $$\eta^{(1)} = \begin{pmatrix} \mu_1 \alpha_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \eta^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & \mu_2 \alpha_2^{-1} \end{pmatrix}.$$

Any effective correspondence with valencies on both sets $\mathscr{P}_1$ and $\mathscr{P}_2$ must have a matrix of the form

$$\xi = \nu_1 \eta^{(1)} + \nu_2 \eta^{(2)}. \tag{13·3}$$

It is not by any means to be taken for granted that $\nu_1$ and $\nu_2$ are both integers (although they are certainly rational); all we can assert is that both or neither must be integral. (And again, in the case of a semi-valency correspondence with valency on $\mathscr{P}_1$, for which $\xi = \nu_1 \eta^{(1)} + \eta$, where $\eta$ is of the form $\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{c}_2 \end{pmatrix}$, $\mathbf{c}_2$ being any anti-multiplier of $\Omega_2$, we still have no guarantee that $\nu_1$ is necessarily an integer.)

Suppose, then, that we wish to define the valencies of a correspondence with a matrix $\xi$ of the form (13·3), on the lines of the first definition given in the last section. We

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should then define our valencies $v_1$ and $v_2$ to be equal to $\nu_1$ and $\nu_2$ respectively. For such a correspondence we have

$$U \approx X + Y + v_1 Q_1^* + v_2 Q_2^*, \tag{13·4}$$

where

$$Q_1^* \approx 2q_1\mu_1 Q_1 \quad \text{and} \quad Q_2^* \approx 2q_2\mu_2 Q_2. \tag{13·5}$$

We must, by the definitions of $\eta^{(1)}$ and $\eta^{(2)}$, have correspondences with every integral valency on either or both of the sets $\mathscr{P}_1$ and $\mathscr{P}_2$. We may possibly, however, have semi-valency correspondences, or double-valency correspondences, with non-integral valencies. The possible existence of 'fractional' valencies in this case is the new feature introduced by the multiple valencies, and is the only way in which the primary sets $\mathscr{P}_1$ and $\mathscr{P}_2$ of our example could influence one another. However, it should be noted that this is, so far, merely a possibility; we have not shown that there are cases in which it actually does arise.

Similarly, we could extend either of the other definitions of valency to this case, remembering that, in order to extend the third definition, two basic curves will be required. The essence of the point made above is that unless we use the second definition of valency we may find that there exist double-valency correspondences of valencies $v_1$ and $v_2$, or semi-valency correspondences with valency $v_1$ on $\mathscr{P}_1$, without the number $v_1$ being a possible value for a correspondence with non-zero valency on $\mathscr{P}_1$ only.

Let us now consider the case where $\mathfrak{F}_A$ is cyclic, and generated by a matrix $\mathbf{a} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$. This simply means that we not only have double-valency correspondences on the sets $\mathscr{P}_1$ and $\mathscr{P}_2$, but also correspondences with valency on the set $\mathscr{P}_1 + \mathscr{P}_2$ (i.e. strict valency correspondences). Our canonical base $C_1, \ldots, C_\rho$ corresponding to this base for $\mathfrak{F}_A$ will give rise to only a single generator $Q_1$ for $\mathfrak{Q}$. Any double-valency correspondence will still have a matrix $\xi$ of the form (13·1), but instead of (13·2) the united curve will be given by

$$U \approx X + Y + (2\kappa_1 q_1 + 2\kappa_2 q_2) Q_1. \tag{13·2′}$$

Apart from this, things are formally the same as in the previous case, but we must remember that in (13·4) the curves $Q_1^*$ and $Q_2^*$ are no longer independent. There is, however, one snag in the first definition of the numerical value of the valency. A correspondence of valency $v$ regarded as a double-valency correspondence will have valencies $(v', v'')$. It may be that $v'$ is different from $v''$, and both are different from $v$. Clearly $v' = v''$ only if $\mu_1 = \mu_2$, and in this case it is easily seen that $v \geqslant v'$. This difficulty does not arise with the second or third definition, but in this latter case we shall need only a single curve, and not two, for defining the valency of either a correspondence of valency on $\mathscr{P}_1 + \mathscr{P}_2$ (i.e. a strict valency correspondence) or a correspondence with valencies on $\mathscr{P}_1$ and $\mathscr{P}_2$.

The generalization of our ideas to correspondences with valencies $v_1, \ldots, v_n$ on primary sets $\mathscr{P}_1, \ldots, \mathscr{P}_n$ is now straightforward. With each primary set $\mathscr{P}_i$ there is associated a curve $Q_i^*$ so that the united curve is given by

$$U \approx X + Y + v_i Q_i^* \quad (i = 1, \ldots, n). \tag{13·4′}$$

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If $\mathscr{P}_{i_1}, \mathscr{P}_{i_2}, \ldots, \mathscr{P}_{i_n}$ are such that there is no correspondence with a valency on any two or more of these sets, then the curves $Q^*_{i_1}, \ldots, Q^*_{i_m}$ are algebraically independent. If, however, there is a correspondence with valency on the set $\mathscr{P}_i + \mathscr{P}_j$, the curves $Q^*_i$ and $Q^*_j$ are dependent. And, if there is a correspondence with a valency on a system of more than two sets, there is also a correspondence with a valency on any pair of them. The corresponding $Q^*_i$ are thus all multiples of a single curve.

## 14. Extension to secondary sets

The further extension of our ideas to secondary sets of cycles is complicated; but as primary sets of cycles are effectively independent of one another, at least as far as the mere existence of valency correspondences is concerned, we can save much wearisome detail, without in any way affecting the essentials of the matter, by considering the apparently rather special problem of a surface whose Riemann matrix has no primary proper submatrix. We shall thus assume that

$$\Omega = \begin{pmatrix} \omega & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega \end{pmatrix}, \tag{14·1}$$

there being $n$ pure submatrices each equal to $\omega$.

Let $\mathscr{S}_1$ be any system of secondary sets consisting of at least one such set; there is no loss of generality (since we may always renumber the sets) in supposing that $\mathscr{S}_1$ consists of the first $n_1$ sets. The remaining system of sets we shall call $\mathscr{S}_2$. We can define a section of $\Omega$ and $\mathfrak{J}_A$ just as in the primary case, and we write

$$\Omega = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix},$$

where $$\Omega_1 = \Omega(\mathscr{S}_1), \quad \Omega_2 = \Omega(\mathscr{S}_2).$$

Any element of the intersection group of $F$ is of the form

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{pmatrix}, \tag{14·2}$$

where $$\mathbf{a}_{11} \in \mathfrak{J}_A(\mathscr{S}_1), \quad \mathbf{a}_{22} \in \mathfrak{J}_A(\mathscr{S}_2).$$

We shall say that $\mathscr{S}_1$ is an *isolated system* of secondary sets (in particular an *isolated secondary set* if it consists of only one set) if $\mathbf{a}_{12} = \mathbf{0}$, $\mathbf{a}_{21} = \mathbf{0}$ for every matrix of $\mathfrak{J}_A$ (i.e. $\mathfrak{J}_A \subseteq \mathfrak{J}_A(\mathscr{S}_1) \oplus \mathfrak{J}_A(\mathscr{S}_2)$).

Every matrix $\xi$ has a decomposition into blocks, of the same size as in (14·2), of the form

$$\xi = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}.$$

If we confine ourselves to that set of point-curve correspondences on $F$ for which $\xi_{12} = \mathbf{0}$, $\xi_{21} = \mathbf{0}$, then the isolated set $\mathscr{S}_1$ can be treated just like a primary set. There is then no difficulty in extending to $\mathscr{S}_1$ results similar to those of Theorems 13·1. There is, however, no question of it being possible to find results analogous to those of

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Theorems 13·2, as we can always find a $\xi$ not in the restricted set which cannot give rise to a valency correspondence $\left(\text{e.g. we may take } \xi \text{ of the form } \begin{pmatrix} 0 & \xi_{12} \\ 0 & 0 \end{pmatrix} \text{ with } \xi_{12} \neq 0\right)$. Thus valency correspondences on isolated sets of cycles present no difficulty, and the theory of the last section on the united curves of such correspondences can be extended immediately to this case.

Non-isolated sets of cycles may, however, present difficulties even in the comparatively easy case in which $\omega$ is simple. The most natural example of this, in the case $n = 2$, is a surface $F$ which is the product of an elliptic curve and itself which we considered in (5). As we pointed out there, we may take bases $\gamma_1, \gamma_2, \gamma_3, \gamma_4$; $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$; $C_1, C_2, C_3$ for the 1- and 3-cycles and the algebraic 2-cycles respectively, so that

$$\mathbf{a}^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{a}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathbf{a}^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

It is clear from the form of $\mathbf{a}^{(3)}$ that the two secondary sets $\mathscr{S}_1$, consisting of $\gamma_1$ and $\gamma_2$, and $\mathscr{S}_2$, consisting of $\gamma_3$ and $\gamma_4$, are not isolated.

The most general anti-multiplier, $\xi$, of $\Omega$ is, again on quoting from (5), of the form

$$\xi = \begin{pmatrix} 0 & \lambda_{11} & 0 & \lambda_{12} \\ -\lambda_{11} & 0 & -\lambda_{12} & 0 \\ 0 & \lambda_{21} & 0 & \lambda_{22} \\ -\lambda_{21} & 0 & -\lambda_{22} & 0 \end{pmatrix}.$$

The correspondence $T$ associated with $\xi$ is a correspondence with non-zero valency on the set $\mathscr{S}_1$ if $\xi\mathbf{a}$ is, for all $\mathbf{a}$, of the form $\begin{pmatrix} k\mathbf{1}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ ($k$ being a non-zero constant which may depend on $\mathbf{a}$), and of semi-valency if no restriction is placed on the blocks in the second row. But

$$\xi\mathbf{a}^{(1)} = \begin{pmatrix} -\lambda_{11} & 0 & 0 & 0 \\ 0 & -\lambda_{11} & 0 & 0 \\ -\lambda_{21} & 0 & 0 & 0 \\ 0 & -\lambda_{21} & 0 & 0 \end{pmatrix}, \quad \xi\mathbf{a}^{(2)} = \begin{pmatrix} 0 & 0 & -\lambda_{12} & 0 \\ 0 & 0 & 0 & -\lambda_{12} \\ 0 & 0 & -\lambda_{22} & 0 \\ 0 & 0 & 0 & -\lambda_{22} \end{pmatrix},$$

$$\xi\mathbf{a}^{(3)} = \begin{pmatrix} -\lambda_{12} & 0 & -\lambda_{11} & 0 \\ 0 & -\lambda_{12} & 0 & -\lambda_{11} \\ -\lambda_{22} & 0 & -\lambda_{21} & 0 \\ 0 & -\lambda_{22} & 0 & -\lambda_{21} \end{pmatrix},$$

from which it is clear that not even semi-valency correspondences of non-zero valency can exist on $F$.

Let us return now to the case considered at the beginning of this section. Any matrix $\mathbf{a}$ of $\mathfrak{F}_A$ is of the form

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \dots & \mathbf{a}_{1n} \\ \mathbf{a}_{21} & & & \\ \vdots & \ddots & & \\ \mathbf{a}_{n1} & & \dots & \mathbf{a}_{nn} \end{pmatrix},$$

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where each $\mathbf{a}_{rs}$ is a multiplier of $\boldsymbol{\omega}$. Any matrix $\boldsymbol{\xi}$ is a similar $n \times n$ array of blocks, each block $\boldsymbol{\xi}_{rs}$ being an anti-multiplier of $\boldsymbol{\omega}$. If $C^*$ is any proper curve of $F$ with $a$-matrix $\mathbf{a}^*$, the transforms of the invariant cycles of $C^*$ in the correspondence, $T^*$, induced on $C^*$ by $T$, are given (5·11) by

$$T^*(\gamma_i^*) \approx \epsilon_{ij}^* \gamma_j^*, \quad \text{where} \quad \boldsymbol{\epsilon}^* = -\boldsymbol{\xi}\mathbf{a}^*.$$

If we split $\boldsymbol{\epsilon}^*$ up into blocks in the same way as $\mathbf{a}^*$ and $\boldsymbol{\xi}$, we see that $T$ is of semi-valency on the first secondary set $\mathscr{S}_1$ of cycles if the leading block is a multiple of the unit matrix, and all other blocks in the first row are zero. The conditions are thus

$$\left.\begin{aligned} \boldsymbol{\xi}_{1r}\mathbf{a}_{r1}^* &= k^*\mathbf{1} \quad (r = 1, \ldots, n), \\ \boldsymbol{\xi}_{1r}\mathbf{a}_{rs}^* &= \mathbf{0} \quad (s = 2, \ldots, n). \end{aligned}\right\} \tag{14·1}$$

As this has to be satisfied for all $C^*$ we have *two* such sets of equations. That these equations are not always solvable appears from the previous example. The equations (14·1) impose conditions only on the first rows of blocks in $\boldsymbol{\xi}$. If these are satisfied and all the other rows of $\boldsymbol{\xi}$ are taken equal to zero we have not merely a correspondence of semi-valency, but actually a valency correspondence on the set $\mathscr{S}_1$.

There is one case where the equations (14·1) are certainly solvable. Suppose that $\boldsymbol{\omega}$ is simple and $\mathfrak{J}_A$ is cyclic. Let $\mathfrak{J}_A$ be generated by

$$\mathbf{a} = \begin{pmatrix} \mu_{11}\boldsymbol{\alpha} & \cdots & \mu_{1n}\boldsymbol{\alpha} \\ \vdots & & \\ \mu_{n1}\boldsymbol{\alpha} & & \mu_{nn}\boldsymbol{\alpha} \end{pmatrix},$$

where $\boldsymbol{\alpha}$ is the unique multiplier of $\boldsymbol{\omega}$. We write this equation in the form $\mathbf{a} = \boldsymbol{\mu} \otimes \boldsymbol{\alpha}$, and as $\mathbf{a}$ is non-singular it is easily seen that $\boldsymbol{\mu}$ is too. Then any matrix $\boldsymbol{\xi}$ must be of the form $\boldsymbol{\nu} \otimes \boldsymbol{\alpha}^{-1}$, and $\boldsymbol{\nu}$ may be a multiple of a quite arbitrary matrix of integers. By block multiplication we see that $\boldsymbol{\xi}\mathbf{a} = (\boldsymbol{\nu}\boldsymbol{\mu}) \otimes \mathbf{1}$. $T$ will thus have a single valency on the first $n_s$ secondary sets if $\boldsymbol{\nu}\boldsymbol{\mu}$ is a multiple of $\begin{pmatrix} \mathbf{1}_{n_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. As $\boldsymbol{\mu}$ is non-singular we can always find $\boldsymbol{\nu}$ so that this is the case.

The ideas of this section extend to the general case of several primary sets of secondary sets. The problems have to be broken down into those of the type we have considered by taking the appropriate primary sections of $\boldsymbol{\Omega}$ and $\mathfrak{J}_A$.

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