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On Jacobi polynomials

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1. The object of this paper is to prove some formulae of Jacobi polynomials including a generating function. The results (2·1)–(2·4), (2·6)–(2·9), (3·1)–(3·4), and (4·1) are believed to be new.

The following notations are employed throughout this paper.

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n, 1+\alpha+\beta+n; \\ 1+\alpha;\end{matrix}\quad \tfrac{1}{2}(1-x)\right], \tag{1·1}$$

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_1F_1\left[\begin{matrix}-n; \\ 1+\alpha;\end{matrix}\quad x\right], \tag{1·2}$$

$$F_2(a; b_1, b_2; c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n}\frac{x^m y^n}{m!\,n!}$$

$$= \frac{1}{\Gamma(a)}\int_0^\infty e^{-t}t^{a-1}\,{}_1F_1(b_1; c_1; xt)\,{}_1F_1(b_2; c_2; yt)\,dt, \tag{1·3}$$

$$|x|+|y|<1, \quad \Re(a)>0,$$

where $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$, $(a)_0 = 1$.

2. We first prove the main result

$$\sum_{m=0}^{n} \frac{(-\beta-n)_m(1+\beta)_{n-m}}{m!\,(n-m)!}\,F_2(a; -m, -(n-m); 1+\alpha, 1+\beta; x, y)$$

$$= \frac{(a)_n(-1)^n(x+y)^n}{(1+\alpha)_n}\,P_n^{(\alpha,\beta)}\left(\frac{y-x}{y+x}\right) \tag{2·1}$$

and deduce from it the following:

$$\sum_{m=0}^{n} \frac{(-\beta-n)_m}{m!}\,P_{n-m}^{(\beta,\gamma+m)}(x) = \frac{(1+\beta+\gamma)_{2n}}{(1+\beta+\gamma)_n}\,\frac{[\frac{1}{2}(x-1)]^n}{n!}, \tag{2·2}$$

$$\sum_{m=0}^{n} \frac{(-\beta-n)_m(1+\beta)_{n-m}}{(1+\alpha)_m(n-m)!}\,P_m^{(\alpha,\gamma-m)}(x) = \frac{(1+\alpha+\gamma)_n(1+\beta)_n[\frac{1}{2}(1-x)]^n}{(1+\alpha)_n\,n!}, \tag{2·3}$$

$$\sum_{m=0}^{n} \frac{(-\beta-n)_m}{(1+\alpha)_m}\,P_m^{(\alpha,\gamma-m)}(x)\,P_{n-m}^{(\beta,\delta+m)}(x)$$

$$= \frac{(2+\alpha+\beta+\gamma+\delta)_{2n}}{(2+\alpha+\beta+\gamma+\delta)_n}\,\frac{[\frac{1}{2}(x-1)]^n}{n!}\,{}_3F_2\left[\begin{matrix}-n, 1+\alpha+\gamma, 1+\alpha+\beta+n; \\ 1+\alpha, 2+\alpha+\beta+\gamma+\delta+n;\end{matrix}\quad 1\right], \tag{2·4}$$

$$\sum_{m=0}^{n} \frac{(-\beta-n)_m}{m!}\left(\frac{1+x}{2}\right)^m P_{n-m}^{(\beta,\delta+m)}(x) = \frac{(\delta+1)_n[\frac{1}{2}(x-1)]^n}{n!}, \tag{2·5}$$

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$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-\beta-n)_m}{m!}P_{n-m}^{(\beta,\gamma+m)}(x)={}_2F_1\left[\begin{matrix}\frac{1}{2}(1+\beta+\gamma),\frac{1}{2}(2+\beta+\gamma);\\ 1+\beta+\gamma;\end{matrix}\quad 2(x-1)\right], \tag{2·6}$$

valid for $|x-1|<\frac{1}{2}$.

$$\sum_{m,n=0}^{\infty}\frac{(-1)^m(1+\beta)_{m+n}}{m!\,(1+\beta)_n}P_n^{(\beta,\gamma+m)}(x)={}_2F_1\left[\begin{matrix}\frac{1}{2}(1+\beta+\gamma),\frac{1}{2}(2+\beta+\gamma);\\ 1+\beta+\gamma;\end{matrix}\quad 2(x-1)\right] \tag{2·7}$$

valid for $|x-1|<\frac{1}{2}$.

$$\sum_{m,n=0}^{\infty}\frac{(-1)^m(1+\beta)_{m+n}}{n!\,(1+\alpha)_m}P_m^{(\alpha,\gamma+n)}(x)={}_3F_2\left[\begin{matrix}\frac{1}{2}(1+\alpha+\gamma),\frac{1}{2}(2+\alpha+\gamma),1+\beta;\\ 1+\alpha+\gamma,1+\alpha;\end{matrix}\quad 2(1-x)\right] \tag{2·8}$$

valid for $|x-1|<\frac{1}{2}$.

$$\sum_{m,n=0}^{\infty}\frac{(-1)^m(1+\beta)_{m+n}}{m!\,(1+\beta)_n}\left(\frac{1+x}{2}\right)^m P_n^{(\beta,\delta+m)}(x)=\left(\frac{2}{3-x}\right)^{\delta+1} \tag{2·9}$$

valid for $|x-1|<2$.

$$\sum_{n=0}^{\infty}P_n^{(\beta,\gamma)}(-1)=2^{-\gamma-1} \tag{2·10}$$

valid for $\Re(\gamma)<0$.

Proof. Due to (1·3), we have for $\Re(a)>0$,

$$\sum_{m=0}^{n}\frac{(-\beta-n)_m(1+\beta)_{n-m}}{m!\,(n-m)!}F_2(a;\,-m,-(n-m);\,1+\alpha,1+\beta;\,x,y)$$

$$=\frac{1}{\Gamma(a)}\sum_{m=0}^{n}\frac{(-\beta-n)_m(1+\beta)_{n-m}}{m!\,(n-m)!}$$

$$\times\int_0^{\infty}e^{-t}t^{a-1}{}_1F_1(-m;\,1+\alpha;\,xt)\,{}_1F_1(-(n-m);\,1+\beta;\,yt)\,dt.$$

The change of order of integration and summation, use of (1·2) and the result ((6), p. 475 (33))

$$\sum_{m=0}^{n}\frac{(-\beta-n)_m}{(1+\alpha)_m}L_m^{(\alpha)}(x)\,L_{n-m}^{(\beta)}(y)=\frac{(-1)^n(x+y)^n P_n^{(\alpha,\beta)}[(y-x)/(y+x)]}{(1+\alpha)_n}$$

then yields

$$\sum_{m=0}^{n}\frac{(-\beta-n)_m(1+\beta)_{n-m}}{m!\,(n-m)!}F_2(a;\,-m,-(n-m);\,1+\alpha,1+\beta;\,x,y)$$

$$=\frac{(-1)^n(x+y)^n P_n^{(\alpha,\beta)}[(y-x)/(y+x)]}{(1+\alpha)_n\,\Gamma(a)}\int_0^{\infty}e^{-t}t^{a+n-1}\,dt$$

$$=\frac{(a)_n(-1)^n(x+y)^n P_n^{(\alpha,\beta)}[(y-x)/(y+x)]}{(1+\alpha)_n}.$$

The change of order of integration and summation is permissible since the series under consideration is finite.

This completes the proof of (2·1).

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Deductions. In (2·1) if we set $x = 0$, $a = 1+\beta+\gamma+n$, $y = \frac{1}{2}(1-x)$ and use (1·1), it reduces to (2·2). On the other hand if we set $y = 0$, $a = 1+\alpha+\gamma$, replace x by $\frac{1}{2}(1-x)$, use (1·1),and the result ((1), p. 3)

$$F(-n, b;\ c;\ 1) = [(c-b)_n]/[(c)_n],$$

it yields (2·3).

When we sum both the sides of (2·2) for n from zero to infinity we easily get the result (2·6).

In (2·3) if we write $\gamma+n$ for γ and sum both the sides for n from zero to infinity it gives

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-\beta-n)_m(1+\beta)_{n-m}}{(1+\alpha)_m(n-m)!}P_m^{(\alpha,\gamma+n-m)}(x) = \sum_{n=0}^{\infty}\frac{(1+\alpha+\gamma+n)_n(1+\beta)_n(1-x)^n}{(1+\alpha)_n 2^n n!}. \tag{2·11}$$

We now shift the index n to $n+m$ on L.H.S. of (2·6) and (2·11) and use the results ((9), p. 240 (1·3), (1·5), (1·25))

$$(a+n)_n = [(a)_{2n}]/[(a)_n];$$

$$(a-n)_n = (-1)^n(1-\alpha)_n;$$

and

$$(a)_{2n} = 2^{2n}(\tfrac{1}{2}a)_n(\tfrac{1}{2}+\tfrac{1}{2}a)_n,$$

we easily obtain the results (2·7) and (2·8) respectively.

In (2·1) we write tx and $(1-t)x$ for x and y respectively, multiply both the sides by $t^{b-1}(1-t)^{a-b-1}$ and integrate with respect to t in the limits from 0 to 1 to get

$$\int_0^1 t^{b-1}(1-t)^{a-b-1} \times \sum_{m=0}^{n}\frac{(-\beta-n)_m(1+\beta)_{n-m}}{m!(n-m)!}F_2(a;\ -m, -(n-m);\ 1+\alpha, 1+\beta;\ tx, (1-t)x)\,dt = \frac{(a)_n(-x)^n}{(1+\alpha)_n}\int_0^1 t^{b-1}(1-t)^{a-b-1}P_n^{(\alpha,\beta)}(1-2t)\,dt.$$

We now express F_2 and the Jacobi polynomial in the series form and evaluate the integrals term by term to obtain

$$\sum_{m=0}^{n}\frac{(-\beta-n)_m(1+\beta)_{n-m}}{m!(n-m)!}\,{}_2F_1\left[\begin{matrix}-m, b;\\ 1+\alpha;\end{matrix}\ x\right]{}_2F_1\left[\begin{matrix}-(n-m), a-b;\\ 1+\beta;\end{matrix}\ x\right] = \frac{(a)_n(-x)^n}{n!}\,{}_3F_2\left[\begin{matrix}-n, b, 1+\alpha+\beta+n;\\ a, 1+\alpha;\end{matrix}\ 1\right], \tag{2·12}$$

valid for $\Re(b) > 0$, $\Re(a-b) > 0$.

In (2·12) if we set $b = 1+\alpha+\gamma$, $a-b = 1+\beta+\delta+n$, replace x by $\frac{1}{2}(1-x)$, then due to (1·1) it reduces to the result (2·4).

Put $\gamma = 0$ in (2·4), use the result ((1), p. 3)* and get the result (2·5), which is a particular case of a result due to Carlitz ((10), p. 376 (4·4)).

On summing both the sides of (2·5) for n from zero to infinity we get the result

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-\beta-n)_m}{m!}\left(\frac{1+x}{2}\right)^m P_{n-m}^{(\beta,\delta+m)}(x) = {}_1F_0(1+\delta;\ -;\ \tfrac{1}{2}(x-1)) = \left(\frac{2}{3-x}\right)^{\delta+1}. \tag{2·13}$$

* Already mentioned.

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When the index n is changed to $n+m$, (2·13) reduces to (2·9). For $x=-1$, (2·9) yields (2·10), which can also be derived from the result ((4), p. 172 (29))

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\,t^n = 2^{\alpha+\beta}\rho^{-1}(1+t+\rho)^{-\beta}(1-t+\rho)^{-\alpha},$$

where $\rho=(1-2xt+t^2)^{\frac{1}{2}}$, for $x=-1$, $t=1$.

3. In this section we obtain the result

$$\sum_{m=0}^{n} \frac{(\gamma)_m}{m!} P_{n-m}^{(\alpha,\beta+m)}(x) = P_n^{(\alpha+\gamma,\beta-\gamma)}(x). \tag{3·1}$$

Proof. The result

$$\sum_{m=0}^{n} \frac{(\gamma)_m (1+\alpha)_{n-m}}{m!\,(n-m)!} F_2(a;\, b,\, -(n-m);\, c,\, 1+\alpha;\, x, y) = \frac{(1+\alpha+\gamma)_n}{n!} F_2(a;\, b,\, -n;\, c,\, 1+\alpha+\gamma;\, x, y) \tag{3·2}$$

can be easily proved by adopting the method used in the proof of (2·1) and using the formula ((8), p. 209 (2))

$$L_n^{(\alpha+\beta)}(x) = \sum_{m=0}^{n} \frac{(\beta)_m L_{n-m}^{(\alpha)}(x)}{m!}$$

instead of ((6), p. 475 (33)).

In (3·2) we replace x and y by tx and $[(1-t)(1-x)]/2$ respectively, express F_2 in the form of series by (1·3), multiply both the sides by $t^{a-\alpha-\beta-n-2}(1-t)^{\alpha+\beta+n}$ and integrate term by term with respect to t in the limits from 0 to 1. The evaluation of the integral then yields the result (3·1).

If we multiply both the sides of (3·1) by t^n and sum for n from zero to infinity, then due to ((4), p. 172 (29))* we obtain

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(\gamma)_m}{m!} P_{n-m}^{(\alpha,\beta+m)}(x)\,t^n = 2^{\alpha+\beta}\rho^{-1}(1+t+\rho)^{\gamma-\beta}(1-t+\rho)^{-\alpha-\gamma}, \tag{3·3}$$

where $\rho=(1-2xt+t^2)^{\frac{1}{2}}$.

On shifting the index n to $n+m$ it becomes

$$\sum_{m,n=0}^{\infty} \frac{(\gamma)_m}{m!} P_n^{(\alpha,\beta+m)}(x)\,t^{m+n} = 2^{\alpha+\beta}\rho^{-1}(1+t+\rho)^{\gamma-\beta}(1-t+\rho)^{-\alpha-\gamma}, \tag{3·4}$$

where $\rho=(1-2xt+t^2)^{\frac{1}{2}}$.

4. Lastly we prove the result

$$\sum_{n=0}^{\infty} F_2(a;\, b,\, -n;\, c, d;\, x, y)\,(z^n/n!) = e^z \Psi_1(a;\, b;\, c, d;\, x,\, -yz). \tag{4·1}$$

As a consequence of (4·1) we obtain the generating function

$$\sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\beta-n)}(x)\,z^n}{(1+\alpha)_n} = e^z\,{}_1F_1(1+\alpha+\beta;\, 1+\alpha;\, \tfrac{1}{2}[(x-1)z]), \tag{4·2}$$

found earlier by Feldheim ((5), p. 120).

* Already quoted.

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Proof. By (1·3) we express F_2 on L.H.S. of (4·1) in the form of an integral, reverse the order of integration and summation, use the formula ((8), p. 209 (4))

$$e^t {}_0F_1(-;\, 1+\alpha;\, -xt) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)\, t^n}{(1+\alpha)_n}$$

and get

$$\frac{e^z}{\Gamma(a)} \int_0^{\infty} e^{-t} t^{a-1} {}_1F_1(b;\, c;\, xt)\, {}_0F_1(-;\, d;\, -yzt)\, dt.$$

We now write ${}_1F_1$ and ${}_0F_1$ in the series form, evaluate the integral and use ((3), p.225 (23)) and get the required result (4·1).

The change of order of integration and summation is justified due to absolute convergence of the series and the integral under consideration ((2), p. 500).

By employing the technique used in the proof of (3·1) and adjusting the parameters one can easily get the result (4·2).

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