# ON RENEWAL THEORY, COUNTER PROBLEMS, AND QUASI-POISSON PROCESSES†

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1. *Introduction.* The power and appropriateness of renewal theory as a tool for the solution of general problems concerning counters has been amply demonstrated by Feller (7), who considered a variety of counter problems and reduced them to special renewal processes. The use of what may be called renewal-type arguments had certainly been made by authors other than Feller (e.g. in § 3 of Domb (3)), but it was only in (7) that the simplicity of the renewal approach to counter problems was recognized and systematically applied. More recently, Hammersley (8) was concerned with the generalization of a counter problem previously studied by Domb (2). This problem may be introduced, mathematically, as follows. Let $\{x_i\}$, $\{y_i\}$ be two independent sequences of independent non-negative random variables which are non-zero with probability one (i.e. two independent *renewal processes*). The $\{x_i\}$ are distributed in a negative-exponential distribution with mean $\lambda^{-1}$, and we write $E_\lambda(x)$ for their distribution function and say $\mathscr{X} \equiv \{x_i\}$ *is a Poisson process* to imply this special property of $\mathscr{X}$; the $\{y_i\}$ have a distribution function‡ $B(x)$ with mean $b_1 \leqslant \infty$. Form the partial sums $X_n = \sum_1^n x_i$, and define $n_t$ to be the greatest integer $k$ such that $X_k \leqslant t$, taking $X_0 = 0$ and $n_t = 0$ if $x_1 > t$. Then define the stochastic process

$$\left.\begin{aligned} \phi(t) &= 0 \quad \text{if} \quad \max\{y_1 - t, y_2 + X_1 - t, y_3 + X_2 - t, \ldots, y_{n_t+1} + X_{n_t} - t\} \geqslant 0, \\ &= 1 \quad \text{otherwise.} \end{aligned}\right\} \tag{1·1}$$

Hammersley's counter problem concerns the stochastic process

$$\left.\begin{aligned} N_t &= \sum_{n=1}^{n_t} \phi(x_n) \quad \text{if} \quad n_t > 0, \\ &= 0 \quad \text{otherwise.} \end{aligned}\right\} \tag{1·2}$$

We shall not elaborate the physical interpretation of the $N_t$ process, of which Hammersley (8) has already given an adequate account. Briefly, the $\{X_n\}$ are co-ordinates of the left-hand end-points of closed intervals of length $\{y_n\}$ placed on the real axis. Points $t$ on the axis are either *covered*, if $\phi(t) = 0$, or *free*, if $\phi(t) = 1$; and the number of left-hand closure points of the resulting covered stretches in $(0, t]$ is $N_t$. As a matter of fact our formulation differs somewhat from Hammersley's in that we suppose that a covered stretch always starts at $t = 0$; also Hammersley supposed his intervals to lie on the circumference of a circle instead of the infinite real axis. The

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‡ All distribution functions are to be considered as continuous to the right.

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formulation given here is more convenient for the renewal theory approach and makes no difference to the asymptotic results in which we are interested.

Domb in (2) considered the particular case in which $B(x) = U(x-\tau)$, where $U(\cdot)$ is the Heaviside unit function; thus the $y_i$ assume the constant value $\tau$ with probability one. This particular case was also treated by Feller (7). Hammersley, essentially, considered the case in which $\tau$, the least solution of the equation $B(x) = 1$, is finite; i.e. he merely supposed the $y_i$ to be bounded. Using special methods of considerable complexity, which depend critically on the boundedness condition, Hammersley deduced the following asymptotic results which hold as $t \to +\infty$:

(A) $$\mathscr{E}N_t \sim \lambda t e^{-\lambda b_1},$$

(B) $$\operatorname{var} N_t \sim \lambda t e^{-\lambda b_1}\left\{1 - 2\int_0^\tau [\lambda e^{-\lambda b_1} - h(z)]\,dz\right\},$$

where $$h(z) = \lambda B(z-0)\exp\left\{-\lambda\int_0^z [1-B(y)]\,dy\right\};$$

(C) $N_t$ is asymptotically normal with the asymptotic mean and variance given in (A) and (B).

Let us write $Z_0 = 0$, $Z_1$ for the least $X_n$ such that $\phi(X_n) = 1$, $Z_2$ for the second smallest $X_n$ such that $\phi(X_n) = 1$, and so on. Then it follows that if we write

$$z_n = Z_n - Z_{n-1} \quad (n \geqslant 1),$$

the sequence $\mathscr{Z} = \{z_n\}$ is a renewal process. This result is obvious physically, but the reader with no faith in intuition can derive it swiftly from (1·1) and (1·2). Let us write $F(\cdot)$ for the distribution function of the $\{z_n\}$, and $\mu_1, \mu_2$ for the first and second moments of $F(\cdot)$, which may or may not be infinite.

Our primary object in this note is to show that, apart from certain minor additions to the general theory, which we develop in § 2, the standard machinery of renewal theory leads directly to results (A), (B) and (C) with no recourse to unnatural boundedness conditions. In fact, once we notice that $N_t$ is the number of renewals in the $\mathscr{Z}$ process up to time $t$, (A) is merely the well-known renewal theorem

$$\frac{\mathscr{E}N_t}{t} \sim \frac{1}{\mu_1} \quad (0 \text{ if } \mu_1 = \infty), \tag{1·3}$$

first proved rigorously by Feller (5), and later by many authors including, in particular, Doob (4), who proved (1·3) directly from the strong law of large numbers. Similarly (B) is an expression of the Renewal Theorem 6 of Smith (10),

$$\frac{\operatorname{var} N_t}{t} \sim \frac{\mu_2 - \mu_1^2}{\mu_1^3} = \sigma^2, \quad \text{say.} \tag{1·4}$$

The asymptotic normality result (C) is only a restatement of the central limit theorem for renewal processes, Feller's proof (6) of which is both brief and elegant. Thus Hammersley's results (A), (B) and (C) will be proved once the values of $\mu_1$ and $\mu_2$ are determined. The manner in which renewal theory simplifies this calculation is shown in

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§ 2. In § 3 we briefly indicate the derivation of (A), (B) and (C) for the counter process $\mathscr{Z}$. Apart from the greater generality of the renewal approach, this method also enables one to avoid the rather fussy discussion of differentials, and is noticeably shorter and simpler.

Results (A) and (B) are not stated above in quite so precise a form as the ones obtained by Hammersley (8), who showed that the $\sim$ sign could be replaced by equality. Hammersley was able to do this partly because of his assumption that the intervals were on the circumference of a circle and partly because of the boundedness assumption $\tau < \infty$. In the general case the asymptotic sign is the most that can be hoped for, but an examination of the consequences of the assumption $\tau < \infty$ has led us to an interesting special renewal process which we call a quasi-Poisson process because of its similarity to the Poisson process. The quasi-Poisson process is discussed at some length in § 4, and it is shown that when $\tau < \infty$ we can replace the $\sim$ sign by equality for all sufficiently large $t$.

Lastly we consider two methods of 'censoring' a renewal process to yield a new renewal process. The first method is introduced at the end of § 4 and may be called the guarantee-censor for reasons we shall there explain. General results are obtained by renewal theory, and it is shown that when the renewal process is quasi-Poisson these become particularly simple. The second method is discussed in § 5 and may be called the paralysis-censor because it corresponds to the effect of an automatic self-paralysis mechanism in a counter, of the type described by Hammersley in § 3 of (8).† Under certain conditions, involving particularly the assumption that the renewal process is quasi-Poisson, we show that the properties of the paralysis-process may be very easily obtained; an operational form of solution is offered for the study of the case in which these conditions do not hold, and is applied to a few examples. It transpires that, while apparently (i.e. according to § 3 of (8)) proposing to study the effect of a paralysis-censor on $\mathscr{Z}$, Hammersley has in fact dealt with the effect of a guarantee-censor. Moreover, a formula for the variance of the number of recorded points which Hammersley gives for this guarantee-censor case is erroneous. The correct formulae for the case of the paralysis-censor are given in § 5, particularly (5·4).

We close this introduction by mentioning that Pollaczek (9) has derived, under certain analyticity conditions, some general formulae appropriate to the counter problem for which $\mathscr{X}$ is not necessarily a Poisson process; he has obtained from these general formulae specific solutions for two special cases. Whilst our calculations will not cover the more general circumstances envisaged by Pollaczek, it should be clear that much of the general discussion of this section, at least, does so.‡ Lastly, we mention that an elaborate treatment of a general class of stochastic processes derived from a Poisson process has recently been given by Takács (12), without making use of renewal theory.§

† In the special case when the renewal process is a Poisson process the paralysis and guarantee censors correspond, respectively, to the familiar Type I and Type II counters of (7).

‡ Pollaczek does not give simple formulae for $\mu_1$ and $\sigma^2$, even for the two special cases mentioned. See, however, our footnote to Example 1 in § 4.

§ *Note added in proof.* This remark is now affected by a paper of Takács which appeared in these Proceedings (52 (1956), 488–98) while the present note was at the press.



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2. *Renewal theory.* Let $\mathscr{T} \equiv \{t_i\}$ be a renewal process, $\{T_n\}$ the associated partial sums, $N_t$ the greatest $k$ such that $T_k \leqslant t$, $F(t)$ the distribution function of the $\{t_i\}$, and $\mu_1$, $\mu_2$ the first two moments of $F(t)$, which may be infinite. Write

$$H(t) = \mathscr{E} N_t, \tag{2·1}$$

$$F^*(s) = \int_{0-}^{\infty} e^{-st}\, dF(t), \tag{2·2}$$

$$H^*(s) = \int_{0-}^{\infty} e^{-st}\, dH(t). \tag{2·3}$$

Then it is well known (10) that $H(t)$ satisfies the renewal equation

$$H(t) = F(t) + \int_{0-}^{t} H(t-u)\, dF(u), \tag{2·4}$$

and that if $F_m(t)$ represents the distribution function of $T_m$ then

$$H(t) = \sum_{1}^{\infty} F_m(t). \tag{2·5}$$

The following relations are now easy and well-known consequences of the last four equations:

$$H^*(s) = \frac{F^*(s)}{1 - F^*(s)}; \tag{2·6}$$

$$F^*(s) = \frac{H^*(s)}{1 + H^*(s)}$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} [H^*(s)]^k. \tag{2·7}$$

These relations between transforms are valid provided the real part of $s$ is sufficiently large.

The following lemma represents the necessity part of a theorem, of which the sufficiency part is the following renewal theorem (Smith (10), Theorem 5):

*If $\mu_2 < \infty$ then, as $t \to \infty$,*

$$H(t) = \frac{t}{\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - 1\right) + o(1). \tag{2·8}$$

LEMMA 1. *If $\mu_1 < \infty$ then a necessary condition for the limit*

$$\beta = \lim_{t=\infty} \left\{H(t) - \frac{t}{\mu_1}\right\}$$

*to exist and be finite is that $\mu_2 < \infty$. When this is so, $\mu_2 = 2\mu_1^2(1+\beta)$.*

*Proof.* Suppose the limit $\beta$ exists and is finite. Then by (2·6) and the Abelian theorem for Laplace–Stieltjes transforms (Widder (13), p. 182) we deduce that

$$\lim_{s=0+} \frac{F^*(s)}{1 - F^*(s)} - \frac{1}{\mu_1 s} = \beta. \tag{2·9}$$

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Since $\mu_1 < \infty$, as $s \to 0+$ we have $1 - F^*(s) = \mu_1 s + o(s)$; and if we combine this fact with (2·9) it is fairly easy to deduce that

$$F^*(s) = 1 - \mu_1 s + \mu_1^2(1+\beta)\, s^2 + o(s^2), \tag{2·10}$$

from which it follows that

$$\lim_{s=0+} \frac{F^*(0) - 2F^*(s) + F^*(2s)}{s^2} = 2\mu_1^2(1+\beta). \tag{2·11}$$

But

$$\frac{F^*(0) - 2F^*(s) + F^*(2s)}{s^2} = \int_{0-}^{\infty} \left\{\frac{1 - e^{-sx}}{sx}\right\}^2 x^2 dF(x) \tag{2·12}$$

$$> \frac{1}{4}\int_{0-}^{\epsilon s^{-1}} x^2 dF(x), \tag{2·13}$$

where $\epsilon$ is chosen so that for all positive $y < \epsilon$ we may have $1 - e^{-y} > \frac{1}{2}y$. If we allow $s \to 0+$, (2·11) and (2·13) combine to prove that $\mu_2 < \infty$. We can deduce the fact that $\mu_2 = 2\mu_1^2(1+\beta)$ from the sufficiency theorem (2·8), or we can use the fact that we now know $\mu_2 < \infty$ to apply dominated convergence to the integral on the right of (2·12). If we do this we observe that

$$\lim_{s=0+} \frac{F^*(0) - 2F^*(s) + F^*(2s)}{s^2} = \int_{0-}^{\infty} x^2 dF(x)$$

$$= \mu_2.$$

The lemma now follows from (2·11).

Notice that, as a consequence of our results, if $\mu_2 = \infty$ then the limit $\beta$ either fails to exist or is infinite.

We now introduce a renewal process of a special type. Let $\{u_n\}$, $\{v_n\}$ be two independent renewal processes; the $\{u_n\}$ have a distribution function $G(\cdot)$, and the $\{v_n\}$ have $E_\lambda(\cdot)$, i.e. $\{v_n\}$ is a Poisson process. If $\mathscr{T}$ has the constructional property $t_n = u_n + v_n$ for all $n$, we shall say $\mathscr{T}$ is a $\lambda$-type renewal process. For a $\lambda$-type process $\mathscr{T}$, $F(t)$ is obviously absolutely continuous, with a frequency function $f(t)$, say; and $H(t)$ admits of a renewal density function $h(t) = H'(t)$. Let us define $T_0 = 0$, and consider the function

$$\psi(t) = 0 \quad \text{if} \quad t \leqslant T_{N_t} + u_{N_t} + 1,$$
$$= 1 \quad \text{otherwise.}$$

Thus, as the renewal process $\mathscr{T}$ develops we can imagine $\psi(t)$ assuming alternately the values zero in the closed intervals $[T_n, T_n + u_{n+1}]$ and unity in the intervening open intervals $(T_n + u_{n+1}, T_{n+1})$. It is convenient to say that $t$ is *free* if $\psi(t) = 1$ and *covered* otherwise. Similarly it is convenient to speak of free or covered intervals.

Write $\pi(t) = \mathscr{E}\psi(t)$ for the probability that $t$ is free. Then it is intuitively clear that $h(t) = \lambda\pi(t)$. This may be proved as follows:

$$\pi(t) = \int_{0-}^{t} e^{-\lambda(t-u)} dG(u) + \int_0^t h(v) \left\{\int_{0-}^{t-v} e^{-\lambda(t-v-u)} dG(u)\right\} dv$$

$$= \frac{1}{\lambda} f(t) + \frac{1}{\lambda}\int_0^t h(v) f(t-v)\, dv$$

$$= \frac{1}{\lambda} h(t),$$



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by the integral equation for renewal density functions (Smith (10)) which is analogous to (2·4). We can now prove:

THEOREM 1. *For any $\lambda$-type renewal process $\mathscr{T}$ there exists the limit*

$$\lim_{t=\infty} \pi(t) = \frac{1}{\lambda\mu_1},$$

*which is to be taken as zero if $\mu_1 = \infty$. If this limit is non-zero, so that $\mu_1 < \infty$, and if*

$$\beta = \lim_{t=\infty} \lambda \int_0^t \left\{\pi(u) - \frac{1}{\lambda\mu_1}\right\} du$$

*exists and is finite, then $\mu_2 < \infty$ and $\mu_2 = 2\mu_1^2(1+\beta)$. However, if $\mu_1 < \infty$ and the limit $\beta$ either does not exist or is infinite, then $\mu_2 = \infty$. Furthermore, if $\mu_2 < \infty$ then*

$$\int_0^\infty \left|\pi(u) - \frac{1}{\lambda\mu_1}\right| du < \infty.$$

*Proof.* The fact that $f(x)$ is the convolution of a negative exponential frequency function and $G(x)$:

$$f(x) = \int_{0-}^{x} \lambda\, e^{-\lambda(x-z)}\, dG(z),$$

involves, as an easily deduced consequence, the fact that $f(x) \to 0$ as $x \to \infty$. Also, by Hölder's inequality, for $1 < p < 2$,

$$f(x) \leqslant \left\{\int_{0-}^{x} \lambda^p\, e^{-\lambda p(x-z)}\, dG(z)\right\}^{1/p} \left\{\int_{0-}^{x} 1^{p'}\, dG(z)\right\}^{1/p'},$$

where $p^{-1} + p'^{-1} = 1$. Hence

$$\{f(x)\}^p \leqslant \int_{0-}^{x} \lambda^p\, e^{-\lambda p(x-z)}\, dG(z),$$

from which it is easy to see that $f(x) \in L_p(0, \infty)$. The convergence of $\pi(t)$ to $(\lambda\mu_1)^{-1}$ is now a consequence of known theorems on the convergence of renewal density (Smith (10), (11)).

The part of Theorem 1 which concerns the limit $\beta$ follows directly from Lemma 1 once it is noticed that

$$H(t) = \int_0^t h(u)\, du = \lambda \int_0^t \pi(u)\, du.$$

The result concerning the absolute convergence of the infinite integral is not needed in the present treatment and is merely stated for completeness in the treatment of $\lambda$-type processes. It is a consequence of Corollary 8·1 of Smith (10).

Theorem 1, which is little more than a pulling together of known renewal theorems, has obvious applications to the counter process $\mathscr{Z}$. We develop these in the next section. However, its field of application is wider than the scope of the present paper. It is of great use in dealing with queueing theory, in particular with the busy interval distribution; this application will appear elsewhere.

3. *The counter process.* As the counter process $\mathscr{Z}$ develops, the function $\phi(t)$ behaves exactly like the function $\psi(t)$ in the last section. It assumes the value zero on the closed covered intervals, and unity on the open free intervals. Evidently, each $z_n$ is the sum

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of $u_n$, the length of the $n$th covered interval, and $v_n$, the length of the $n$th free interval. Further, by a well-known property of the $E_\lambda(\cdot)$ distribution function, the $\{v_n\}$ will be distributed in accordance with $E_\lambda(\cdot)$; for the $v_n$ measure the distance from the end of a closed covered interval to the next renewal in a Poisson process. It is thus evident that $\mathscr{Z}$ is a $\lambda$-type renewal process. Theorem 1 shows that once $\pi(t)$ is determined the calculation of $\mu_1$ and $\mu_2$ (and hence $\sigma^2$) is simple.

A particularly easy example is afforded by the counter process $\mathscr{Z}$ for which $B(y) = U(y-\tau)$, where $0<\tau<\infty$. As stated in § 1, this problem has been tackled by Domb, Feller and Hammersley in various ways and with varying degrees of complexity. But it is obvious that for this particular process $\pi(t) = e^{-\lambda\tau}\,U(t-\tau)$, since no free instant can occur in $[0,t]$ and an instant $t>\tau$ is free if and only if no $X_n$ falls in $[t-\tau,t]$. Thus we deduce at once from Theorem 1 that

$$\left.\begin{aligned}\mu_1 &= \lambda^{-1}\,e^{\lambda\tau},\\ \sigma^2 &= \lambda\,e^{-\lambda\tau}\,(1-2\lambda\tau\,e^{-\lambda\tau}).\end{aligned}\right\} \tag{3·1}$$

From (3·1) the results (A), (B) and (C) follow easily via (1·3) and (1·4). Moreover, we see that for this particular process (adopting the notation used for $\mathscr{T}$),

$$H^*(s) = \lambda\,e^{-\lambda\tau}\frac{e^{-s\tau}}{s}, \tag{3·2}$$

and so, by (2·7), we have

$$F^*(s) = \sum_{k=1}^{\infty}(-1)^{k+1}\lambda^k\,e^{-k\lambda\tau}\frac{e^{-ks\tau}}{s^k}. \tag{3·3}$$

It follows from (3·3), by a routine interpretation of $e^{-as}$ as a shift operator and of $s^{-n}$ as the Laplace–Stieltjes transform of $t^nU(t)/n!$, that

$$F(t) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{\lambda^k\,e^{-k\lambda\tau}\,(t-k\tau)^k}{k!}\,U(t-k\tau). \tag{3·4}$$

The right-hand-side of (3·4) is very simply related to the 'tapered exponential function' which arose in the course of Hammersley's investigation (8).

For the case of a quite general $B(y)$ the determination of $\pi(t)$ is hardly any more difficult than it was in the special case we have just discussed. For it is apparent that if $n_t = k$ we have $k$ intervals with left-hand closure points independently and uniformly distributed over $(0,t)$, so that

$$\operatorname{prob}\{\phi(t)=1 \mid n_t=k\} = B(t-0)\left\{\frac{1}{t}\int_0^t B(\theta)\,d\theta\right\}^k,$$

whence

$$\begin{aligned}h(t) = \lambda\pi(t) &= \lambda\sum_{k=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^k}{k!}\operatorname{prob}\{\phi(t)=1 \mid n_t=k\}\\ &= \lambda B(t-0)\exp\left\{-\lambda\int_0^t[1-B(\theta)]\,d\theta\right\}.\end{aligned} \tag{3·5}$$

THEOREM 2. *For the counter-process $\mathscr{Z}$, $\mu_1$ is finite if and only if $[1-B(\cdot)]\in L_1(0,\infty)$. If $\mu_1$ is finite, it is given by the equation*

$$\lambda\mu_1 = e^{\lambda b_1}, \tag{3·6}$$

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*and $\mu_2$ is finite if and only if*

$$\lambda B(t) \exp\left\{-\lambda \int_0^t [1 - B(\theta)]\, d\theta\right\} - \lambda e^{-\lambda b_1} \tag{3·7}$$

*belongs to $L_1(0, \infty)$. If $\mu_2$ is finite, then $\sigma^2$ is given by the expression*

$$\lambda e^{-\lambda b_1}\left\{1 + 2\int_0^\infty [h(t) - \lambda e^{-\lambda b_1}]\, dt\right\}, \tag{3·8}$$

*where $h(t)$ is given in* (3·5).

*Proof.* By Theorem 1 and (3·5) we see that $\mu_1$ is finite if and only if

$$\lim_{t=\infty} B(t-0) \exp\left\{-\lambda \int_0^t [1 - B(\theta)]\, d\theta\right\}$$

is not zero. The fact that $B(\cdot)$ is non-decreasing and bounded by unity shows that $\mu_1 < \infty$ if and only if $[1 - B(\theta)] \in L_1(0, \infty)$. This in turn implies that $B(\infty) = 1$, and if we observe that

$$b_1 = \int_0^\infty \{1 - B(\theta)\}\, d\theta,$$

then (3·6) follows. The rest of Theorem 2 is an immediate consequence of (3·5), Theorem 1, and the simple relation

$$\sigma^2 = \mu_1^{-1}(1 + 2\beta). \tag{3·9}$$

From (3·6) and (3·8), the results (A), (B) and (C) follow as before. The present theorem provides some insight into the way in which results (A), (B) and (C) may fail and is clearly valid when $\tau = \infty$. Notice, however, that if $\tau < \infty$ then $h(t)$ is constant for all $t \geqslant \tau$. (This fact was used by Hammersley.) We shall make use of this remark in the next section.

The argument leading to (3·5) may be compared with that leading to equation (5) in (8).

4. *Quasi-Poisson processes.* We return for this section to the discussion of the quite general renewal process $\mathscr{T}$ of § 2, not necessarily supposing it to be of the $\lambda$-type. In (10) the residual lifetime at $y$ is defined as $T_{N_y+1} - y$ (this is the *forward delay* of (1)) and

$$Y(y;\, x) = \text{prob}\,\{T_{N_y+1} - y \leqslant x\}. \tag{4·1}$$

When $\mathscr{T}$ happens to be a Poisson process, a number of pleasant properties hold. In particular, $h(z)$ is constant for all $z$; and $Y(y;\, x)$ is independent of $y$. In § 3 we saw how to construct arbitrarily many renewal processes for which $h(z)$ is constant for all $z$ exceeding some fixed $\tau < \infty$, and the question naturally presents itself as to whether $Y(y;\, x)$ is independent of $y \geqslant \tau$ for such processes. The next theorem answers this question in the affirmative, and for this reason we call these special renewal processes *quasi-Poisson* processes. We call $\tau$ the *index*.

Note that in our discussion $\mu_1^{-1}$ is to be interpreted as zero if $\mu_1 = \infty$.

THEOREM 3. *A necessary and sufficient condition for $Y(y;\, x)$ to be independent of $y$ for all $y \geqslant \tau$ is that $H(x)$ be linear in $x$ for all $x \geqslant \tau$.*

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*Proof.* Evidently, from (2·1) and (1·3), if $H(x)$ is linear for all large $x$ then it must be of the form $\mu_1^{-1}x+\gamma$, where $\gamma$ is some constant. We observe that

$$\begin{aligned} Y(y;\,x) &= \{F(x+y)-F(y)\}+\sum_{k=1}^{\infty}\int_{0-}^{y}\{F(x+y-z)-F(y-z)\}\,dF_k(z) \\ &= \{F(x+y)-F(y)\}+\int_{0-}^{y}\{F(x+y-z)-F(y-z)\}\,dH(z) \\ &= H(x+y)-H(y)-\int_{y-0}^{x+y}F(x+y-z)\,dH(z) \\ &= \int_{0-}^{x}\{1-F(x-z)\}\,d_zH(y+z). \end{aligned} \tag{4·2}$$

The third step in this chain of equations is deduced from the renewal equation (2·4).

To prove the sufficiency part of the theorem we suppose $y\geqslant\tau$, so that

$$d_zH(y+z)=\mu_1^{-1}dz.$$

Thus (4·2) gives at once $$Y(y;\,x)=\frac{1}{\mu_1}\int_0^x\{1-F(z)\}\,dz, \tag{4·3}$$

and shows that $Y(y;\,x)$ is independent of $y$ for $y\geqslant\tau$. Notice that (4·3) agrees with the known result ((10), Theorem 4)

$$\lim_{y=\infty}Y(y;\,x)=\frac{1}{\mu_1}\int_0^x\{1-F(z)\}\,dz, \tag{4·4}$$

which holds for any renewal process.

To prove the necessity part of the theorem we choose any $y\geqslant\tau$ and deduce from (4·2) and (4·4) that

$$\frac{1}{\mu_1}\int_0^x\{1-F(x-z)\}\,dz=\int_0^x\{1-F(x-z)\}\,d_zH(y+z). \tag{4·5}$$

Let $\alpha$ be any number such that $F(\alpha)<1$; we can clearly do this with $\alpha>0$. Then a straightforward measure-theoretic argument based on (4·5) shows that $H(y+z)$ is absolutely continuous for all $0\leqslant z<\alpha$, and possesses a derivative equal almost everywhere in this range to $\mu_1^{-1}$. The arbitrariness of $y$ completes the proof.

We notice that, if $\mu_1<\infty$, Theorem 3 shows that the limit $\beta$ of Lemma 1 will always exist and be finite, since it will be given by $H(\tau)-\mu_1^{-1}\tau$, and $H(t)$ is finite for all $t$ and any renewal process. Thus $\mu_2<\infty$. In fact rather more is true for a quasi-Poisson process, as the next theorem shows. However, before we prove this theorem we must first show that it is impossible to have $\mu_1=\infty$ for a quasi-Poisson process, unless it be of a degenerate kind.

LEMMA 2. *If $\mathscr{T}$ is quasi-Poisson with index $\tau$ and if $\mu_1=\infty$, then $F(x)=F(0)<1$ for all finite $x$.*

*Proof.* Let us notice first the meaning of the statement that $F(x)=F(0)$ for all finite $x$. We cannot have $F(0)=1$, by definition of a renewal process. Thus the $t_i$ are zero with probability $F(0)$ and infinite with probability $1-F(0)$. Such a quasi-Poisson process we shall call *degenerate*.

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Suppose there exists an $x$ such that $F(x) > F(0)$. Then there exist $a$, $b$ both strictly positive and such that $F(b) - F(a) > 0$. From this fact we see that $H(nb) - H(na) > 0$ for all integers $n$, so that $H(x)$ cannot be constant for all sufficiently large $x$. But if $\mu_1 = \infty$ and the process $\mathscr{T}$ is quasi-Poisson then it follows from Theorem 3 that $H(x)$ is constant for all large $x$. This contradiction proves the lemma.

THEOREM 4. *If $\mathscr{T}$ is a (non-degenerate) quasi-Poisson process with index $\tau$, then there exists a $\delta > 0$ such that the integral $F^*(s)$ of (2·2) is convergent for all $\Re s > -\delta$. Consequently all the moments of $F(x)$ are finite.*

*Proof.* By Lemma 2 we may suppose $\mu_1 < \infty$; and since, by Theorem 3, $H(x)$ is given by $\mu_1^{-1}x + \gamma$ for all $x \geqslant \tau$, we observe that

$$H^*(s) = \int_{0-}^{\tau} e^{-sx}\, dH(x) + \frac{e^{-\tau s}}{\mu_1 s}. \tag{4·6}$$

Hence, from (2·6), $F^*(s)$ can be written as a ratio of integral functions,

$$F^*(s) = \frac{\mu_1 s \int_{0-}^{\tau} e^{-sx}\, dH(x) + e^{-\tau s}}{\mu_1 s + \mu_1 s \int_{0-}^{\tau} e^{-sx}\, dH(x) + e^{-\tau s}}. \tag{4·7}$$

Now it follows from Widder ((13), p. 58) that the singularity, if any, of $F^*(s)$ which is nearest the point $s = 0$ in the complex $s$-plane must be located on the real axis and is, in fact, where the real axis is cut by the axis of convergence of (2·2). It cannot be in the open half-plane $\Re s > 0$; and must be at a zero of the integral function in the denominator of (4·7). This zero is obviously not at $s = 0$ and so must be located at some point $s = -\delta$, where $\delta > 0$. This proves the theorem.

A further pleasant feature of the Poisson process, in addition to those already referred to, is that the cumulants of the distribution of $N_t$ are directly proportional to $t$. An analogous result holds for the quasi-Poisson process, although it is not so simple. In the following theorem we consider only the mean and variance of $N_t$:

THEOREM 5. *If $\mathscr{T}$ is a (non-degenerate) quasi-Poisson process with index $\tau$, then for all $t \geqslant \tau$,*

$$\mathscr{E}N_t = \frac{t}{\mu_1} + \beta;$$

*while for all $t \geqslant 2\tau$,*

$$\operatorname{var} N_t = \sigma^2 t - \left(\frac{2\mu_3}{3\mu_1^3} - \frac{5\mu_2^2}{4\mu_1^4} + \frac{\mu_2}{2\mu_1^2}\right),$$

*where $\mu_3$ is the third moment (necessarily finite) of $F(\cdot)$.*

*Proof.* We know already from Theorem 3 that for all $t \geqslant \tau$, $H(t) = \mu_1^{-1}t + \gamma$, and, since all the moments of $F(\cdot)$ are finite, we can now identify the constant $\gamma$ with the limit $\beta$ of Lemma 1. Next we remark that by equation (7·6) of (10),

$$\mathscr{E}\{N_t\}^2 = H(t) + 2\int_{0-}^{t} H(t-z)\, dH(z). \tag{4·8}$$

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It is a simple matter to verify that the right-hand side of (4·8) is a quadratic function of $t$ for all $t \geqslant 2\tau$. To determine the appropriate coefficients we observe that the Laplace transform of (4·8) is

$$\int_0^\infty e^{-st}\mathscr{E}\{N_t\}^2\,dt = \frac{F^*(s)}{s\{1-F^*(s)\}} + \frac{2\{F^*(s)\}^2}{s\{1-F^*(s)\}^2}$$
$$= \frac{2}{\mu_1^2 s^3} + \left(\frac{2\mu_2}{\mu_1^3} - \frac{3}{\mu_1}\right)\frac{1}{s^2} - \left(\frac{2\mu_3}{3\mu_1^3} - \frac{3\mu_2^2}{2\mu_1^4} + \frac{3\mu_2}{2\mu_1^2} - 1\right)\frac{1}{s} + o(1), \qquad (4\cdot9)$$

where $o(1) \to 0$ as $s \to 0+$. Thus, knowing that, for large $t$, $\mathscr{E}\{N_t\}^2$ is a quadratic function of $t$, we may infer from (4·9) that for $t > 2\tau$,

$$\mathscr{E}\{N_t\}^2 = \frac{t^2}{\mu_1^2} + \left(\frac{2\mu_2}{\mu_1^3} - \frac{3}{\mu_1}\right)t - \left(\frac{2\mu_3}{3\mu_1^3} - \frac{3\mu_2^2}{2\mu_1^4} + \frac{3\mu_2}{2\mu_1^2} - 1\right). \qquad (4\cdot10)$$

If we combine (4·10) with the result we have already obtained for $H(t) = \mathscr{E}N_t$, the variance of $N_t$ turns out to be as stated in the theorem.

We complete our present discussion of quasi-Poisson processes with the following:

THEOREM 6. *If $\mathscr{T}$ is a quasi-Poisson process with zero index, then there exists a $\lambda > 0$ and a $p$, $0 \leqslant p < 1$, such that for $x \geqslant 0$,*

$$F(x) = (1-p)(1-e^{-\lambda x}) + p.$$

*Proof.* Suppose $H(x) = \mu_1^{-1}x + \gamma$ for all $x \geqslant 0$. Clearly $\gamma \geqslant 0$, since $H(x)$ must be non-negative. Then

$$H^*(s) = \frac{1}{\mu_1 s} + \gamma,$$

and so, by (2·6) and some elementary manipulation,

$$F^*(s) = \frac{(1+\gamma)^{-1}}{1+(1+\gamma)\mu_1 s} + \frac{\gamma}{1+\gamma}.$$

Inverting this Laplace–Stieltjes transform yields

$$F(x) = \frac{1}{(1+\gamma)}\left\{1 - \exp\left(-\frac{x}{\mu_1(1+\gamma)}\right)\right\} + \frac{\gamma}{1+\gamma},$$

and this completes our proof.

From any renewal process $\mathscr{T}$ we may derive a new process ${}^g\mathscr{T}$ as follows. Choose a constant $\varpi > 0$. Define ${}^gT_0 = 0$, and ${}^gT_1$ as the least $T_n$ such that $T_n - T_{n-1} > \varpi$. Then define ${}^gT_2$ as the second smallest $T_n$ such that $T_n - T_{n-1} > \varpi$, and so on. It is evident that if ${}^gt_n = {}^gT_n - {}^gT_{n-1}$, then ${}^g\mathscr{T} \equiv \{{}^gt_n\}$ is a new renewal process (we shall identify all functions, etc., associated with ${}^g\mathscr{T}$ by the prefix $g$). The process ${}^g\mathscr{T}$ would arise if, in studying a physical realization of $\mathscr{T}$ we refuse to count renewals if they are within a distance (on the time-scale) $\varpi$ of the preceding renewal (whether this was counted or not). More vividly, ${}^g\mathscr{T}$ would arise if the renewals carried a 'maker's guarantee' to replace without charge the renewal if it needs replacement in less that $\varpi$ time units from its moment of installation. Then the renewal instants ${}^gT_n$ of ${}^g\mathscr{T}$ represent only those renewals in $\mathscr{T}$ which cost anything, and ${}^gN_t$ will be the number of renewals actually paid for in the period $(0, t]$.

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The simplest approach to an understanding of ${}^{g}\mathscr{T}$ appears to lie in the calculation of ${}^{g}H(t)$, and this is rendered straightforward by the following observation. At most only one ${}^{g}T_n$ can fall in an interval $(t, t+h]$ if $h \leqslant \varpi$ and $t$ has any non-negative value. Thus the probability that one ${}^{g}T_n$ does fall in $(t, t+h]$ can be equated to the expectation expression ${}^{g}H(t+h) - {}^{g}H(t)$. Let us take $t \geqslant \varpi$. Then

$$\begin{aligned}
{}^{g}H(t+h) - {}^{g}H(t) &= F(t+h) - F(t) + \sum_{k=1}^{\infty} \int_{0-}^{t-\varpi} \{F(t+h-z) - F(t-z)\}\, dF_k(z) \\
&\qquad + \sum_{k=1}^{\infty} \int_{t-\varpi-0}^{t+h-\varpi} \{F(t+h-z) - F(\varpi)\}\, dF_k(z) \\
&= F(t+h) - F(t) + \int_{0-}^{t-\varpi} \{F(t+h-z) - F(t-z)\}\, dH(z) \\
&\qquad + \int_{t-\varpi-0}^{t+h-\varpi} \{F(t+h-z) - F(\varpi)\}\, dH(z) \\
&= F(t+h) + \int_{0-}^{t+h-\varpi} F(t+h-z)\, dH(z) - F(\varpi)\, H(t+h-\varpi) \\
&\qquad - F(t) - \int_{0-}^{t-\varpi} F(t-z)\, dH(z) + F(\varpi)\, H(t-\varpi). \qquad (4{\cdot}11)
\end{aligned}$$

It will be noticed that the right-hand side of (4·11) is of the form $\Psi(t+h) - \Psi(t)$, from which we deduce immediately that (4·11) is valid for all $h > 0$. If we use the fact that ${}^{g}H(\varpi) = 0$ we can now deduce from (4·11) that for all $t \geqslant \varpi$,

$$\begin{aligned}
{}^{g}H(t) &= F(t) + \int_{0-}^{t-\varpi} F(t-z)\, dH(z) - F(\varpi)\, H(t-\varpi) - F(\varpi) - F(\varpi)\, H(0) + F(\varpi)\, H(0) \\
&= H(t) - \int_{t-\varpi-0}^{t} F(t-z)\, dH(z) - F(\varpi) - F(\varpi)\, H(t-\varpi). \qquad (4{\cdot}12)
\end{aligned}$$

THEOREM 7. *In the renewal process ${}^{g}\mathscr{T}$ derived from $\mathscr{T}$, ${}^{g}\mu_1$ is finite if and only if $\mu_1$ is finite and $F(\varpi) < 1$. If ${}^{g}\mu_1$ is finite then it is given by*

$$ {}^{g}\mu_1 = \frac{\mu_1}{1 - F(\varpi)}, \qquad (4{\cdot}13)$$

*and ${}^{g}\mu_2$ is finite if and only if $\mu_2$ is finite. If ${}^{g}\mu_2$ is finite, we have*

$$ {}^{g}\sigma^2 = \left(\frac{\mu_1}{{}^{g}\mu_1}\right)^2 \sigma^2 - \frac{1}{{}^{g}\mu_1}\left(1 - \frac{\mu_1}{{}^{g}\mu_1}\right) + \frac{2}{\mu_1\, {}^{g}\mu_1} \int_0^{\varpi} z\, dF(z). \qquad (4{\cdot}14)$$

*Proof.* We will use the consequence of Theorem 1 of Smith (10), that

$$ \lim_{t=\infty} \int_{t-\varpi-0}^{t} F(t-z)\, dH(z) = \frac{1}{\mu_1} \int_0^{\varpi} F(z)\, dz, \qquad (4{\cdot}15)$$

where the right-hand side of (4·15) is to be taken as zero if $\mu_1 = \infty$.

Thus, from (4·12) and (1·3),

$$\begin{aligned}
\frac{1}{{}^{g}\mu_1} &= \lim_{t=\infty} \frac{{}^{g}H(t)}{t} = \lim_{t=\infty} \frac{H(t) - F(\varpi)\, H(t-\varpi)}{t} \\
&= \frac{1 - F(\varpi)}{\mu_1}. \qquad (4{\cdot}16)
\end{aligned}$$

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Equation (4·16) proves that part of our theorem which refers to ${}^{\varrho}\mu_1$ and (4·13). If we now suppose ${}^{\varrho}\mu_1 < \infty$, we observe next that by (4·12) and for all $t \geqslant \tau$,

$$ {}^{\varrho}H(t) - \frac{t}{{}^{\varrho}\mu_1} = \left\{H(t) - \frac{t}{\mu_1}\right\} - F(\varpi)\left\{H(t-\varpi) - \frac{t-\varpi}{\mu_1}\right\} + \frac{\varpi F(\varpi)}{\mu_1} - F(\varpi) - \int_{t-\varpi-0}^{t} F(t-z)\,dH(z). \quad (4\cdot17) $$

From (4·17) we deduce, via Lemma 1 and (4·15), that

$$ {}^{\varrho}\beta = \{1 - F(\varpi)\}\beta + \frac{\varpi F(\varpi)}{\mu_1} - F(\varpi) - \frac{1}{\mu_1}\int_0^{\varpi} F(z)\,dz; \quad (4\cdot18) $$

so that ${}^{\varrho}\beta$ is finite if and only if $\beta$ is finite. Hence ${}^{\varrho}\mu_2$ is finite if and only if $\mu_2$ is finite. Also, since ${}^{\varrho}\sigma^2 = {}^{\varrho}\mu_1^{-1}(1 + 2{}^{\varrho}\beta)$, and since

$$ \frac{1}{\mu_1}\int_0^{\varpi} F(z)\,dz = \frac{\varpi F(\varpi)}{\mu_1} - \frac{1}{\mu_1}\int_0^{\varpi} z\,dF(z), $$

the formula (4·14) is derivable from (4·18) by elementary manipulations.

*Corollary* 7. *If $\mathscr{T}$ is quasi-Poisson with index $\tau$, then ${}^{\varrho}\mathscr{T}$ is quasi-Poisson with index $\tau + \varpi$.*

*Proof.* If $H(t)$ is linear in $t$ for all $t \geqslant \tau$, say $H(t) = \mu_1^{-1}t + \gamma$, then we have from (4·12) that for all $t \geqslant \tau + \varpi$,

$$ {}^{\varrho}H(t) = \mu_1^{-1}t + \gamma - \mu_1^{-1}\int_0^{\varpi} F(z)\,dz - F(\varpi) - F(\varpi)\{\mu_1^{-1}t - \mu_1^{-1}\varpi + \gamma\}; $$

i.e. ${}^{\varrho}H(t)$ is linear. This proves the Corollary.

It should now be clear that, if $\mu_2 < \infty$, results (A), (B) and (C) will hold for the ${}^{\varrho}\mathscr{T}$ process, and that (A) and (B) will be replaceable by the exact results of Theorem 5 if $\mathscr{T}$ is quasi-Poisson.

We close this section by giving two examples.

*Example* 1. If $\mathscr{T}$ is Poisson with a distribution function $F(\cdot) = E_\lambda(\cdot)$, then we find that $\mu_1 = \lambda^{-1}$, $\sigma^2 = \lambda$, $F(\varpi) = 1 - e^{-\lambda\varpi}$ and Theorem 7 yields the results:

$$ {}^{\varrho}\mu_1 = \lambda^{-1}e^{\lambda\varpi}, $$
$$ {}^{\varrho}\sigma^2 = \lambda e^{-\lambda\varpi}(1 - 2\lambda\varpi e^{-\lambda\varpi}), $$

which are identical with the equations (3·1). Reflexion will show that when the underlying $\mathscr{T}$ process is Poisson the process ${}^{\varrho}\mathscr{T}$ is identical to the special counter process $\mathscr{Z}$ for which $B(y) = U(y - \varpi)$. Thus the present example represents another method of considering the example considered in § 3.†

*Example* 2. Let us next suppose $\mathscr{T} = \mathscr{Z}$, where $\mathscr{Z}$ is the counter process of § 3 with $B(y) = U(y - \tau)$. It should be obvious that if $\varpi \leqslant \tau$ then ${}^{\varrho}\mathscr{Z} \equiv \mathscr{Z}$, and so we may imagine $\varpi > \tau$. For $\mathscr{Z}$ we have already the appropriate values of $\mu_1$, $\sigma^2$ and $F(t)$ in (3·1)

† In fact, if we take $\mathscr{Z} \equiv \mathscr{T}$, with general $F(\cdot)$, then $\mathscr{Z} \equiv {}^{\varrho}\mathscr{Z}$ is the more general counter process envisaged by Pollaczek, for the case $B(y) = U(y - \varpi)$. Theorem 7 thus provides $\mu_1$ and $\sigma^2$ for a useful part of Pollaczek's wider class of counter processes.

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and (3·4). Thus the calculation of ${}^g\mu_1$ and ${}^g\sigma^2$ follows easily from (4·13) and (4·14) by elementary manipulations. However, the calculation of the integral

$$\int_0^\varpi z\,dF(z), \tag{4·19}$$

which is required by (4·14), can be simplified by the use of transforms in the following way. By the convolution property of Laplace–Stieltjes transforms, the Laplace–Stieltjes transform of

$$\int_0^t (t-z)\,dF(z) \tag{4·20}$$

is simply $s^{-1}F^*(s)$, which by (3·3) is

$$\sum_{k=1}^{\infty} (-1)^{k+1}\lambda^k e^{-k\lambda t}\left(\frac{e^{-kst}}{s^{k+1}}\right). \tag{4·21}$$

But we recognize (4·21) as the transform of

$$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{\lambda^k e^{-k\lambda\tau}}{(k+1)!}(t-k\tau)^{k+1}\,U(t-k\tau) = -\lambda^{-1}e^{\lambda\tau}F(t+\tau)+t. \tag{4·22}$$

From the equivalence of (4·20) and (4·22) we therefore deduce that

$$\int_0^\varpi z\,dF(z) = \lambda^{-1}e^{\lambda\tau}F(\varpi+\tau)-\varpi[1-F(\varpi)]. \tag{4·23}$$

Elementary computation based on (4·23) then gives

$$\left.\begin{aligned} {}^g\mu_1 &= \lambda^{-1}e^{\lambda\tau}\{1-F(\varpi)\},\\ {}^g\sigma^2 &= \lambda\,e^{-\lambda\tau}\{1-F(\varpi)\}\left\{1-2F(\varpi)+2F(\varpi+\tau)-\frac{2(\varpi+\tau)}{\mu_1}[1-F(\varpi)]\right\}. \end{aligned}\right\} \tag{4·24}$$

Let us say that we have derived ${}^g\mathscr{T}$ from $\mathscr{T}$ by application of a guarantee-censor. Then Hammersley in § 4 of (8) is actually studying the effect of a guarantee-censor on the counter process $\mathscr{Z}$, and not, as he apparently supposes, the effect of paralysis. The justification for this contention lies in the critical four lines of text following equation (47) of (8). It is therefore to be expected that our (4·24) will agree with Hammersley's (59) and (60). Unfortunately, while they agree with respect to ${}^g\mu_1$, they indicate quite different values for ${}^g\sigma^2$. A simple test reveals that Hammersley's value for ${}^g\sigma^2$ must be incorrect. For if we put $\tau = 0$, then $\mathscr{Z} = \mathscr{X}$, the basic Poisson process (since all the 'intervals' involved in the construction of the counter process $\mathscr{Z}$ have shrunk to points). Thus we reduce our problem to the easier one considered in Example 1 above, for which ${}^g\sigma^2$ is simply $\lambda e^{-\lambda\varpi}(1-2\lambda\varpi e^{-\lambda\varpi}) = {}^g\sigma_0^2$, say. If in (4·24) we put $\tau = 0$ and notice that for this case $1-F(\varpi) = e^{-\lambda\varpi}$, then it is easy to see that ${}^g\sigma^2 = {}^g\sigma_0^2$, correctly. Hammersley's (60) fails to pass this test, it gives a value for ${}^g\sigma^2$ which is quite different from ${}^g\sigma_0^2$ and we must therefore conclude that his formula is incorrect.†

† In comparing the results of the present work with those of Hammersley (8), it is useful to observe that $e^*(-a, b) = 1-F(\varpi)$, $e^*(-a, b+1) = 1-F(\varpi+\tau)$.

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5. *Paralysis.*† Let $\mathscr{T}$ be the renewal process of § 2, not necessarily quasi-Poisson, and let $\mathscr{W} \equiv \{w_i\}$ be an independent renewal process with associated distribution function $C(\cdot)$ and moments $c_1, c_2, \ldots$. Let us write ${}^pT_0 = 0$, ${}^pT_1$ for the least $T_n$ which exceeds $w_1$, ${}^pT_2$ for the least $T_n$ which exceeds ${}^pT_1 + w_2$, and, in a general way, ${}^pT_n$ for the least $T_n$ which exceeds ${}^pT_{n-1} + w_n$. Then it follows that if we write ${}^pt_1 = {}^pT_1$, ${}^pt_n = {}^pT_n - {}^pT_{n-1}$ $(n > 1)$, the sequence $\{{}^pt_n\}$ is a renewal process ${}^p\mathscr{T}$. We shall employ the notations ${}^pF(\cdot)$, ${}^p\mu_1$, etc. with regard to ${}^p\mathscr{T}$, and say ${}^p\mathscr{T}$ is derived from $\mathscr{T}$ by a paralysis-censor.

When $\mathscr{T}$ is quasi-Poisson, ${}^p\mathscr{T}$ can be particularly easy to study.

THEOREM 8. *If $\mathscr{T}$ is quasi-Poisson with index $\tau$, and if $C(\tau) = 0$, then*

$$ {}^pF^*(s) = C^*(s)\left\{\frac{1 - F^*(s)}{\mu_1 s}\right\}, \tag{5·1} $$

*and so, for $n = 1, 2, \ldots$,*

$$ {}^p\mu_n = \sum_0^\infty \binom{n}{r} \frac{c_{n-r}\mu_{r+1}}{(r+1)\mu_1}, \tag{5·2} $$

*where the right-hand side of* (5·2) *may be infinite.*

*Proof.* Let $\kappa = N_{w_1} + 1$. Then it is evident that ${}^pt_1 = T_\kappa = (T_\kappa - w_1) + w_1$. But, with probability one, $w_1 \geqslant \tau$. Thus $\xi = T_\kappa - w_1$, the *residual lifetime* in the $\mathscr{T}$ process at $w_1$, has the distribution function

$$ F_{(1)}(x) = \int_0^x \frac{1 - F(z)}{\mu_1}\, dz, $$

which is independent of $w_1$. Hence ${}^pt_1$ may be considered the sum of two independent random variables $\xi$ and $w_1$, with distribution functions $F_{(1)}(\cdot)$ and $C(\cdot)$ respectively. Equation (5·1) then follows from the two equations

$$ \left.\begin{aligned} F^*_{(1)}(s) &= \int_{0-}^\infty e^{-sx}\, dF_{(1)}(x) = \frac{1 - F^*(s)}{\mu_1 s}, \\ {}^pF^*(s) &= C^*(s)\, F^*_{(1)}(s). \end{aligned}\right\} \tag{5·3} $$

From (5·3) we can find the moments of $F_{(1)}(\cdot)$ by expansion as a Taylor series in $s$. We find that the $r$th moment of $F_{(1)}(\cdot)$ is $\mu_{r+1}/(r+1)\mu_1$. Hence

$$ \begin{aligned} \mathscr{E}(\xi + w_1)^n &= \sum_0^n \binom{n}{r} (\mathscr{E}\xi^r)\,(\mathscr{E}w_1^{n-r}) \\ &= \sum_0^n \binom{n}{r} \frac{\mu_{r+1}}{(r+1)\mu_1} c_{n-r}, \end{aligned} $$

as was to be proved.

As an example of the application of Theorem 8, let us take $\mathscr{T} = \mathscr{Z}$, with

$$ B(y) = U(y - \tau) \quad \text{and} \quad C(w) = U(w - \varpi). $$

Then the conditions of Theorem 8 are satisfied, provided $\varpi > \tau$. In fact ${}^p\mathscr{Z}$ is the counter process with paralysis appropriate to the blood-cell counter problem

† In this section we consider the effect of the censoring procedure discussed by Hammersley in § 3 of (8).

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considered in (8). Calculations based on the values of $\mu_1$, $\mu_2$ and $F^*(s)$ which are given (or implied) by equations (3·1) and (3·3) then yield

$$ {}^pF^*(s) = \sum_0^\infty (-1)^k (\lambda e^{-\lambda\tau})^{k+1} \frac{e^{-(k\tau+\varpi)s}}{s^{k+1}}, $$

whence
$$ {}^pF(x) = \sum_0^\infty (-1)^k (\lambda e^{-\lambda\tau})^{k+1} \frac{(x-k\tau-\varpi)^{k+1}}{(k+1)!} U(t-k\tau-\varpi). $$

We also find that

$$ \left.\begin{aligned} {}^p\mu_1 &= (\varpi-\tau)+\lambda^{-1}e^{\lambda\tau}, \\ {}^p\sigma^2 &= \frac{\lambda e^{-\lambda\tau}(1-2\lambda\tau e^{-\lambda\tau})}{\{1+\lambda(\varpi-\tau)e^{-\lambda\tau}\}^3}. \end{aligned}\right\} \tag{5·4} $$

From (5·4) the results (A), (B) and (C) follow at once. However, ${}^p\mathscr{Z}$ is not a quasi-Poisson process, so that (A) and (B) are not exact for all large $t$.

When the special conditions of Theorem 8 fail to hold the calculation of ${}^pF(\cdot)$, ${}^p\mu_1$ and ${}^p\sigma^2$ becomes much more complicated. We conclude this note with the development of some operational formulae which may be used in these more difficult circumstances. To simplify our calculations we shall assume all distribution functions to be absolutely continuous. (We shall, however, be quite prepared to apply our results to cases where they are not!) If $C(\cdot) = E_\lambda(\cdot)$ then we have

$$ {}^pf(x) = \int_0^x \lambda e^{-\lambda z}\left\{f(x)+\int_0^z f(x-z_1)\,h(z_1)\,dz_1\right\}dz, \tag{5·5} $$

which simplifies after an integration by parts to the equation

$$ {}^pf(x) = f(x) - e^{-\lambda x}h(x) + \int_0^x f(x-z)\,[e^{-\lambda z}h(z)]\,dz. \tag{5·6} $$

(We have also made use of the renewal integral equation connecting $f(x)$ and $h(x)$.) If we write

$$ f^*(s) = \int_0^\infty e^{-sx} f(x)\,dx, $$

$$ h^*(s) = \int_0^\infty e^{-sx} h(x)\,dx, $$

for Laplace transforms, then (5·6) gives quickly

$$ \begin{aligned} {}^pf^*(s) &= f^*(s) - h^*(s+\lambda) + f^*(s)\,h^*(s+\lambda) \\ &= \frac{h^*(s)-h^*(s+\lambda)}{1+h^*(s)}, \end{aligned} \tag{5·7} $$

since $f^*(s) = h^*(s)/[1+h^*(s)]$.

If $c(x) = C'(x)$ were quite general, (5·5) would be replaced by

$$ f(x) = \int_0^x c(z)\left\{f(x)+\int_0^z f(x-z_1)\,h(z_1)\,dz_1\right\}dz. \tag{5·8} $$

Let us suppose that we can represent $c(x)$ in the form

$$ c(x) = \int_E \lambda e^{-\lambda x}\,d\phi(\lambda), \tag{5·9} $$

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where $E$ is some set in the complex open half-plane $\Re\lambda > 0$ and $\phi(\cdot)$ is some measure defined on $E$ for which the integral (5·9) has a meaning. We shall suppose also that the integral (5·9) is absolutely convergent at $x = 0$, to justify the changing of the order of certain multiple integrations. Thus we have from (5·9) that

$$1 = \int_E d\phi(\lambda), \tag{5·10}$$

and we may infer from (5·7) and the linearity of (5·8) in $c(x)$ that in the general case

$$^p f^*(s) = \frac{h^*(s) - \bar{h}^*(s)}{1 + h^*(s)}, \tag{5·11}$$

where

$$^p \bar{h}^*(s) = \int_E h^*(s+\lambda)\, d\phi(\lambda). \tag{5·12}$$

We consider some illustrative examples of the use of (5·11).

*Example* 1. If

$$c^*(s) = \int_0^\infty e^{-sx} c(x)\, dx,$$

then under certain conditions, for some $\gamma > 0$,

$$c(x) = \frac{1}{2\pi i} \int_{-\gamma - i\infty}^{-\gamma + i\infty} e^{xs} c^*(s)\, ds. \tag{5·13}$$

(This inversion will be valid, in particular, if $c^*(s)$ is analytic in $\Re s > -\delta$ for some $\delta > 0$ and if $c^*(s)$ is small enough at $\infty$ for the integral to be absolutely convergent.) If we put $s = -\lambda$ in (5·13) we deduce a representation (5·9) for $c(x)$ in which $E$ is the line joining $\gamma - i\infty$ to $\gamma + i\infty$ and

$$d\phi(\lambda) = \frac{c^*(-\lambda)}{2\pi i \lambda} d\lambda. \tag{5·14}$$

Hence

$$\bar{h}^*(s) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{h^*(s+\lambda)\, c^*(-\lambda)}{\lambda} d\lambda. \tag{5·15}$$

*Example* 2. As a special case and a check on the last example, let us suppose that $\mathscr{T}$ is quasi-Poisson with index $\tau$. Then, as in (4·6),

$$h^*(s) = I^*(s) + \frac{e^{-\tau s}}{\mu_1 s},$$

where $I^*(s)$ is an integral function which is $O(e^{-\tau s})$ in the half-plane $\Re s \leqslant 0$. Let us suppose that $c(x) = \delta(x - \varpi)$, where $\varpi > \tau$ and $\delta(x)$ is the Dirac delta function. Thus $c^*(s) = e^{-\varpi s}$, and (5·15) reduces to

$$\bar{h}^*(s) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \left\{ I^*(s+\lambda) + \frac{e^{-\tau(s+\lambda)}}{\mu_1(s+\lambda)} \right\} \frac{e^{\varpi\lambda}}{\lambda} d\lambda.$$

The integrand of this contour integral is meromorphic, having only two poles, both simple; one is at $\lambda = 0$ and the other at $\lambda = -s$, which may be supposed in the half-plane $\Re\lambda < 0$. The integral may be evaluated by contour integration in a familiar way

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by allowing $\gamma \to 0+$ (and providing an indentation at 0), and by then adding to the contour an infinitely large semi-circle joining $-i\infty$ to $+i\infty$ in the half-plane $\Re s \leqslant 0$. We deduce that

$$\bar{h}^*(s) = I^*(s) + \frac{e^{-\tau s}}{\mu_1 s} - \frac{e^{-s\varpi}}{\mu_1 s},$$

$$= h^*(s) - \frac{e^{-s\varpi}}{\mu_1 s}.$$

If we insert this determination of $\bar{h}^*(s)$ into (5·10) we discover that

$${}^p f^*(s) = \frac{e^{-s\varpi}}{\mu_1 s}\left\{\frac{1}{1+h^*(s)}\right\}$$

$$= e^{-s\varpi}\left\{\frac{1-f^*(s)}{\mu_1 s}\right\},$$

in agreement with Theorem 8.

*Example* 3. Suppose that we can represent $c(x)$ in the form

$$c(x) = \sum_\nu c_\nu e^{-\lambda_\nu x}, \tag{5·16}$$

where the summation may be finite or infinite, where the $\lambda_\nu$ are complex numbers such that $\Re\lambda_\nu > 0$, and where $\Sigma |c_\nu| < \infty$. Clearly (5·16) could (but will not) be represented in the form of (5·9), and we infer that

$$\bar{h}^*(s) = \sum_\nu \frac{c_\nu}{\lambda_\nu} h^*(s+\lambda_\nu).$$

Thus

$${}^p f^*(s) = \frac{h^*(s) - \sum_\nu \frac{c_\nu}{\lambda_\nu} h^*(s+\lambda_\nu)}{1+h^*(s)},$$

and provided an analytic form for $h^*(s)$ is known, the calculation of the lower moments of ${}^p f^*(s)$ follows from a straightforward expansion as a power series in $s$. It is apparent that these moments will involve the derivatives of $h^*(s)$ at the points $\lambda_\nu$.

As a closing remark, let us notice that (5·11) can be thrown into the following rather more vivid form,

$${}^p f^*(s) = f^*(s)\left\{1 - \frac{\bar{h}^*(s)}{h^*(s)}\right\}.$$

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