Mathematical Proceedings of the Cambridge Philosophical Society

VOL. 107 MARCH 1990 PART 2

Math. Proc. Camb. Phil. Soc. (1990), **107**, 193

Printed in Great Britain

A note on the subnormal structure of general linear groups

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(*Received* 27 *February* 1989)

The purpose of this note is to improve results of J. S. Wilson [**12**] and L. N. Vaserstein [**10**] concerning the subnormal structure of the general linear group $G = \mathrm{GL}(n, R)$ of degree $n \geqslant 3$ over a commutative ring R. To do this we sharpen results of J. S. Wilson [**12**], A. Bak [**1**] and L. N. Vaserstein [**10**] on subgroups normalized by a relative elementary subgroup. It should be said also that (especially for the case $n = 3$) our proof is very much simpler than that of [**12**, **10**]. To formulate our results let us recall some notation.

As usual e is the identity matrix and e_{ij} is a standard matrix unit, that is, a matrix with 1 in position (i,j) and zeros elsewhere. By $t_{ij}(\xi)$ for $i \neq j$ and $\xi \in R$ we denote the corresponding elementary transvection $e + \xi e_{ij}$. The group $E(n, R)$ generated by all the elementary transvections is called the elementary subgroup of G. Now let I be an ideal in R. One defines the elementary group $E(n, I)$ of level I as the subgroup generated by all the elementary transvections $t_{ij}(\xi)$, with $\xi \in I, i \neq j$, and the relative elementary group $E(n, R, I)$ of level I as the normal closure of $E(n, I)$ in $E(n, R)$, i.e. the smallest normal subgroup in $E(n, R)$ containing $E(n, I)$. Let $\phi_I : \mathrm{GL}(n, R) \to \mathrm{GL}(n, R/I)$ be the reduction homomorphism modulo I. By the principal congruence subgroup $\mathrm{GL}(n, R, I)$ and the full congruence subgroup $C(n, R, I)$ of level I one means the kernel of ϕ_I and the full preimage of the centre of $\mathrm{GL}(n, R/I)$ under ϕ_I respectively. The symbol 'R' in $\mathrm{GL}(n, R, I)$ has a mere ritual significance since this group is independent of the particular choice of an ambient ring R with 1, but the group $C(n, R, I)$ does depend on R.

It is clear from the definitions that $\mathrm{GL}(n, R, I)$ and $C(n, R, I)$ are always normal in G and it was first shown by A. A. Suslin [**5**] that $E(n, R, I)$ is also normal in G whenever R is commutative and $n \geqslant 3$. Conversely, it was proven by J. S. Wilson [**12**] for $n \geqslant 4$ and by I. Z. Golubchik [**4**] for $n \geqslant 3$ that the standard description of subgroups in $\mathrm{GL}(n, R)$ normalized by $E(n, R)$ holds, that is, for any such subgroup H there exists a uniquely determined ideal I of R such that $E(n, R, I) \leqslant H \leqslant C(n, R, I)$.

These papers were the most important break-through in the theory of linear

https://doi.org/10.1017/S030500410006847X Published online by Cambridge University Press

groups over rings since the fundamental contribution of H. Bass [2] who had obtained analogous results 'at the stable level', that is, for sufficiently large n (we cannot go far back into the history of this field; see [9] and the references therein). Since then other proofs of these results and many generalizations have been obtained by E. Abe, Z. I. Borewicz, I. Z. Golubchik, W. van der Kallen, S. G. Khlebutin, V. I. Kopeiko, A. W. Mason, B. R. McDonald, A. V. Mikhalev, E. B. Plotkin, A. V. Stepanov, A. A. Suslin, G. Taddei, M. S. Tulenbaev, L. N. Vaserstein, the author, W. C. Waterhouse, E. I. Zelmanov and others (see, for example, [3, 6–8, 11] and a survey by A. E. Zalesskii [13]).

Let us further recall that a subgroup H in G is said to be subnormal of defect (or depth) d if there is a chain

$$H = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_d = G$$

of length $d+1$ and if no such chain of lesser length exists (here $G_{i-1} \trianglelefteq G_i$ means that G_{i-1} is a normal subgroup of G_i). In this case we write $H \trianglelefteq {}^dG$ (following [12]). Then one of the principal results of [12] may be stated as follows. Let R be a commutative ring, $n \geqslant 3$, F a subgroup of $G = \mathrm{GL}(n,R)$ containing $E(n,R)$ and $H \trianglelefteq {}^dF$ a subnormal subgroup in F of defect d. Then there exists an ideal $I \trianglelefteq R$ such that

$$E(n,R,I^m) \leqslant H \leqslant C(n,R,I),$$

where $m = m(n,d)$ depends only on n and d. For $n \geqslant 4$, J. S. Wilson himself gave the estimate $m \leqslant \frac{1}{6}(7^d-1)$. In [10] L. N. Vaserstein improved this to $m \leqslant \frac{1}{5}(6^d-1)$ for all $n \geqslant 3$. He also gave a better lower bound for m, namely $m \geqslant 2^d-1$. It would be of course desirable to compute the exact value of m. Here we make a further step in this direction. Our chief goal in this note is to prove the following result.

THEOREM 1. *Let R be a commutative ring, $n \geqslant 3$, $E(n,R) \leqslant F \leqslant \mathrm{GL}(n,R)$ and $H \trianglelefteq {}^dF$. Then*

$$E(n,R,I^m) \leqslant H \leqslant C(n,R,I)$$

for some ideal $I \trianglelefteq R$ and $m \leqslant \frac{1}{4}(5^d-1)$.

This theorem is an immediate consequence of the following

THEOREM 2. *Let R be a commutative ring, $n \geqslant 3$ and H a subgroup in $G = \mathrm{GL}(n,R)$ normalized by the relative elementary group $E(n,R,I)$ for an ideal $I \trianglelefteq R$. Then there exists an ideal $A \trianglelefteq R$ such that*

$$E(n,R,I^5A) \leqslant H \leqslant C(n,R,A).$$

J. S. Wilson gave I^6A, (for $n \geqslant 4$), A. Bak gave $I^{24}A$ and L. N. Vaserstein gave I^6A as the bound on the left-hand side here. Our proof of this theorem is based on a pulling-out method which has been employed in [3] to present a simplified proof for the Wilson–Golubchik theorem (this method has also many further applications to the study of subgroups in G; see [3]).

Let $J(H)$ and $L(H)$ be the upper level and the lower level of a subgroup H respectively. They are defined as follows: $J(H)$ is the smallest ideal I of R such that $H \leqslant C(n,R,I)$ and $L(H)$ is the largest ideal I of R such that $E(n,R,I) \leqslant H$. In these terms Theorem 1 asserts that $J(H)^m \subseteq L(H)$ for $m = \frac{1}{4}(5^d-1)$ and Theorem 2 asserts that $I^5J(H) \subseteq L(H)$.

Now we commence the proof of Theorem 2. Let us take an arbitrary matrix

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$a = (a_{ij})$ from H. We first want to show that $t_{pq}(I^5 a_{kh}) \subseteq H$ for any $p \neq q$ and $k \neq h$. To do this take arbitrary $\alpha, \beta, \gamma, \lambda, \mu \in I$ and set

$$u = t_{12}(-\alpha a'_{3r})\, t_{13}(\alpha a'_{2r}) \in E(n, I),$$

where a'_{ij} is the entry of the inverse matrix a^{-1} in position (i,j). Then the commutator $b = [a, u] = aua^{-1}u^{-1}$ belongs to H and a direct computation shows that

$$b = (e + \sum \alpha a_{i1} \Delta_{rj} e_{ij})\, u^{-1}$$

where the sum is taken over $i, j = 1, \ldots, n$ and $\Delta_{rj} = a'_{2r} a'_{3j} - a'_{3r} a'_{2j}$. It is completely clear that $\Delta_{rj} = -\Delta_{jr}$ and $\Delta_{rr} = 0$. The matrix $b = b(r)$ depends on r, and $\Delta_{rr} = 0$ implies that $b_{ir} = \delta_{ir}$ for all $i = 1, \ldots, n$. For $k \neq r$ set

$$c = c(r, k) = [b, t_{rk}(\beta)] \in H.$$

A straightforward computation shows that

$$c = e + \sum \alpha\beta a_{k1} \Delta_{rj} e_{rj}.$$

Now choose $s \neq r$. If $p \neq r$ and $q \neq s$ then

$$d_q = t_{rq}(\alpha\beta\gamma a_{k1} \Delta_{rs}) = [c, t_{sq}(\gamma)] \in H$$

and moreover

$$t_{rs}(\alpha\beta\gamma\lambda\, a_{k1} \Delta_{rs}) = [d_q, t_{qs}(\lambda)] \in H.$$

Now let us distinguish between two cases, namely $n \geqslant 4$ and $n = 3$. For $n \geqslant 4$ let us fix some $t \neq r, k, p$. Then the formulae

$$v = t_{pt}(\alpha\beta\gamma\lambda\, a_{k1} \Delta_{rs}) = [t_{pr}(\lambda), d_t] \in H,$$

$$t_{pq}(\alpha\beta\gamma\lambda\mu\, a_{k1} \Delta_{rs}) = [v, t_{tq}(\mu)] \in H$$

conclude the proof of the desired inclusions for $q = t$ and $q \neq t$ respectively. It only remains to observe that our statement follows from the inclusions $t_{pq}(I^5 a_{k1} \Delta_{rs}) \subseteq H$ since the ideal generated by all the minors Δ_{rs} equals R.

For $n = 3$ almost all the necessary inclusions may be obtained by the same arguments as before, but a few of them need some further reasoning. Let us first concentrate on the element a_{21}. We have

$$c(1, 2) = e + \alpha\beta a_{21} \Delta_{12} e_{12} + \alpha\beta a_{21} \Delta_{13} e_{13}$$

and by the preceding arguments we can only conclude that $t_{31}(I^6 a_{21} \Delta_{12}) \subseteq H$. But one sees immediately that $a_{21} \Delta_{12} = -a_{31} \Delta_{13}$, and starting from the matrix

$$c(1, 3) = e + \alpha\beta a_{31} \Delta_{12} e_{12} + \alpha\beta a_{31} \Delta_{31} e_{13}$$

instead of $c(1,2)$ we get the necessary inclusion $t_{31}(I^5 a_{21} \Delta_{12}) \subseteq H$. Quite analogously from $c(1,2)$ we can only get the inclusion $t_{21}(I^6 a_{21} \Delta_{13}) \subseteq H$, but using the matrix

$$c(3, 2) = e + \alpha\beta a_{21} \Delta_{31} e_{31} + \alpha\beta a_{21} \Delta_{32} e_{32}$$

we can see that again $t_{21}(I^5 a_{21} \Delta_{13}) \subseteq H$. This reasoning shows that here one also has $t_{pq}(I^5 a_{21} \Delta_{rs}) \subseteq H$, and hence $t_{pq}(I^5 a_{21}) \subseteq H$, which concludes our analysis for a_{21}. It remains only to notice that taking in the preceding discussion index h instead of 1 and indices f, g such that f, g, h are pairwise distinct instead of 2, 3 we obtain the desired inclusions $t_{pq}(I^5 a_{kh}) \subseteq H$.

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Now lemma 10 from [**10**] shows that above inclusions together with normality of $E(n, R, I)$ in G imply that $L(H) \geq I^5 J(H)$. This concludes the proof of Theorem 2. Now Theorem 1 follows by the same kind of argument as in [**12**] or [**10**].

We do not believe that further reduction of the bound in Theorems 1 and 2 is probable. It would be interesting to confirm this faith with appropriate examples. Results analogous to those presented here are also valid for an almost commutative ring R, that is a ring finitely generated as a module over its centre. One should only modify the above arguments in style of [**7**, **8**].

It is perhaps worth mentioning that the results of this note have been generalized by E. B. Plotkin and the author to arbitrary Chevalley groups. Our proof follows the lines of the simplified proof for structure theorems of G. Taddei and L. N. Vaserstein [**6**, **11**] found recently by E. B. Plotkin and the author and by A. V. Stepanov.

I would like to express my warmest thanks to Prof. Z. I. Borewicz for numerous discussions and to Prof. L. N. Vaserstein who generously supplied me with necessary offprints and manuscripts.

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