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On equilibrium states for partially hyperbolic horseshoes

I. RIOS† and J. SIQUEIRA‡

† *Instituto de Matemática e Estatística, Universidade Federal Fluminense, Rua Prof. Marcos Waldemar de Freitas Reis, S/N - Bloco H, 4o Andar, 24210-201 Niterǿi, Brazil*
(e-mail: rios@mat.uff.br)

‡ *Centro de Matemática da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal*
(e-mail: jaqueline.rocha@fc.up.pt)

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Abstract. We prove the existence and uniqueness of equilibrium states for a family of partially hyperbolic systems, with respect to Hölder continuous potentials with small variation. The family comes from the projection, on the center-unstable direction, of a family of partially hyperbolic horseshoes introduced by Díaz *et al* [Destroying horseshoes via heterodimensional cycles: generating bifurcations inside homoclinic classes. *Ergod. Th. & Dynam. Sys.* **29** (2009), 433–474]. For the original three-dimensional system we consider potentials with small variation, constant on local stable manifolds, obtaining existence and uniqueness of equilibrium states.

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1. *Introduction*

The thermodynamic formalism, inspired by statistical mechanics, is a part of ergodic theory that studies the behavior of invariant measures with regular Jacobians, called

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equilibrium states. We formally define an equilibrium state with respect to a potential as follows.

Definition 1.1. Consider a continuous map $G : \Lambda \to \Lambda$ on a compact metric space Λ. We say that a G-invariant probability measure μ is an *equilibrium state* for G with respect to a potential $\phi : \Lambda \to \mathbb{R}$ if it satisfies

$$h_\mu(G) + \int \phi \, d\mu = \sup_\eta \left\{ h_\eta(G) + \int \phi \, d\eta \right\},$$

where the supremum is taken over all G-invariant probability measures.

The theory was introduced by Sinai in the 1970s, and since then it has been studied extensively by many authors who have made great advances in the area. In [**24**], Sinai initiated the study of equilibrium states for Anosov diffeomorphisms, which was completed by Bowen and Ruelle (see [**5**, **20**, **21**]). Sinai, Ruelle and Bowen considered a system consisting of a uniformly expanding map and a Hölder continuous potential. For that system they proved that it has a unique equilibrium state. Their strategy was to use a finite Markov partition to build a conjugacy with a Markov shift of finite type. With some adjustments, the proof extends to the case of uniformly hyperbolic maps.

In the context of non-uniformly hyperbolic maps (in dimension greater than one), we mention the important contributions made by Sarig (see [**22**, **23**]), Buzzi and Sarig [**8**], who studied the existence and uniqueness of Gibbs measures and equilibrium states for countable Markov shifts, Buzzi *et al* [**7**], who studied piecewise expanding maps, and Oliveira and Viana [**17**] and Varandas and Viana [**25**], who studied non-uniformly expanding maps.

Among those contributions, [**17**] is the most closely related to the present work. There the authors proved the existence and uniqueness of equilibrium states for a family of non-uniformly expanding maps associated with a class of Hölder continuous potentials with small variation. In their work, they considered finite and non-generating Markov partitions. Some rectangles are uniformly expanding and others could have some contractive behavior. They used the notion of *hyperbolic times* introduced by Alves *et al* in [**2**] to construct special Gibbs-type measures.

Later, Leplaideur *et al* [**15**] studied the existence and properties of equilibrium states, for a family of partially hyperbolic horseshoes introduced by Díaz *et al* in [**12**]. The three-dimensional horseshoes are simultaneously at the boundary of the set of uniformly hyperbolic systems and the set of persistently non-hyperbolic ones. In [**15**], the authors studied the ergodic measures with support in the limit set, showing that there is a spectral gap on the set of central Lyapunov exponents. As a consequence, there is a phase transition for the potential $t \cdot \log |J^c|$, where J^c is the (smooth) Jacobian of the system restricted to the (one dimensional) central direction and $t \geq 0$. In fact, for a special value t_0, there are two mutually singular equilibrium states. For $t > t_0$, the equilibrium state is unique, and it is a Dirac measure. They proved the existence of equilibrium states for every t (for every continuous potential, in fact), but the question of uniqueness for $0 \leq t < t_0$ was left open. In [**11**], the authors explored geometric and ergodic properties of a class of horseshoes with similar dynamical properties.

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Still in the context of the partially hyperbolic horseshoes introduced in [**12**], Arbieto and Prudente [**3**] proved that, for Hölder continuous potentials depending only on the unstable direction (thus constant on the center-stable one), the associated equilibrium state is unique. This includes constant potentials (proving the uniqueness of measures of maximal entropy), but not the family studied in [**15**].

Here we consider the same family of three-dimensional horseshoes, taking the inverse map. We start by defining a two-dimensional abstract map, obtained by projecting the system on two center-unstable leaves (center-stable leaves for the original map). We prove the existence and uniqueness of equilibrium states associated with Hölder continuous potentials with small variation, where by small we mean smaller than a fixed constant explicitly given. This part is strongly inspired by the work of Oliveira and Viana [**17**], but there are many extra difficulties to circumvent in our case: the degree of the map is not constant, and the number of expanding rectangles of the Markov partition is too small.

Back to the three-dimensional problem, we address the question of uniqueness of equilibrium states for a special class of Hölder continuous potentials that depend only on the center-stable direction (complementing the direction considered in [**3**]), and whose variation has the same bound as before. This class includes the family studied in [**15**], which was the motivation of this work, for small values of the parameter t.

The results in this paper are stated in the opposite order of their proofs. The first result, Theorem A, concerns the three-dimensional problem, and, in general terms, is as follows.

THEOREM A. *Let F be the partially hyperbolic horseshoe studied in* [**15**]*, and let ϕ be a Hölder continuous potential with small variation, supported on the horseshoe. Assume that ϕ does not vary on the $1/2$-local unstable manifolds. Then there exists a unique equilibrium state for the system F with respect to the potential ϕ.*

Consider a two-dimensional map G defined as the projection of the partially hyperbolic horseshoe F^{-1} on two center-unstable leaves, crossing the horseshoe at the top and bottom components (a precise definition will be given later). The next result is central to the entire work, and can be roughly stated as follows.

THEOREM B. *Let ϕ be a Hölder continuous potential with small variation, defined on the non-wandering set of G. Then there exists a unique equilibrium state for the system G with respect to ϕ.*

In order to prove Theorem B we study the associated Ruelle–Perron–Frobenius operator $\mathcal{L}_\phi$ and its dual $\mathcal{L}_\phi^*$. Theorem C states properties of these operators. It has an intrinsic interest, and it will also be used in the proof of Theorem B.

THEOREM C. *Let λ be the spectral radius of the transfer operator $\mathcal{L}_\phi$. There exists a probability measure ν and a positive function h satisfying*

$$\mathcal{L}_\phi^* \nu = \lambda \nu \quad \text{and} \quad \mathcal{L}_\phi h = \lambda h.$$

See §2 for precise statements of Theorems A, B and C.

This paper is organized as follows. In §2, we introduce the family $F = F_{\lambda,\sigma}$ of horseshoes constructed in the paper [**12**] and the projected maps $G = G_{\lambda,\sigma}$. We also state the main results of our work in a precise way.

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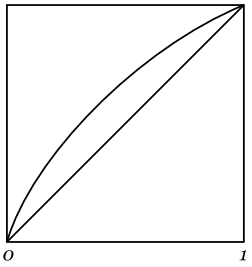


FIGURE 1. Function f.

In §3 we prove Theorem C. We establish the existence of a reference measure and prove the abundance of hyperbolic times for almost every point with respect to it. We find an eigenfunction for the transfer operator and state some of its properties. In §4 we prove the existence and uniqueness of the equilibrium state, which is a non-lacunary Gibbs measure. Finally, in §5, we prove Theorem A. We use the equilibrium state for the map G to construct an equilibrium state supported on the three-dimensional horseshoe, and prove the uniqueness.

2. *Definitions and main results*

We now describe a family of three-dimensional horseshoes introduced by Díaz *et al* in [**12**]. Consider $R = [0, 1] \times [0, 1] \times [0, 1] \subset \mathbb{R}^3$ and let

$$\tilde{R}_0 = [0, 1] \times [0, 1] \times [0, 1/6] \quad \text{and} \quad \tilde{R}_1 = [0, 1] \times [0, 1] \times [5/6, 1].$$

Define

$$F_0(x, y, z) := F_{|\tilde{R}_0}(x, y, z) = (\rho x, f(y), \beta z),$$

where $0 < \rho < 1/3$, $\beta > 6$ and $f(y) = 1/(1 - (1 - 1/y)e^{-1})$. See the graph of the function f in Figure 1.

We also define

$$F_1(x, y, z) = F_{|\tilde{R}_1}(x, y, z) = (\tfrac{3}{4} - \rho x, \sigma(1 - y), \beta_1(z - \tfrac{5}{6})),$$

where $0 < \sigma < 1/3$ and $3 < \beta_1 < 4$.

Then, for $X \in R$, we have

$$F(X) = \begin{cases} F_0(X) & \text{if } X \in \tilde{R}_0, \\ F_1(X) & \text{if } X \in \tilde{R}_1. \end{cases} \tag{1}$$

Points $X \in R$ outside $\tilde{R}_0 \cup \tilde{R}_1$ will be mapped injectively outside R. Note that F depends on the parameters ρ, β, β_1 and σ.

Figure 2 depicts the construction of the map F as a horseshoe map. In [**12**], Díaz *et al* showed that for fixed parameters satisfying the condition above, the non-wandering set for F is partially hyperbolic, with one-dimensional central direction, parallel to the y-axis. The vertical direction is expanding and the horizontal direction, parallel to the x-axis is contracting. The y-axis direction has both behaviors.

Remark 2.1. Note that the definition of F does not depend on the z coordinate, both in $\tilde{R}_0$ and $\tilde{R}_1$. This implies that the Jacobian of F restricted to the central direction (parallel

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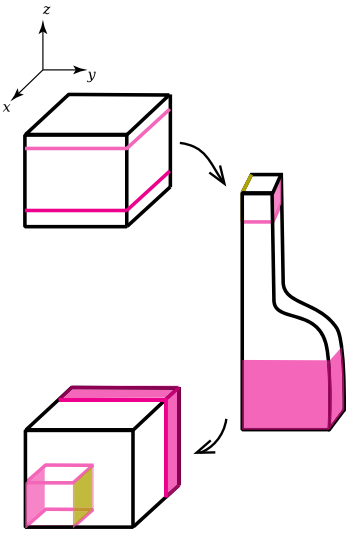


FIGURE 2. The map F.

to the y-axis) does not depend on the z-coordinate in these regions, being constant on the $1/2$-local strong unstable manifolds.

We are now able to establish the first main result. Let $\omega = (1+\sqrt{5})/2$.

THEOREM A. *Let $F : \tilde{R}_0 \cup \tilde{R}_1 \to R$ be the three-dimensional partially hyperbolic horseshoe defined as above. Let $\varphi : \tilde{R}_0 \cup \tilde{R}_1 \to \mathbb{R}$ be a Hölder continuous potential with $\sup \varphi - \inf \varphi < (\log \omega)/2$. Assume that φ does not depend on the z-coordinate in each set $\tilde{R}_0$ and $\tilde{R}_1$. Then there exists a unique equilibrium state for the system F with respect to the potential φ.*

Now we define the map G, which is related to the projection of F^{-1} in two center-stable planes (see §5). We consider an abstract space consisting of three rectangles R_1, R_2 and R_3 defined as follows:

$$R_1 = [0, \rho] \times [0, 1] \times \{0\},$$
$$R_2 = [3/4 - \rho, 3/4] \times [0, \sigma] \times \{0\},$$
$$R_3 = [0, \rho] \times [2, 3] \times \{5/6\}.$$

The rectangles are inside two planes called P_0 and P_1 as in Figure 3. Compare with Figure 5. Let $g_0 : [0, 1] \to \mathbb{R}$, $g_0(y) = f^{-1}(y)$ and let $g_1 : [0, 1] \to \mathbb{R}$ be defined by $g_1(y) = 1 - \sigma^{-1} y$. See their graphs in Figure 4.

Let $\alpha = \rho^{-1}$. We define a map $G : \bigcup_{i=1}^{3} R_i \to P_0 \cup P_1$ by its restrictions G_i to each rectangle R_i as follows:

$$G_1(x, y, z) = (\alpha x, g_0(y), 0),$$
$$G_2(x, y, z) = (\alpha(3/4 - x), g_1(y), 5/6),$$
$$G_3(x, y, z) = (\alpha x, g_0(y), 0).$$

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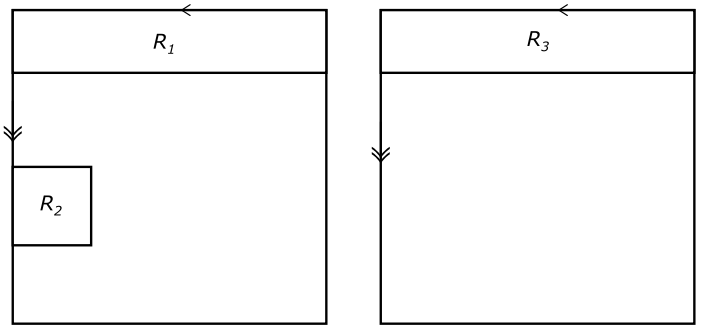


FIGURE 3. Rectangles R_1, R_2 and R_3.

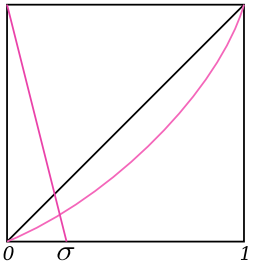


FIGURE 4. Functions g_0 and g_1.

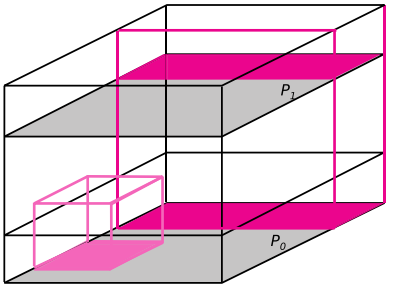


FIGURE 5. Horizontal planes.

The behavior of the map G is similar on R_1 and R_3. In these rectangles, the vertical direction is expanding. In the horizontal direction G sends the points from the right side to the left, except for the extreme points whose x coordinates are fixed. In the rectangle R_2 both directions are expanding.

We consider Λ the maximal invariant set of the union of the rectangles:

$$\Lambda := \bigcap_{n\in\mathbb{N}} G^{-n}\left(\bigcup_{i=1}^{3} R_i\right),$$

and from now on we call G the restriction $G|_\Lambda : \Lambda \to \Lambda$.

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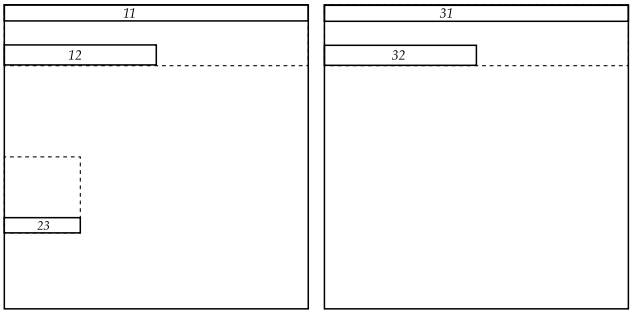


FIGURE 6. Second generation.

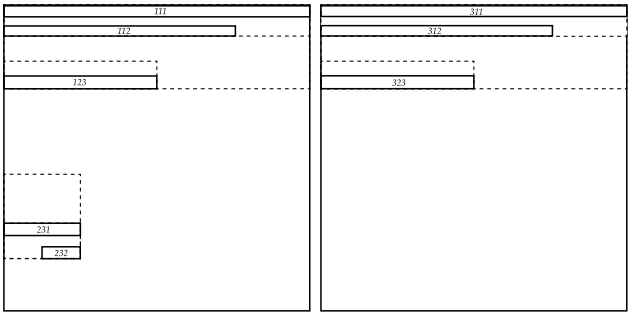


FIGURE 7. Third generation.

Let A be the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

and consider Σ_A the finite-type subshift with transition matrix A:

$$\Sigma_A = \{\Theta = (\theta_0\theta_1\theta_2 \cdots) \in \{1, 2, 3\}^{\mathbb{N}} \mid A_{\theta_i\theta_{i+1}} = 1\}.$$

According to this, the following transitions are allowed:

$$1 \to 1, 2$$
$$2 \to 3$$
$$3 \to 1, 2.$$

A sequence in Σ_A will be called *admissible*. Finite sequences respecting the transition rules will be called *admissible words*.

The map G is semi-conjugated to the shift σ on Σ_A. This result is an adaptation of a result in [**12**] (see Lemma 5.2 therein).

Note that the points belonging to the rectangles 1 and 2 have two pre-images, while points in the rectangle 3 have just one pre-image.

In Figures 6 and 7, the steps in the generation of the set Λ are depicted. Note that Λ is not a Cantor set: it contains infinitely many line segments. We point out that G is not

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conjugated to the shift σ. The entire segment $[0, 1]$ is associated to a unique sequence on Σ_A: the constant sequence $(\cdots 1, 1, 1 \cdots)$.

The topological entropy of the subshift σ is given by

$$h_{\text{top}}(\sigma) = \log\left(\frac{1+\sqrt{5}}{2}\right) = \log \omega.$$

By the semiconjugacy we have $h_{\text{top}}(G) \geq \log \omega$. We are ready to state the next main result of this paper.

THEOREM B. *Let $G : \Lambda \to \Lambda$ be as above and let $\phi : \Lambda \to \mathbb{R}$ be a Hölder continuous potential satisfying* $\sup \phi - \inf \phi < (\log \omega)/2$. *Then there exists a unique equilibrium state for the system G with respect to the potential ϕ.*

We point out that our class of potentials includes the zero potential, constant potentials, and $t \cdot \|\partial G/\partial y\|$, for small values of $t \geq 0$, related to the potential J^c studied in [**15**]. In particular, for $t = 0$ we have a unique measure of maximal entropy, both for F and for G. For F, this result was obtained before in [**3**]. There the authors consider potentials that are invariant on center-stable leaves, and this class contains the constant potentials.

2.1. *Ruelle–Perron–Frobenius operator.* Let $\mathcal{C}(\Lambda)$ be the set of real continuous functions on Λ. For a Hölder continuous potential ϕ we define a transfer operator $\mathcal{L}_\phi : \mathcal{C}(\Lambda) \to \mathcal{C}(\Lambda)$, called the *Ruelle–Perron–Frobenius operator*, which associates to each $\psi \in \mathcal{C}(\Lambda)$ a function $\mathcal{L}_\phi(\psi) : \Lambda \to \mathbb{R}$ by

$$\mathcal{L}_\phi \psi(x) = \sum_{y \in G^{-1}(x)} e^{\phi(y)} \psi(y). \tag{2}$$

For each $n \in \mathbb{N}$ we consider the Birkhoff sum $S_n\phi(x) = \sum_{j=0}^{n-1} \phi(G^j(x))$. Note that, iterating the operator, we obtain, for each $n \in \mathbb{N}$,

$$\mathcal{L}_\phi^n \psi(x) = \sum_{y \in G^{-n}(x)} e^{S_n\phi(y)} \psi(y). \tag{3}$$

We also consider the dual operator of $\mathcal{L}_\phi$, $\mathcal{L}_\phi^* : \mathcal{M}(\Lambda) \to \mathcal{M}(\Lambda)$, that satisfies

$$\int \psi \, d\mathcal{L}_\phi^* \eta = \int \mathcal{L}_\phi(\psi) \, d\eta,$$

for every $\psi \in \mathcal{C}(\Lambda)$ and every $\eta \in \mathcal{M}(\Lambda)$. Now we can establish the third main theorem of this paper. Consider ϕ as in the statement of the Theorem B.

THEOREM C. *Let λ be the spectral radius of the transfer operator $\mathcal{L}_\phi$. There exist a probability measure $\nu \in \mathcal{M}(\Lambda)$ and a positive function $h : \Lambda \to \mathbb{R}$ which satisfy*

$$\mathcal{L}_\phi^* \nu = \lambda \nu \quad \text{and} \quad \mathcal{L}_\phi h = \lambda h.$$

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3. *Proof of Theorem C*

We consider $K = \{f \in \mathcal{C}(\Lambda) \mid f \geq 0\}$ the cone of continuous non-negative functions on Λ. Then K is a normal cone which gives a partial order on $\mathcal{C}(\Lambda)$. Since $\mathcal{L}_\phi$ is a positive and bounded operator, we get the following result from Mazur's theorem (see [**13**]).

PROPOSITION 3.1. *If λ is the spectral radius of $\mathcal{L}_\phi$, then there exists a probability measure ν satisfying*

$$\mathcal{L}_\phi^* \nu = \lambda \nu.$$

Thus, Proposition 3.1 gives the existence of an eigenmeasure of the dual of the transfer operator, proving the first part of Theorem C. To prove the second part, the strategy will be to construct a sequence of functions that has an accumulation point which is an eigenfunction for the transfer operator.

First we establish a lower bound on the eigenvalue of $\mathcal{L}_\phi^*$.

LEMMA 3.1. *The spectral radius of $\mathcal{L}_\phi$, $r(\mathcal{L}_\phi) = \lambda$, satisfies*

$$\lambda \geq e^{\inf \phi + \log \omega} \quad \text{where } \omega = \frac{1+\sqrt{5}}{2}.$$

Proof. By definition $\lambda = r(\mathcal{L}_\phi) = \limsup \|\mathcal{L}_\phi^n\|^{1/n}$. Therefore,

$$\|\mathcal{L}_\phi^n 1(x)\| = \sum_{y \in G^{-n}(x)} e^{S_n \phi(y)} \geq \#\{G^{-n}(x)\} \cdot e^{n \inf \phi}.$$

Then, we have

$$\lambda \geq \limsup [\#\{G^{-n}(x)\} \cdot e^{n \inf \phi}]^{1/n} = e^{\inf \phi} e^{\log \omega} = e^{\inf \phi + \log \omega}$$

since $\limsup_{n \to \infty} (1/n) \log \#\{G^{-n}(x)\} = \log \omega$. □

To overcome the lack of hyperbolicity, we introduce the next definitions, which are key tools for the estimates to be done.

3.1. *Hyperbolic times and hyperbolic cylinders.* The concept of *hyperbolic time*, which was introduced by Alves in [**1**] and extensively studied in [**2**] by Alves *et al* is defined as follows.

Definition 3.1. We say that $n \in \mathbb{N}$ is a (G,c)-*hyperbolic time* for $x \in \Lambda$, for some $c > 0$ if

$$\prod_{j=k}^{n-1} \|DG(G^j(x))^{-1}\|^{-1} \geq e^{2c(n-k)}, \quad 0 \leq k \leq n-1.$$

Remark 3.1. A useful property of hyperbolic times was given in [**2**]. Roughly, they proved that if n is a hyperbolic time for x, there exists a neighborhood of the nth iterate of x for which every inverse branch is exponentially contracting. Moreover, the radius $\epsilon > 0$ of the neighborhood depends only on c and on the map.

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For the precise statement and its proof, see [**2**].

Given $n \in \mathbb{N}$ and n symbols $a_j \in \{1, 2, 3\}$ we define the *cylinder set of length n* by

$$\mathcal{R}^n(a_0, \ldots, a_{n-1}) = \{x \in \Lambda \mid x \in R_{a_0}, G(x) \in R_{a_1}, \ldots, G^{n-1}(x) \in R_{a_{n-1}}\}.$$

When there is no risk of confusion, we denote $\mathcal{R}^n$ to refer to $\mathcal{R}^n(a_0, \ldots, a_{n-1})$.

We say that a cylinder set $\mathcal{R}^n(a_0, \ldots, a_{n-1})$ is a *hyperbolic cylinder* if, for all $x \in R^n$, we have that n is a hyperbolic time for x. Our goal is to prove that the map G admits many hyperbolic cylinders. Intuitively, points that visit more often the expanding region of the system (the rectangle R_2) have better expansion rates.

For each $\gamma \in (0, 1)$ and $n \geq 1$, let $I(\gamma, n)$ be the set of admissible words of length n such that the proportion of symbols 1 and 3 is at least γ. More precisely, if the admissible word $(a_0, a_1, \ldots, a_{n-1}) \in I(\gamma, n)$, we have

$$\#\{0 \leq j \leq n-1 \mid a_j = 1, 3\} > \gamma n.$$

We can associate words in $I(\gamma, n)$ with cylinders sets that may fail to be hyperbolic. We will study the exponential growth of the cardinality of the sets $I(\gamma, n)$. We can estimate $\#I(\gamma, n)$ by counting the number of admissible words of length n with at most $(1-\gamma)n$ symbols 2.

By definition, to compose a word with k symbols 2 we have to follow the rules below:

- a symbol 2 is always followed by a 3;
- a symbol 3 is followed by 1 or 2;
- a symbol 1 is followed by 1 or 2.

This means that, once we have distributed the symbols 2, the other symbols, except possibly for the first one, are already determined. Following these rules the number of words of length n with k symbols 2 is less than or equal to $2\binom{n-k}{k}$. The factor 2 comes from the fact that if the first symbol in the word is not a 2, we have two choices: 1 or 3. The other term is the number of different ways of distributing the 2s (followed by 3s). According to this we can estimate:

$$\#I(\gamma, n) \leq \sum_{k=0}^{(1-\gamma)n} 2\binom{n-k}{k}. \tag{4}$$

LEMMA 3.2. *For each $n \in \mathbb{N}$, and $0 < k < n$ we have*

$$\binom{n-k}{k} \leq \frac{e^k}{\sqrt{2\pi k} \cdot e^{1/(12k+1)}} \cdot \left(\frac{n}{k} - 1\right)^k.$$

Proof. In fact,

$$\begin{aligned}\binom{n-k}{k} &= \frac{(n-k)!}{k!(n-2k)} = \frac{(n-k)(n-k-1)\cdots(n-k-(k+1))}{k!} \\ &\leq \frac{(n-k)^k}{k!}.\end{aligned}$$

By Stirling's formula, see [**4**], we have

$$k! = \sqrt{2\pi k}\left(\frac{k}{e}\right)^k e^{\lambda_k} > \sqrt{2\pi k}\left(\frac{k}{e}\right)^k e^{1/(12k+1)}.$$

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Then

$$\binom{n-k}{k} \le \frac{(n-k)^k}{\sqrt{2\pi k}(k/e)^k e^{1/(12k+1)}} = \frac{e^k}{\sqrt{2\pi k}\cdot e^{1/(12k+1)}} \cdot \left(\frac{n}{k}-1\right)^k,$$

which finishes the proof of the lemma. □

LEMMA 3.3. *For n large enough and $0 < \alpha < (\sqrt{2}-1)/2$ we have*

$$\binom{n-k}{k} < \binom{n-(k+1)}{k+1} \quad \text{for all } k \le [\alpha n].$$

Proof. In fact, we have that the inequality

$$\binom{n-k}{k} = \frac{(n-k)!}{k!\cdot(n-2k)!} < \frac{(n-(k+1))!}{(k+1)!(n-2(k+1))!} = \binom{n-(k+1)}{k+1}$$

holds, if and only if

$$\frac{(n-k)\cdot(k+1)}{(n-2k)\cdot(n-2k-1)} < 1.$$

If we have $n^2 - 4nk - n + 4k^2 + 2k > 0$, then the previous inequality is equivalent to the following:

$$-5k^2 + (5n-3)k + (-n^2+2n) < 0.$$

Since this is always true, it is enough to obtain conditions for $n^2 - 4nk - n + 4k^2 + 2k > 0$. By taking $k \le [\alpha n]$ we get

$$n^2 - 4kn \ge n^2 - 4\alpha n^2 - n > 4\alpha^2 n^2 + 2\alpha n \ge 4k^2 + 2k,$$

where the first and the last inequalities always hold. To get the second inequality, it is enough to prove that

$$n^2(-4\alpha - 4\alpha^2 + 1) - n(1-2\alpha) > 0.$$

Taking $0 < \alpha < (\sqrt{2}-1)/2$, the coefficient of n^2 is positive, and the inequality is true for n large enough. □

Let $0 < \alpha < (\sqrt{2}-1)/2$ and consider $\alpha = \alpha(\gamma) = 1-\gamma$. Then

$$\#I(\gamma, n) \le 2[\alpha n]\binom{n-[\alpha n]}{[\alpha n]}.$$

Applying Lemma 3.2 we obtain

$$\#I(\gamma, n) \le \alpha n \cdot \frac{e^{\alpha n}}{\sqrt{2\pi\alpha n}\cdot e^{1/(12\alpha n+1)}} \cdot \left(\frac{n}{\alpha n}-1\right)^{\alpha n} \le \alpha n \cdot \frac{e^{\alpha n}}{\sqrt{2\pi\alpha n}\cdot e^{1/(12\alpha n+1)}} \cdot \left[\left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right]^n.$$

Note that, if we make $\gamma \to 1$, we have $\alpha \to 0$ and hence $((1-\alpha)/\alpha)^\alpha$ converges to 1. Then we can establish the following proposition.

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PROPOSITION 3.2. *If $\beta = (\log \omega)/2$, then there exists $0 < \gamma < 1$ which satisfies*

$$\#I(\gamma, n) \le e^{\beta n} \quad \textit{for } n \textit{ large enough.}$$

Proof. We consider

$$\begin{aligned} f(\alpha, n) &:= \frac{1}{n} \ln\left(\alpha n \frac{e^{\alpha n}}{\sqrt{2\pi\alpha n} e^{1/(12\alpha n+1)}} \left(\frac{1-\alpha}{\alpha}\right)^{\alpha n} \right) \\ &= \frac{1}{n}\left(\frac{\ln \alpha + \ln n}{2} - \frac{1}{12\alpha n + 1} - \frac{\ln(2\pi)}{2} \right) \\ &\quad + \alpha + \alpha \ln(1-\alpha) - \alpha \ln \alpha. \end{aligned}$$

Taking the limit in n we have

$$h(\alpha) := \lim_{n\to\infty} f(\alpha, n) = \alpha + \alpha \ln(1-\alpha) - \alpha \ln \alpha = \alpha + \alpha \ln\left(\frac{1-\alpha}{\alpha}\right).$$

Since $\lim_{\alpha\to 0} h(\alpha) = 0$ and $h((\sqrt{2}-1)/2) > 0.48$, there exists $\alpha_0 \in (0, (\sqrt{2}-1)/2)$ with $h(\alpha_0) = (\log(\omega))/4$ since $(\log \omega)/4 < 0.3$. We have that $f(\alpha, n)$ converges to $h(\alpha)$ uniformly on any compact subset of $]0, 1[$. Then, for n sufficiently large, $|f(\alpha_0, n) - h(\alpha_0)| < (\log \omega)/4$.

Thus, $f(\alpha_0, n) < (\log \omega)/2$ and, therefore, there exists $\gamma = \gamma(\alpha_0)$ satisfying the lemma. □

We point out that if γ satisfies

$$\sigma^{-(1-\gamma)} \cdot \left(\frac{1}{e}\right)^{\gamma} > 1, \tag{5}$$

there is $c > 0$ such that

$$\sigma^{-(1-\gamma)} \cdot \left(\frac{1}{e}\right)^{\gamma} > e^{4c} > 1, \tag{6}$$

for that it is enough to consider $0 < \gamma < (\log(\sigma^{-1}))/(\log(\sigma^{-1} + 1))$.

3.2. *Positive proportion of hyperbolic times.* From now on we will fix $c > 0$ as in equation (6) and we will omit c and G in the notation, calling (G, c)-hyperbolic time simply a hyperbolic time.

In our setting we can prove that good admissible words are associated with cylinder sets with positive proportion of hyperbolic times. For this, we use the well-known Pliss lemma (see [**19**]).

PROPOSITION 3.3. *There exists $\theta > 0$ such that for every $n \ge 1$ and every admissible word $(a_0, \ldots, a_{n-1}) \notin I(\gamma, n)$ there are $l > \theta n$ and numbers $1 \le n_1 < \cdots < n_l \le n$ such that $R^{n_j}[i_0, \ldots, i_{n_j-1}]$ is hyperbolic for all $j = 1, \ldots, l$.*

Proof. Given $i = 1, 2, 3$, consider $b_i = \inf\{\log \|DG(x)^{-1}\|^{-1} \mid x \in R_i\}$. Since we have in each rectangle the following:

$$\begin{aligned} \|DG(x)^{-1}\| &= \sigma \quad \text{if } x \in R_2, \\ \|DG(x)^{-1}\| &\le e \quad \text{if } x \in R_1 \text{ or } R_3, \end{aligned}$$

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we can estimate b_i by

$$b_i \geq \begin{cases} \log\left(\frac{1}{e}\right) = -1 & \text{if } i = 1, 3, \\ \log(\sigma^{-1}) & \text{if } i = 2. \end{cases}$$

Defining $a_t = b_{i_t}$ where $t = 0, \ldots, n-1$, and since $(i_0, \ldots, i_{n-1})$ belongs to the complement of $I(\gamma, n)$, we have

$$\begin{aligned} a_1 + \cdots + a_{n-1} &= \sum_{t=0}^{n-1} b_{i_t} \geq (1-\gamma)n \log(\sigma^{-1}) + \gamma n(-1) \\ &> 4cn \end{aligned}$$

where the last inequality follows from the choice of c (see equation (6)).

Let $A = \sup_{x\in\Lambda}\{\log \|DG^{-1}(x)\|^{-1}\}$. Then $a_t \leq A$ for every $t = 0, \ldots, n-1$. We consider $c_1 = 2c$ and $c_2 = 4c$ and then by the Pliss lemma, there exist $l > \theta n$ with $\theta = 2c/(A - 2c)$ and l integer numbers $1 \leq n_1 < \cdots < n_l \leq n$ for which the following inequality holds

$$a_k + \cdots + a_{n_j} \geq 2c(n_j - k) \quad \text{for all } 0 \leq k \leq n_j - 1, \ \ j = 1, \ldots, l.$$

Therefore, for each $x \in R^{n_j}$ we have

$$\begin{aligned} \log\left(\prod_{i=k}^{n_j-1} \|DG(G^i(x)^{-1})\|^{-1}\right) &= \sum_{i=k}^{n_j-1} \log \|DG(G^i(x)^{-1})\|^{-1} \\ &\geq \sum_{i=k}^{n_j-1} a_i \geq 2c(n_j - k). \end{aligned}$$

Thus, n_j is a hyperbolic time for x. Since this holds for all $x \in R^{n_j}$ and all $j = 1, \ldots, l$ we prove the result. □

We will prove a type of distortion condition on Birkhoff sums of the potential ϕ. In the case of a system given by a hyperbolic map and a Hölder continuous function, this property holds for every cylinder, and is referred to as the Bowen property. In our setting, we prove that it holds for hyperbolic cylinders.

PROPOSITION 3.4. *There exists a constant $K_0 > 0$ such that if x, y belong to a hyperbolic cylinder R^n, the following holds:*

$$|S_n\phi(x) - S_n\phi(y)| \leq K\, d(x, y)^\delta \leq K_0.$$

Proof. Assume ϕ is Hölder continuous with constants (C, δ). Then, given $x, y \in R^n$, a hyperbolic cylinder, we have

$$\begin{aligned} |S_n\phi(x) - S_n\phi(y)| &= \left|\sum_{j=1}^{n} \phi(G^{n-j}(x)) - \phi(G^{n-j}(y))\right| \\ &\leq \sum_{j=1}^{n} |\phi(G^{n-j}(x)) - \phi(G^{n-j}(y))| \\ &\leq C \sum_{j=1}^{n} d(G^{n-j}(x), G^{n-j}(y))^\delta. \end{aligned}$$

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Since x and y belong to the same hyperbolic cylinder, by Remark 3.1 for each $1 \le j \le n$ we have

$$d(G^{n-j}(x), G^{n-j}(y))^{\delta} \le e^{-jc}.$$

Then, there is a constant $K_0 > 0$ which depends only on δ, c and C which satisfies

$$|S_n\phi(x) - S_n\phi(y)| \le K_0. \qquad \square$$

3.3. *Jacobians and conformal measures.* In this section we define a Jacobian of a probability measure. Then we derive some properties for a Jacobian of the reference measure ν obtained as an eigenmeasure for the transfer operator. This will allow us to prove that the closure of the set of points that belong to infinitely many hyperbolic cylinders has full measure with respect to ν.

Definition 3.2. Let η be a probability measure on Λ. We say that a function $J_\eta G$ is a *Jacobian* of η with respect to G if it satisfies

$$\eta(G(A)) = \int_A J_\eta G \, d\eta,$$

for all measurable sets A such that $G|_A$ is injective.

In other words, given the measure η, $G_*\eta(B) = \eta(G(B))$ for all measurable subsets $B \in \Lambda$.

The Radon–Nikodym theorem ensures that the Jacobian is essentially unique. Further, it coincides with the Radon–Nikodym derivative of the measure $G_*\eta$ with respect to η, $J_\eta G = dG_*\eta/d\eta$. We will get a useful property regarding the Jacobian with respect to an iterate of G.

PROPOSITION 3.5. *If the Jacobian of μ with respect to G, $J_\mu G$, exists, then for each $n \in \mathbb{N}$ the map G^n also has a Jacobian and it is given by*

$$J_\mu G^n = \prod_{j=0}^{n-1} J_\mu G \circ G^j.$$

Proof. For each function $\psi : \Lambda \to \mathbb{R}$ we define the support of ψ by $\operatorname{supp}(\psi) = \{x \in \Lambda \mid \psi(x) \neq 0\}$.

For topological spaces, one can establish another definition of Jacobian which is equivalent to that given in the Definition 3.2. We say that $J_\mu G$ is a Jacobian of G with respect to μ, if for any open set U satisfying that $G|_U$ is injective, and for all $\psi \in \mathcal{C}(\Lambda)$ such that $\operatorname{supp}(\psi) \subset G(U)$, the following holds:

$$\int_{G(U)} \psi \, d\mu = \int_U (\psi \circ G) J\mu G \, d\mu.$$

Using this definition we can prove the proposition. We only need to treat the case $n = 2$ since the general case follows by induction. Given an open set U, suppose that $G|_U$ is

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injective and ψ is such that $\operatorname{supp}(\psi) \subset G^2(U)$. We have to show that $J_\mu G^2 = J_\mu G \cdot J_\mu G \circ G$. In fact, we have

$$\begin{aligned}\int_{G^2(U)} \psi \, d\mu &= \int_{G(U)} (\psi \circ G) J_\mu G \, d\mu \\ &= \int_U [[(\psi \circ G) \cdot J_\mu G] \circ G] \cdot J_\mu G \, d\mu \\ &= \int_U [(\psi \circ G^2) \cdot (J_\mu G \circ G)] \cdot J_\mu G \, d\mu. \end{aligned}$$ □

In the next lemma we compute the Jacobian of the dual operator eigenmeasure.

PROPOSITION 3.6. *If the spectral radius of $\mathcal{L}_\phi$, $\lambda = r(\mathcal{L}_\phi)$, is an eigenvalue associated with an eigenmeasure ν, then we have*

$$J_\nu G = \lambda e^{-\phi}.$$

Proof. Given A which satisfies $G_{|A}$ is injective, we have to show that $\nu(G(A)) = \int_A \lambda e^{-\phi} \, d\nu$. We have

$$\begin{aligned}\int_A \lambda e^{-\phi} \, d\nu &= \int e^{-\phi} \cdot \chi_A \, d\lambda\nu \\ &= \int e^{-\phi} \cdot \chi_A \, d\mathcal{L}_\phi^* \nu \\ &= \int \mathcal{L}_\phi(e^{-\phi} \cdot \chi_A) \, d\nu. \end{aligned}$$

Let χ_A denote the characteristic function on A. By the injectivity of G on A we get

$$\begin{aligned}\mathcal{L}_\phi(e^{-\phi} \cdot \chi_A)(x) &= \sum_{y \in G^{-1}(x)} e^{\phi(y)} \cdot e^{-\phi(y)} \cdot \chi_A(y) \\ &= \sum_{y \in G^{-1}(x)} \chi_A(y) = \chi_{G(A)}(x). \end{aligned}$$

Thus,

$$\int_A \lambda e^{-\phi} \, d\nu = \int \chi_{G(A)}(x) \, d\nu = \nu(G(A)).$$ □

A measure which has a Jacobian is said to be conformal with respect to its Jacobian or a J-conformal measure. In the case of a reference measure ν associated with a Perron–Frobenius operator and a potential ϕ we simply say that ν is a *conformal measure* when its Jacobian depends on ϕ as in the last proposition. For more about conformal measures see e.g. [**10**].

Combining Propositions 3.6 and 3.5 we obtain the following result.

LEMMA 3.4. *There exists $K_1 > 0$ satisfying*

$$K_1^{-1} \leq \frac{J_\nu G^n(x)}{J_\nu G^n(y)} \leq K_1,$$

for all $x, y \in R^n$, where R^n is a hyperbolic cylinder.

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Proof. Given $n \in \mathbb{N}$, by Proposition 3.5 the Jacobian of ν with respect to G^n can be calculated using the Jacobian of ν with respect to G. Thus, by Proposition 3.6 for each $x \in \Lambda$:

$$\begin{aligned} J_\nu G^n(x) &= \prod_{j=0}^{n-1} J_\mu G \circ G^j = \prod_{j=0}^{n-1} \lambda e^{-\phi(G^j(x))} \\ &= \lambda^n e^{-\phi(x)} \cdot e^{-\phi(G(x))} \cdots e^{-\phi(G^{n-1}(x))} \\ &= \lambda^n e^{-S_n\phi(x)}. \end{aligned}$$

Consider R^n a hyperbolic cylinder of length n and $x, y \in R^n$. By Proposition 3.4:

$$\left|\log\left(\frac{J_\nu G^n(x)}{J_\nu G^n(y)}\right)\right| = |S_n\phi(x) - S_n\phi(y)| \le K_0.$$

To finish the proof, consider $K_1 = e^{K_0}$. □

Given $n \in \mathbb{N}$, let B^n be the collection of cylinders $R^n(a_0, \ldots, a_{n-1})$ generated by words $(a_0, \ldots, a_{n-1}) \notin I(\gamma, n)$.

Let P^n be the collection of cylinders generated by words of length n belonging to $I(\gamma, n)$.

PROPOSITION 3.7. *The measure of the cylinders in P^n, with respect to ν, decreases exponentially when n goes to infinity. Therefore, ν almost every point $x \in \Lambda$ belongs to elements of B^n for infinitely many values of n.*

Proof. Let $n \in \mathbb{N}$ and let $R^n \in P^n$ be a cylinder. Then we have

$$\begin{aligned} \nu(G^n(R^n)) &= \int_{R^n} J_\nu G^n \, d\nu \\ &= \int_{R^n} \lambda^n e^{-S_n\phi} \, d\nu \\ &\ge \lambda^n e^{-n \sup \phi} \nu(R^n). \end{aligned}$$

Since $\nu(G^n(R^n)) \le 1$ it follows that $\nu(R^n) \le \lambda^{-n} e^{n \sup \phi}$. If we take the sum over all cylinders generated by words $(a_0, \ldots, a_{n-1}) \in I(\gamma, n)$ and use Proposition 3.2, we obtain

$$\sum_{I(\gamma,n)} \nu(R^n) \le \lambda^{-n} e^{\sup \phi + \beta}.$$

On the other hand, Lemma 3.1 gives us $\lambda \ge \exp(\inf \phi + \log \omega)$. Since $\beta = \log(\omega)/2$, we prove the result. □

We use the notation R_h^n to refer to hyperbolic cylinders of length n. In Proposition 3.3 we obtained plenty of hyperbolic times associated with words which do not belong to the set $I(\gamma, n)$.

Let $\mathcal{H}$ be the closure of the set of points $x \in \Lambda$ which belong to infinitely many hyperbolic cylinders R_h^n. Next, we prove that $\mathcal{H}$ is a full measure set with respect to ν.

PROPOSITION 3.8. *Let $n_1(x)$ be the first hyperbolic time of x. We have that n_1 is integrable and $\nu(\mathcal{H}) = 1$.*

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Proof. By Proposition 3.3, for each $x \in B^n$ we have $n_1(x) \leq n$. Therefore, if x is such that $n_1(x) > n$ we have $x \in P^n$. Then

$$\int n_1 \, d\nu = \sum_{n=0}^{\infty} \nu(\{x : n_1(x) > n\}) \leq 1 + \sum_{n=1}^{\infty} \nu(P^n).$$

According to Proposition 3.7, the measure of the sets in P^n decreases exponentially with n, which implies that the sum in the last inequality is finite. Then, the first hyperbolic time is integrable with respect to ν. Since B^n consists of cylinders generated by words in the complement of $I(\gamma, n)$, Proposition 3.3 guarantees that every point belongs to at least l hyperbolic cylinders, where $l > \theta n$. In addition, by Proposition 3.7, for n large enough, ν almost every point belongs to a set in B^n. Thus, ν almost every point x in Λ belongs to infinitely many hyperbolic cylinders, and $\nu(\mathcal{H}) = 1$. □

For each $x \in \mathcal{H}$, let $n_k(x)$ be the kth hyperbolic time of x. Consider also θ as in Proposition 3.3. According to the proof of Proposition 3.8 we have the following corollary.

COROLLARY 3.1. *For ν every point $x \in \Lambda$ and n large enough the following holds:*

$$\#\{k \geq 1 : n_k(x) \leq n\} \geq \theta n.$$

In the next proposition we estimate the measure of hyperbolic cylinders with respect to ν.

PROPOSITION 3.9. *There exists a constant $K_2 > 0$ such that for every $x \in R_h^n$, a hyperbolic cylinder of length n, and every $n \in \mathbb{N}$, we have:*

$$K_2^{-1} \leq \frac{\nu(R^n)}{\exp(S_n\phi(x) - Pn)} \leq K_2,$$

where $P = \log \lambda$.

Proof. Given $n \in \mathbb{N}$, $R^n = R^n(a_0, a_1, \ldots, a_{n-1})$ and $x \in R^n$. By Proposition 3.1 we have

$$(\mathcal{L}_\phi^*)^n \nu = \lambda^n \nu \quad \text{for all } n \in \mathbb{N}.$$

And then, we obtain

$$\begin{aligned} e^{Pn}\nu(R^n) &= e^{Pn} \int \chi_{R^n} \, d\nu \\ &= \frac{e^{Pn}}{\lambda^n} \int \mathcal{L}_\phi^n(\chi_{R^n}) \, d\nu \\ &= \int \mathcal{L}_\phi^n(\chi_{R^n}) \, d\nu, \end{aligned}$$

where χ_{R^n} denotes the characteristic function on R^n and $e^P = \lambda$. According to the transfer operator definition, we can write

$$\mathcal{L}_\phi^n \chi_{R^n}(x) = \sum_{y \in G^{-n}(x)} e^{S_n\phi(y)} \chi_{R^n}(y) = \sum_{y \in G^{-n}(x) \cap R^n} e^{S_n\phi(y)}.$$

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Therefore, given $n \in \mathbb{N}$ for each $x \in R^n$ there exists a unique $y \in G^{-n}(x) \cap R^n(x)$. We define $A(n) = \{y \in G^{-n}(x) \cap R^n\}$ and then

$$\left| \frac{\nu(R^n)}{\exp(S_n\phi(x) - Pn)} \right| = \left| \frac{\int_{A(n)} e^{S_n\phi}\, d\nu}{e^{S_n\phi(x)}} \right|.$$

By Proposition 3.4 there exists a constant $K_0 > 0$ satisfying

$$|S_n\phi(x) - S_n\phi(y)| < K_0,$$

for all x, y in the same hyperbolic cylinder, which is enough to finish the proof. □

3.4. *Transfer operator eigenfunction.* In this section we will show the existence of an eigenfunction of the Ruelle–Perron–Frobenius transfer operator. We will, therefore, show the second part of Theorem C and finish its proof. In addition, we will show that any eigenfunction is positive and Hölder continuous.

THEOREM 3.1. *There exists a positive eigenfunction $h : \Lambda \longrightarrow \mathbb{R}$ for the operator $\mathcal{L}_\phi$ associated with the eigenvalue $\lambda = r(\mathcal{L}_\phi)$:*

$$\mathcal{L}_\phi h = \lambda h.$$

We define, for each $n \in \mathbb{N}$, the function $T_n : \Lambda \longrightarrow \mathbb{R}$ by

$$T_n(x) = \sum_{\substack{y \in G^{-n}(x) \cap R^n \\ R^n \text{ hyperbolic}}} e^{S_n\phi(y)} \quad \text{for all } x \in \Lambda.$$

We also define, for each $n \in \mathbb{N}$,

$$h_n = \frac{1}{n} \sum_{i=1}^{n-1} \lambda^{-i} T_i.$$

Consider the following sequence of real numbers

$$Z_n = \sum_{R^n \text{ hyperbolic}} e^{\sup\{S_n\phi(x) | x \in R^n\}}.$$

Note that for every $x \in \Lambda$ we have

$$T_n(x) \le Z_n.$$

LEMMA 3.5. *There exists a constant $K_3 > 0$ satisfying*

$$\lambda^{-n} T_n \le \lambda^{-n} Z_n \le K_3 \quad \textit{and} \quad K_3^{-1} \le \frac{1}{n} \sum_{j=0}^{n-1} \lambda^{-j} Z_j.$$

Proof. The first inequality on the left-hand side is trivial because $\lambda > 0$. So, it is enough to get an upper bound for $\lambda^{-n} Z_n$. Given $n \in \mathbb{N}$, R_h^n a hyperbolic cylinder and $y \in R_h^n$ by Proposition 3.9 we have

$$K_2^{-1} \nu(R^n) \le \lambda^{-n} e^{S_n\phi(y)} \le K_2 \nu(R^n).$$

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By taking the supremum over every $y \in R^n$ and taking the sum over all hyperbolic cylinders R^n of length n, we get

$$K_2^{-1} \sum_{R^n \text{ hip.}} \nu(R^n) \le \sum_{R^n \text{ hip.}} \lambda^{-n} e^{\sup S_n \phi} \le K_2 \sum_{R^n \text{ hip.}} \nu(R^n). \tag{7}$$

Thus,

$$\lambda^{-n} Z_n \le K_2 \cdot \nu\Big(\bigcup_{R^n \text{ hip}} R^n \Big) \le K_2.$$

Now, to prove the second part of the lemma, we consider the first inequality in (7), to obtain

$$K_2^{-1} \frac{1}{n} \sum_{j=0}^{n-1} \nu\Big(\bigcup_{R^j \text{ hip}} R^j \Big) \le \frac{1}{n} \sum_{j=0}^{n-1} \lambda^{-j} Z_j.$$

For every $j \in \mathbb{N}$ let $\mathcal{R}_j := \bigcup_{R^j_{\text{hip}}} R^j$ be the union of hyperbolic cylinders of length j. According to Corollary 3.1, for almost every point $x \in \Lambda$ and n large enough, we have

$$\begin{aligned} \frac{1}{n} \sum_{j=0}^{n-1} \nu(\mathcal{R}_j) &= \int \int \chi_{\mathcal{R}_j}(x)\, d\nu(x)\, dm_n(j) \\ &= \int \int \chi_{\mathcal{R}_j}(x)\, dm_n(j)\, d\nu(x) \ge \theta > 0. \end{aligned}$$

Note that we use Fubini's theorem in the second equality. To finish the proof take $K_3 = K_2/\theta$. □

We will prove that $\{h_n\}$ is an equicontinuous family. Let δ be the Hölder constant of the potential ϕ in Theorem B.

LEMMA 3.6. *There exists a constant $K_4 > 0$ such that for every $n \in \mathbb{N}$, the function $\lambda^{-n} T_n$ is Hölder continuous with constants K_4 and δ.*

Proof. Since the rectangles R_i, $i = 1, 2$ or 3, are mutually disjoint, it is enough to consider points in the same rectangle. Let $n \in \mathbb{N}$ and $x_1, x_2 \in R_i$. Then $|\lambda^{-n} T_n(x_1) - \lambda^{-n} T_n(x_2)|$ is equal to

$$\left| \lambda^{-n} \Big(\sum_{y \in G^{-n}(x_1) \cap R_h^n} e^{S_n \phi(y)} - \sum_{z \in G^{-n}(x_2) \cap R_h^n} e^{S_n \phi(z)} \Big) \right|.$$

We group terms whose pre-images of x_1 and x_2 belong to the same hyperbolic cylinder. Thus, if $y \in G^{-n}(x_1)$ and $z \in G^{-n}(x_2)$ belong to the same hyperbolic cylinder of length n, by Proposition 3.4

$$|S_n \phi(y) - S_n \phi(z)| < K_0 d(x_1, x_2)^{\delta}.$$

On the other hand, we can write

$$\begin{aligned} |e^{S_n \phi(y)} - e^{S_n \phi(z)}| &\le e^{\max\{S_n \phi(y), S_n \phi(z)\}} |S_n \phi(y) - S_n \phi(z)| \\ &\le e^{\max\{S_n \phi(y), S_n \phi(z)\}} K_0 d(x_1, x_2)^{\delta}. \end{aligned}$$

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By definition of Z_n

$$\sum_{y\in G^{-n}(x_1)\cap R_h^n} e^{S_n\phi(y)} - \sum_{z\in G^{-n}(x_2)\cap R_h^n} e^{S_n\phi(z)}$$
$$\leq d(x_1, x_2)^\delta K_0 \sum_{z,y\in R_h^n} e^{\max\{S_n\phi(y), S_n\phi(z)\}} \leq d(x_1, x_2)^\delta K_0 Z_n.$$

Thus

$$|\lambda^{-n}T_n(x_1) - \lambda^{-n}T_n(x_2)| \leq \lambda^{-n} Z_n K_0 d(x_1, x_2)^\delta \leq K_4 d(x_1, x_2)^\delta,$$

where we consider K_3 as in Lemma 3.5 and $K_4 = K_0 K_3$. □

PROPOSITION 3.10. *The sequence $|\lambda^{-k}(\mathcal{L}_\phi T_k - T_{k+1})|$ converges uniformly to zero when k goes to infinity.*

Proof. Given $k \geq 0$ and $x \in \Lambda$, by definition we have

$$\mathcal{L}_\phi T_k(x) = \sum_{y\in G^{-1}(x)} e^{\phi(y)} T_k(y)$$
$$= \sum_{y\in G^{-1}(x)} e^{\phi(y)} \sum_{z\in G^{-k}(y)\cap R_h^k} e^{S_k\phi(z)}.$$

We consider the set A_{k+1} of cylinders $R^{k+1}[a_0, a_1, \ldots, a_k]$ that are not hyperbolic, such that $R^k[a_1, \ldots, a_k]$ are hyperbolic. In other words, A_{k+1} is the set of every cylinder of length $k+1$, such that considering the k last symbols which generate the cylinder, we have that the cylinder of length k is hyperbolic but the cylinder of length $k+1$ is not hyperbolic.

Note that there is no $0 \leq i \leq k$ that is a hyperbolic time for every element of R^{k+1}. Otherwise, by concatenation, R^{k+1} would be hyperbolic, which does not happen. By definition of T_k

$$|\lambda^{-k}(\mathcal{L}_\phi T_k - T_{k+1})| \leq \left|\lambda^{-k} \sum_{\substack{z\in G^{-1}(y)\cap A \\ A\in A_{k+1}}} e^{S_{k+1}\phi(z)}\right|$$
$$\leq |\lambda^{-k} \cdot \#A_{k+1} \cdot e^{(k+1)\sup\phi}|.$$

However, $A_{k+1} \subset \{R^{k+1}(a_0, \ldots, a_k)\backslash(a_0, \ldots, a_k) \in I(\gamma, k+1)\}$, where γ is given by Proposition 3.2. In fact, since no cylinder $R^i(a_0, \ldots, a_i)$ is hyperbolic, the proportion of indices $i \in \{0, \ldots k+1\}$ such that $(a_0, \ldots, a_{k+1})$ is hyperbolic is zero, thus smaller than θ. See Proposition 3.1. Then, by Proposition 3.2 we have

$$\#A_{k+1} \leq e^{\beta(k+1)} \quad \text{for } k \text{ large enough.}$$

Remember that we take $\beta = \log(\omega)/2$. From Lemma 3.1, we have that $\lambda \geq e^{\inf\phi + \log\omega}$, thus

$$|\lambda^{-k}(\mathcal{L}_\phi T_k - T_{k+1})| \leq e^{-k(\inf\phi+\log\omega)} \cdot e^{\beta(k+1)} \cdot e^{(k+1)\sup\phi}$$
$$= e^{-k(-\beta-(\sup\phi-\inf\phi)+\log\omega)} \cdot e^{\beta+\sup\phi}.$$

Since $\sup\phi - \inf\phi < \beta$ we have that the last term in the inequality converges exponentially to zero when k goes to ∞. □

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Recall that

$$h_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} \lambda^{-i} T_i(x) \quad \text{for all } x \in \Lambda.$$

According to Lemma 3.6 every function $\lambda^{-n}T_n$ is $(K_4, \delta)-$ Hölder continuous, but since h_n is the average of the functions $\lambda^{-n}T_n$ we have that h_n is also (K_4, δ) Hölder continuous and, then, the family $(h_n)_{n\in\mathbb{N}}$ is equicontinuous.

PROPOSITION 3.11. *The sequence of functions $(h_n)_{n\in\mathbb{N}}$ has at least one accumulation point. Moreover, any accumulation point h satisfies*

$$\mathcal{L}_\phi h = \lambda h.$$

Proof. Lemma 3.5 ensures that the h_n is uniformly bounded, since it is also equicontinuous, by the Ascoli–Arzelá theorem we have existence of an accumulation point. Let h denote one accumulation point:

$$h = \lim_{k\to\infty} h_{n_k}.$$

Then we have

$$\begin{aligned}
\mathcal{L}_\phi h &= \mathcal{L}_\phi\Big(\lim_{k\to\infty} h_{n_k}\Big) = \lim_{k\to\infty} \mathcal{L}_\phi(h_{n_k}) \\
&= \lim_{k\to\infty} \frac{1}{n_k}[\mathcal{L}_\phi T_0 + \lambda^{-1}\mathcal{L}_\phi T_1 + \cdots + \lambda^{-n_k+1}\mathcal{L}_\phi T_{n_k-1}] \\
&= \lim_{k\to\infty}\Bigg[\frac{1}{n_k}[\lambda(T_0 + \cdots + \lambda^{n_k-1}T_{n_k-1}) - \lambda T_0 + \lambda^{-n_k+1}T_{n_k}] \\
&\quad - \frac{1}{n_k}\sum_{i=1}^{n_k-1} \lambda^{-i}(\mathcal{L}_\phi T_i - T_{i+1})\Bigg].
\end{aligned}$$

By Proposition 3.10 the last term converges to zero. Since, by Lemma 3.5, $\lambda^{-n}T_n$ is uniformly bounded, we have

$$\mathcal{L}_\phi h = \lim_{k\to\infty} \frac{1}{n_k}[\lambda(T_0 + \cdots + \lambda^{n_k-1}T_{n_k-1})] = \lambda h. \qquad \square$$

LEMMA 3.7. *There exists a constant $K_5 > 0$ such that, for every $n \in \mathbb{N}$ and for every $x \in \Lambda$,*

$$Z_n \le K_5 T_n(x).$$

Proof. Let $x \in \Lambda$ and $n \ge 1$, and fix $m > 3$. Note that each rectangle R_i, $i = 1, 2$ or 3, covers Λ after, at most, three iterations. Let $y, z \in G^{-n}(x) \cap R_h^n$. By Proposition 3.4, since y, z belong to the same hyperbolic cylinder, we have

$$e^{S_n\phi(z)} \le e^{K_0} e^{S_n\phi(y)}.$$

On the other hand, one can write

$$S_n\phi(z) = S_{n+m}\phi(z) - S_m\phi(G^n(z)) \le S_{n+m}\phi(z) - m \cdot \inf \phi.$$

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Fixing y and taking the supremum over z in the hyperbolic cylinder R^n, we get

$$e^{\sup S_n\phi|_{R^n}} \le e^{K_0} e^{-m\cdot\inf\phi} e^{S_{n+m}\phi(y)}.$$

Note that each cylinder of length $n+m$ is included in a cylinder of length n. Adding over every hyperbolic cylinder of length n, we obtain

$$Z_n \le e^{K_0} e^{-m\cdot\inf\phi} \sum e^{S_{n+m}\phi(y)} \le e^{K_0} e^{-m\cdot\inf\phi} T_{n+m}(x).$$

Comparing $T_{n+m}(x)$ and $T_n(x)$, we have

$$\begin{aligned} T_{n+m}(x) &= \sum_{y\in G^{-(n+m)}(x)\cap R_h^{n+m}} e^{S_{n+m}\phi(y)} \\ &\le \sum_{y\in G^{-(n+m)}(x)\cap R_h^{n+m}} e^{S_n\phi(G^m(y))+S_m\phi(y)} \\ &\le \log(\omega) e^{m\sup\phi} \sum_{z\in G^{-(n)}(x)\cap R_h^{n}} e^{S_n\phi(z)} \\ &\le \log(\omega) e^{m\sup\phi} T_n(x). \end{aligned}$$

Let $K_5 = \log(\omega) e^{K_0 + m(\sup\phi - \inf\phi)}$. Thus,

$$Z_n \le K_5 T_n(x) \quad \text{for all } x \in \Lambda. \qquad \square$$

Using Lemma 3.5, we have

$$K_3^{-1} \le \frac{1}{n}\sum_{j=0}^{n-1} \lambda^{-j} Z_j \le K_5 \frac{1}{n}\sum_{j=0}^{n-1} \lambda^{-j} T_j(x) = K_5 h_n(x).$$

We just proved the following lemma.

LEMMA 3.8. *The eigenfunction h of the transfer operator obtained as an accumulation point of the sequence $(h_n)_{n\in\mathbb{N}}$ satisfies $h > 0$.*

In this section, we obtained an eigenfunction for the transfer operator, and an eigenmeasure for its dual, both associated with the same eingenvalue, and this ends the proof of Theorem C. Moreover, we proved that the eigenfunction h is positive.

4. *Equilibrium states for the map G*

Consider the probability measure $\mu := h\nu$, where ν is the reference measure from the previous section. More precisely, μ is defined on each Borelian A by

$$\mu(A) = \int_A h \, d\nu.$$

Since $h > 0$, the measures μ and ν are equivalent. We will prove that μ is an equilibrium state for G with respect to the potential ϕ. Some of the statements here are analogous to those in [**16**, **15**], we will adapt them to our context, omitting some of the proofs.

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4.1. *Properties of the measure $\mu = h\nu$.*

LEMMA 4.1. *The probability measure μ is invariant under G.*

Proof. We will show that every $\psi \in \mathcal{C}^0(\Lambda)$ satisfies

$$\int \psi \circ G \, d\mu = \int \psi \, d\mu.$$

In fact

$$\begin{aligned}
\int \psi \circ G \, d\mu &= \int (\psi \circ G) \cdot h \, d\nu && (8)\\
&= \frac{1}{\lambda} \int (\psi \circ G) \cdot h \, d\mathcal{L}^*\nu && (9)\\
&= \frac{1}{\lambda} \int \mathcal{L}_\phi[(\psi \circ G) \cdot h] \, d\nu && (10)\\
&= \frac{1}{\lambda} \int \psi \cdot \mathcal{L}_\phi(h) \, d\nu && (11)\\
&= \int \psi \cdot h \, d\nu = \int \psi \, d\mu. && (12)
\end{aligned}$$

Note that (9) and (10) follow from the fact that ν is an eigenmeasure and h is an eigenfunction. We get (11) from the definition of the transfer operator. To finish, (12) follows again from the fact that h is an eigenfunction and by the definition of μ. □

We say that a sequence (a_j) of integers is *non-lacunary* if it is increasing and satisfies $a_{j+1}/a_j \to 1$.

Definition 4.1. An invariant probability measure η is said to be a non-lacunary Gibbs measure if there exist a constant $K > 0$ and a real number P satisfying

$$K^{-1} \leq \frac{\eta(R^{a_j}(x))}{\exp(S_{a_j}\phi(x) - Pa_j)} \leq K,$$

for η almost every x and for a non-lacunary sequence $a_j = a_j(x)$, where $R^{a_j}(x)$ is the cylinder of length a_j containing x.

PROPOSITION 4.1. *The sequence of hyperbolic times $n_k(x)$ is a non-lacunary sequence for μ almost every $x \in \Lambda$.*

Proof. According to Proposition 3.8 we have that n_1 is ν-integrable. Since h is bounded, n_1 is μ-integrable. In addition, μ is invariant and $\mu(\mathcal{H}) = 1$, by Proposition 3.8. Following [**17**, Proposition 3.8], we have that $n_k(x)$ is non-lacunary for μ almost every point. □

PROPOSITION 4.2. *The probability μ is an invariant non-lacunary Gibbs measure.*

Proof. Combining Proposition 3.9 and the fact that h is bounded, both from below and above, we have that there exists $K > 0$ satisfying

$$K^{-1} \leq \frac{\mu(R^n)}{\exp(S_n\phi(x) - Pn)} \leq K,$$

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for every hyperbolic cylinder R^n and every $x \in R^n$. By Proposition 4.1, the sequence of hyperbolic times is a non-lacunary sequence and, then, the proposition is proved. □

LEMMA 4.2. *The Jacobian of μ with respect to G is given by*

$$J_\mu G = \frac{h \circ G}{h} J_\nu G.$$

In particular, $J_\mu G = \lambda((h \circ G)/h)e^{-\phi}$.

Proof. Given a Borelian A where G is injective, we have to show that

$$\mu(G(A)) = \int_A \frac{h \circ G}{h} J_\nu G.$$

According to Proposition 3.6 we have $J_\nu G = \lambda e^{-\phi}$. Consider $(\psi_n)_{\{n\geq 1\}}$ a sequence of continuous functions on Λ such that $\psi_n \to \chi_A\ \nu$ almost every point and $\|\psi_n\| \leq 2$ for all $n \geq 1$ where $\|\psi_n\| = \sup_{|x|=1} |\psi(x)|$. Thus,

$$\mathcal{L}_\phi(e^{-\phi}\psi_n)(x) = \sum_{y \in G^{-1}(x)} e^{\phi(y)} e^{-\phi(y)} \psi_n(y) = \sum_{y \in G^{-1}(x)} \psi_n(y).$$

Note that the last term converges to $\chi_{G(A)}(x)$ for μ almost every point x, since A is such that G is injective on A.

By the dominated convergence theorem we have

$$\mu(G(A)) = \int_\Lambda \chi_{G(A)}\, d\mu = \lim_{n\to+\infty} \int \mathcal{L}_\phi(e^{-\phi}\psi_n)\, d\mu.$$

Therefore, by Theorem C, we obtain

$$\begin{aligned}
\mu(G(A)) &= \lim_{n\to+\infty} \int_\Lambda \mathcal{L}_\phi(e^{-\phi}\psi_n)\, d\mu \\
&= \lim_{n\to+\infty} \int_\Lambda \mathcal{L}_\phi(e^{-\phi}\psi_n) h\, d\nu \\
&= \lim_{n\to+\infty} \int_\Lambda \mathcal{L}_\phi((h \circ G)e^{-\phi}\psi_n)\, d\nu \\
&= \lim_{n\to+\infty} \int_\Lambda (h \circ G)(e^{-\phi}\psi_n)\, d\mathcal{L}_\phi^* \nu \\
&= \lim_{n\to+\infty} \int_\Lambda \lambda(h \circ G)(e^{-\phi}\psi_n)\, d\nu \\
&= \int_\Lambda \lambda(h \circ G) \cdot e^{-\phi} \cdot \chi_A\, d\nu \\
&= \int_A \lambda \cdot \frac{(h \circ G)}{h} \cdot e^{-\phi}\, d\mu,
\end{aligned}$$

which finishes the proof of the lemma. □

Let α be a partition of Λ and define for each $n \in \mathbb{N}$ a new partition

$$\alpha_n = \bigvee_{j=0}^{n-1} G^{-j}(\alpha).$$

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Namely, an element $\mathcal{P} \in \alpha_n$ is written as

$$\mathcal{P} = P_{i_0} \cup G^{-1}(P_{i_1}) \cup \cdots \cup G^{-1}(P_{i_{n-1}})$$

where $P_{i_k} \in \alpha$ for every $i_k = 0, \ldots, n-1$.

Fix $c > 0$ as in equation (6) and ϵ as in Remark 3.1.

LEMMA 4.3. *Let η be a invariant probability measure whose Lyapunov exponents are larger than $8c$, and α a partition with diameter smaller than ϵ. Then, for η almost every point $x \in \Lambda$, $\alpha_n(x)$ goes to zero when $n \to \infty$. In particular, α is a G-generating partition with respect to η.*

The proof is analogous to [**16**, Lemma 4.3]. We want to verify that the Lyapunov exponents of $\mu = h\nu$ satisfy the conditions of the last lemma.

LEMMA 4.4. *Let $(a_n) \in \Sigma_A$ be a recurrent sequence for the shift σ with $a_0 = 2$. Then there exist $k \in (0, 1)$ and an increasing subsequence of times $(a_{n_j})_{j \geq 0}$ such that*

$$|DG^{n_j}_{|E^c}(x)| \geq C \cdot k^j$$

where x belongs to the cylinder of length n_j generated by the first n_j terms of the sequence (a_n).

This statement is adapted from [**15**, Lemma 3.2] and its proof is analogous.

LEMMA 4.5. *The Lyapunov exponents of $\mu = h\nu$ are larger than $8c$.*

Now we state a version of the Roklin's formula proved by Oliveira and Viana in [**16**] to compute the metric entropy of G with respect to some measure, by integrating the log of the Jacobian of this measure with respect to G. See [**16**] for the proof.

Roklin's formula. If η is an invariant probability measure and there exists a G-generated partition with respect to η, then we have

$$h_\eta(G) = \int \log(J_\eta G)\, d\eta.$$

PROPOSITION 4.3. *If λ is the spectral radius of the transfer operator $\mathcal{L}_\phi$, then the following holds*

$$h_\mu(G) + \int \phi\, d\mu = \log \lambda = P.$$

Proof. Lemmas 4.5 and 4.3 imply that there exists a G-generating partition. Then, according to the Roklin's formula we have

$$h_\mu(G) = \int \log(J_\mu G)\, d\mu.$$

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On the other hand, Lemma 4.2 shows that $J_{\mu G} = \lambda((h \circ G)/h)e^{-\phi}$, which gives us

$$\begin{aligned} h_\mu(G) + \int \phi \, d\mu &= \int \log(J_\mu G) \, d\mu + \int \phi \, d\mu \\ &= \int \log\left(\lambda \frac{h \circ G}{h} e^{-\phi}\right) d\mu + \int \phi \, d\mu \\ &= \int \log \lambda \, d\mu + \int \log(h \circ G) \, d\mu - \int \log h \, d\mu \\ &= \int \log \lambda \, d\mu = \log \lambda = P. \end{aligned}$$

The fourth equality follows from the fact that the probability measure is invariant under G. □

We have to check that μ is an equilibrium state for the system (G, ϕ). Actually, any expanding non-lacunary Gibbs measure is an equilibrium state. This proof was given in [**17**] (combining Proposition 6.1 and Corollary 6.7) and still holds in our context. We will give the idea here. We first remember that

$$\sup\left\{h_\eta(G) + \int \phi \, d\eta\right\} = P_{\text{top}}(G, \phi) = \sup\{P_{\text{top}}(G, \Lambda \backslash \mathcal{H}), P_{\text{top}}(G, \mathcal{H})\},$$

where the last equality is a result due Pesin in [**18**], and $P_{\text{top}}(G, \Lambda\backslash\mathcal{H})$ denotes the topological pressure relative to the set $\Lambda\backslash\mathcal{H}$. Then, using the last proposition it is enough to prove that $P > P_{\text{top}}(G, \Lambda\backslash\mathcal{H})$ and $P = P_{\text{top}}(G, \mathcal{H})$. For $P > P_{\text{top}}(G, \Lambda\backslash\mathcal{H})$ see [**17**, Proposition 6.4]. To see that P coincides with the topological pressure relative to the set $\mathcal{H}$, we construct a family of partitions $\mathcal{U}^m$ of $\mathcal{H}$ formed by hyperbolic cylinders and whose diameters go to zero as $m \to \infty$. If $U \in \mathcal{U}^m$ is a hyperbolic cylinder of length n, then, by Proposition 4.2, we have

$$K^{-1} \leq \sum_{U \in \mathcal{U}} \exp(S_n \phi(U) - Pn) \leq K.$$

Note that changing P for an arbitrary $\alpha \in \mathbb{R}$, the second inequality above fails to be true for every n if $\alpha \neq P$. Then from the definition of topological pressure relative to a set, we have $P = P_{\text{top}}(G, \mathcal{H})$. Hence, μ is an equilibrium state for the system (G, ϕ).

4.2. *Uniqueness.* In this section we finish the proof of Theorem B, proving the uniqueness of the equilibrium state for the system (G, ϕ). For this we will compute the Jacobian of an equilibrium state. Applying Jensen's formula from basic calculus to the logarithm function, we have the following useful lemma.

LEMMA 4.6. *Let p_i and q_i be positive real numbers for each $i = 1, \ldots, n$. The next inequality holds:*

$$\frac{\sum_{i=1}^n p_i \log q_i}{\sum_{i=1}^n p_i} \leq \log\left(\frac{\sum_{i=1}^n p_i q_i}{\sum_{i=1}^n p_i}\right).$$

Moreover, the equality holds if and only if the q_i are all equal for every $i = 1, \ldots, n$.

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PROPOSITION 4.4. *Let η be an equilibrium state for the system (G, ϕ) of the Theorem B. The Jacobian of η is given by*

$$J_\eta G = \frac{h \circ G}{h} \lambda e^{-\phi}.$$

Proof. Since $\mu = h\nu$ is an equilibrium state (according to the previous section) it maximizes the topological pressure. If η is also an equilibrium state, we have

$$h_\eta(G) + \int \phi \, d\eta = h_\mu(G) + \int \phi \, d\mu = \log \lambda.$$

Note that the last equality follows from Lemma 4.3. Define the function g on Λ by

$$g(x) = \frac{h(x)e^{\phi(x)}}{\lambda(h \circ G)(x)}.$$

We want to show that $J_\eta G = 1/g$. By the definition of g we have

$$\begin{aligned} h_\eta(G) + \int \log g \, d\eta &= h_\eta(G) + \int \log\left(\frac{h(x)e^{\phi(x)}}{\lambda(h \circ G)(x)}\right) d\eta \\ &= h_\eta(G) + \int \phi \, d\eta - \log \lambda \\ &\quad + \int \log(h) - \log(h \circ G) \, d\eta \\ &= \int \log(h) - \log(h \circ G) \, d\eta = 0. \end{aligned}$$

We point out that the last equality follows from the invariance of η. On the other hand, using Roklin's formula we have

$$h_\eta G = \int \log J_\eta G \, d\eta.$$

Substituting in the previous equation, we have

$$\begin{aligned} 0 = h_\eta(G) + \int \log g \, d\eta &= \int \log\left(\frac{g}{(J_\eta G)^{-1}}\right) d\eta \\ &= \int \sum_{y \in G^{-1}(x)} (J_\eta G(y))^{-1} \log\left(\frac{g(y)}{(J_\eta G(y))^{-1}}\right) d\eta. \end{aligned}$$

Since η is G-invariant it yields

$$\sum_{y \in G^{-1}(x)} \frac{1}{J_\eta G(y)} = 1, \quad \eta \text{ almost every } x \in \Lambda. \tag{13}$$

Note that any $x \in \Lambda$ has, at most, two pre-images. Let $f_n(x)$ be the number of pre-images of x by G^n. Consider $y_i \in G^{-n}(x)$ with $i = 1, \ldots, f_n(x)$ and take

$$p_i = \frac{1}{J_\eta G(y_i)} \quad \text{and} \quad q_i = \frac{g(y_i)}{(J_\eta G)^{-1}(y_i)}.$$

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By equation (13), we have $\sum p_i = 1$. Applying Lemma 4.6 we have

$$\sum_{i=1}^{f_n(x)} p_i \log q_i \leq \log\left(\sum_{i=1}^{f_n(x)} p_i q_i\right)$$
$$= \log\left(\sum_{i=1}^{f_n(x)} g(y_i)\right) = \log 1 = 0.$$

Integrating with respect to η

$$0 = \int \sum_{i=1}^{f_n(x)} p_i \log q_i \, d\eta(x) \leq \int \log\left(\sum_{i=1}^{f_n(x)} p_i q_i\right) d\eta(x) = 0.$$

Then the equality holds. So Lemma 4.6 also guarantees that the q_i are equal. Thus, there exists a constant which depends on x, such that

$$q_i = \frac{g(y)}{(J_\eta G(y))^{-1}} = c(x) \quad \text{for all } y \in G^{-1}(x).$$

By (13), again, we have

$$1 = \frac{\sum_{y\in G^{-1}(x)} g(y)}{\sum_{y\in G^{-1}(x)} (J_\eta G(y))^{-1}} = c(x).$$

Then g coincides with $(J_\eta G)^{-1}$ in every pre-image of points in an η full measure set. Finally, the invariance of η implies that they coincide for η almost every point. □

As in the last Proposition let η be an equilibrium state for (G, ϕ). From Lemma 3.8 we have that $h > 0$, so the measure given by $\tau = (1/h)\eta$ is well-defined. We will prove that τ is an eigenmeasure for the dual transfer operator.

LEMMA 4.7. *Let η be an equilibrium state for the system (G, ϕ). Then the measure $\tau = (1/h)\eta$ is an eigenmeasure for $\mathcal{L}_\phi^*$ associated with the spectral radius of $\mathcal{L}_\phi$:*

$$\mathcal{L}_\phi^* \tau = \lambda \tau.$$

Proof. We will show that every continuous function $\psi : \Lambda \to \mathbb{R}$ satisfies

$$\int \psi \, d\mathcal{L}_\phi^* \tau = \lambda \int \psi \, d\tau.$$

In fact,

$$\int \psi \, d\mathcal{L}_\phi^* \tau = \int \mathcal{L}_\phi \psi \, d\tau$$
$$= \int \mathcal{L}_\phi \psi \cdot \frac{1}{h} \, d\eta$$
$$= \int \sum_{y\in G^{-1}(x)} \left[e^{\phi(y)} \psi(y) \cdot \frac{1}{h(G(y))} \right] d\eta(x).$$

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Since η is an equilibrium state, Proposition 4.4 ensures that for every y the Jacobian of η satisfies

$$\frac{e^{\phi(y)}}{h(G(y))} = \lambda \cdot \frac{1}{J_\eta G(y) h(y)}. \tag{14}$$

Thus,

$$\begin{aligned}\sum_{y\in G^{-1}(x)} \frac{1}{J_\eta G(y)} &= \sum_{y\in G^{-1}(x)} \frac{e^{\phi(y)}\cdot h(y)}{\lambda h(G(y))} \\ &= \frac{\sum_{y\in G^{-1}(x)} e^{\phi(y)}\cdot h(y)}{\lambda h(x)} = \frac{\mathcal{L}_\phi h(x)}{\lambda h(x)} = 1.\end{aligned}$$

The last equality comes from the fact that h is an eigenfunction of $\mathcal{L}_\phi$. According to the equation (14), we have

$$\begin{aligned}\int \psi \, d\mathcal{L}_\phi^* \tau &= \int \sum_{y\in G^{-1}(x)} \lambda \frac{\psi(y)}{J_\eta G(y) h(y)} \, d\eta(x) \\ &= \lambda \int \sum_{y\in G^{-1}(x)} \frac{1}{J_\eta G(y)} \frac{\psi(y)}{h(y)} \, d\eta(x) \\ &= \lambda \int \sum_{y\in G^{-1}(x)} \frac{\psi(y)}{h(y)} \, d\eta \\ &= \lambda \int \frac{\psi}{h} \, d\eta = \lambda \int \psi \, d\tau,\end{aligned}$$

which finishes the proof. □

As consequence of this result, any equilibrium state associated with (G, ϕ) is equivalent to an eigenmeasure so it is an expanding measure, i.e. it gives full measure to the set $\mathcal{H}$.

PROPOSITION 4.5. *Consider the system (G, ϕ) as in Theorem B. Any equilibrium state of G associated with the potential ϕ is a non-lacunary Gibbs probability measure.*

Proof. Let η be an equilibrium state. By Lemma 4.7 the measure $\tau = (1/h)\eta$ is an eigenmeasure of the dual operator $\mathcal{L}_\phi^*$ associated with the eigenvalue λ. Therefore, τ is a non-lacunary Gibbs measure. On the other hand, we can write $\eta = h\tau$. So Proposition 3.9 guarantees that τ is a non-lacunary Gibbs measure. □

LEMMA 4.8. *Let μ_1 and μ_2 be expanding non-lacunary Gibbs probability measures. Then μ_1 and μ_2 are equivalent.*

This result is proven in [**17**]. In our context, it follows easier since the rectangles have empty boundaries.

We are now able to finish the proof of Theorem B. Let η be an ergodic equilibrium state for the system (G, ϕ). Then η is expanding, and Proposition 4.5 ensures that η is a non-lacunary Gibbs invariant measure. On the other hand, the probability measure $\mu = h\nu$ (where h and ν are given by Theorem C) is also expanding and an invariant non-lacunary Gibbs measure. But Lemma 4.8 ensures that two expanding non-lacunary Gibbs

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probability measures are equivalent. Then μ is ergodic and we have $\mu = \eta$. This concludes the proof of Theorem B.

5. *Equilibrium states for the horseshoe*

In this section we prove Theorem A. Some of the ideas in this section are based in the constructions in [**9**]. Let $G : \Lambda \to \Lambda$ be the map defined in §2. Consider the horseshoe map $F : \Omega \to \Omega$, also defined in §2, and its inverse F^{-1}.

We define a projection of the parallelepipeds $\tilde{R}_0$ and $\tilde{R}_1$ onto the planes P_0 and P_1, $\pi : \tilde{R}_1 \cup \tilde{R}_2 \to P_0 \cup P_1$ by

$$\pi(x, y, z) := \begin{cases} (x, y, 0) & \text{if } (x, y, z) \in \tilde{R}_0, \\ (x, y, \frac{5}{6}) & \text{if } (x, y, z) \in \tilde{R}_1. \end{cases}$$

It is straightforward to check that π is continuous, surjective and

$$\pi \circ F^{-1} = G \circ \pi.$$

LEMMA 5.1. *For each $X \in \Lambda$ we have $h(F^{-1}, \pi^{-1}(X)) = 0$.*

Proof. Since the inverse horseshoe is contracting in the direction of the z-axis, for each $X \in \Lambda$ its restriction to the fiber $\pi^{-1}(X)$,

$$F^{-1}|_{\pi^{-1}(X)} : \pi^{-1}(X) \to \pi^{-1}(G(X)),$$

is a contraction. The number $S(n, \epsilon, \pi^{-1}(X))$ which is the maximum of the cardinalities of the (n, ϵ)-separated sets in $\pi^{-1}(X)$ is constant and it depends only on ϵ (and not on n). Thus,

$$h(F^{-1}, \pi^{-1}(X)) = \lim_{\epsilon \to 0} \lim_{n \to 0} \frac{1}{n} \log S(n, \epsilon, \pi^{-1}(X)) = 0. \qquad \square$$

Since the projection π is actually a semiconjugacy between the inverse horseshoe F^{-1} and the map G, we have (see [**5**])

$$h_{\text{top}}(F^{-1}) \leq h_{\text{top}}(G) + \sup\{h(F^{-1}, \pi^{-1}(X)),\ X \in \Lambda\}.$$

By Lemma 5.1 we get

$$h_{\text{top}}(F^{-1}) \leq h_{\text{top}}(G).$$

On the other hand, because π is a semiconjugacy, we have the other inequality immediately (see [**6**]), which gives us

$$h_{\text{top}}(F^{-1}) = h_{\text{top}}(G). \tag{15}$$

Thus, we have proved that the topological entropies of these two maps are equal.

5.1. *Existence.*

Consider the system (G, ϕ), with G as defined in §2 and ϕ a Hölder continuous potential with small variation

$$\sup \phi - \inf \phi < \frac{\log(\omega)}{2}.$$

Let μ_ϕ be the unique equilibrium state associated with the system (G, ϕ) obtained in Theorem B. Let $\mathcal{A}_\Lambda$ be the Borel σ-algebra in Λ and define $\mathcal{A}_0 := \pi^{-1}(A_\Lambda)$.

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Thus, $\mathcal{A}_0$ is a σ-algebra on the horseshoe, whose elements are given by $A = \pi^{-1}(B)$, where B is a Borelian. Note that $\mathcal{A}_0 \subset F^{-1}(\mathcal{A}_0)$. If for each $n \in \mathbb{N}$ we define $\mathcal{A}_n := F^{-n}(\mathcal{A}_0)$, we have an increasing sequence of σ-algebras

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_n \subset \cdots$$

For each n define a probability measure $\mu_n^\star : \mathcal{A}_n \to [0, 1]$ by

$$\mu_n^\star(F^{-n}(A_0)) = \mu_\phi(\pi(A_0)) \quad \text{for all } A_0 \in \mathcal{A}_0.$$

Note that $\mu_n^\star$ depends also on ϕ. Using the semiconjugacy it is straightforward to prove that each $\mu_n^\star$ is invariant under F^{-1}. Finally, we consider $\mathcal{A} := \bigcup_{n=0}^{\infty} \mathcal{A}_n$, which is a σ-algebra on Ω.

Define $\mu^\star : \mathcal{A} \to [0, 1]$ by

$$\mu^\star(A) = \mu_n^\star(A) \quad \text{if } A \in \mathcal{A}_n.$$

It is easy to see that $\mu^\star$ is well-defined. We have to prove that μ is σ-additive. For this we will use the following criterium from classical measure theory.

PROPOSITION 5.1. *Let $\mathcal{A}$ be a σ-algebra on Ω and $\mu^\star : \mathcal{A} \to [0, 1]$ satisfying*

$$\mu^\star(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \mu^\star(A_i)$$

for every finite family (A_i) of disjoints sets in $\mathcal{A}$. If there exists a family $\mathcal{C} \subset \mathcal{A}$ which satisfies the following.

(1) *The class $\mathcal{C}$ is compact: if $C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots$ are elements of the family $\mathcal{C}$ then $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$.*

(2) *The class $\mathcal{C}$ has the approximation property: for all $A \in \mathcal{A}$ we have*

$$\mu^\star(A) = \sup\{\mu^\star(C) \mid C \subset A,\ C \in \mathcal{C}\};$$

then $\mu^\star$ is a probability measure on $\mathcal{A}$.

We will construct a family $\mathcal{C}$ which satisfies the conditions (i) and (ii) in the above proposition.

We consider K_Λ the collection of all compacts on Λ and we define $K_0 := \pi^{-1}(K_\Lambda)$, which is a subfamily of the σ-algebra $\mathcal{A}_0$.

We define $\mathcal{C} := \bigcup_{n=0}^{\infty} F^{-n}(K_0)$ which is contained in $\mathcal{A}$ and it is a compact class with the approximation property.

(1) Let $C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots$ be sets of $\mathcal{C}$. For every $i \in \mathbb{N}$ there exists $D_i \subset K_\Lambda$ and there exists $n_i \in \mathbb{N}$ such that $C_i = F^{-n_i}(\pi^{-1}(D_i))$. Since $\pi^{-1}(D_i)$ is a closed set in a compact space Λ, $\pi^{-1}(D_i)$ is compact. By the continuity of F^{-1}, $(C_i)_{i\in\mathbb{N}}$ is a family of nested compact sets, therefore $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$.

(2) Let $A \in \mathcal{A}$. Using the regularity of the measure μ_ϕ on Λ we have $\mu_\phi(B) = \sup\{\mu_\phi(D) \mid D \subset B,\ D \in K_\Lambda\}$ for all $D \in \mathcal{A}_\Lambda$. By the definition of $\mathcal{C}$, for all $C \in \mathcal{C}$ there exist $D \in K_\Lambda$ and $n \in \mathbb{N}$ such that $C = F^{-n}(\pi^{-1}(D))$. Thus $\mu^\star(C) = \mu^\star(F^{-n}(\pi^{-1}(D))) = \mu_\phi(\pi(\pi^{-1}(D))) = \mu_\phi(D)$. Then

$$\sup\{\mu_\phi(D) \mid D \subset B,\ D \in K_\Lambda\} = \sup\{\mu_\star(C) \mid C \subset A,\ C \in \mathcal{C}\}.$$

To finish it is enough to note that for all $A \in \mathcal{A}$, there exists $B \in K_\Lambda$ such that $\mu^\star(A) = \mu_\phi(B)$.

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Then, by the criterion, we proved that $\mu^\star$ is a probability measure on $\mathcal{A}$. Since every $\mu_n^\star$ is invariant under F^{-1} the measure $\mu^\star$ is also F^{-1}-invariant.

LEMMA 5.2. *The extension of the σ-algebra $\mathcal{A}$ coincides with the Borel σ-algebra on Ω.*

Proof. Since π is continuous we have that $\mathcal{A}$ is contained in the Borel σ-algebra. It is enough to prove that for all $X \in \Omega$ there exists a fundamental system of neighborhoods of X contained in $\mathcal{A}$.

Let $X \in \Omega$ and remember that $\rho < 1$ is the contraction factor in the direction of the z-axis. For each $n \in \mathbb{N}$, by the uniform continuity of G^n there exists δ_n such that $d(z, w) < \delta_n$ implies $d(G^n(y), G^n(z)) < 1/n$.

We define $B_n(X) = F^{-n}(\pi^{-1}(\mathcal{B}(\pi(F^n(X)), \delta_n)))$. Therefore, for every $n \in \mathbb{N}$ and $X \in B_n(X)$, we have diam $B_n(X) = \rho^n + 1/n$ which goes to zero when n goes to infinity. □

LEMMA 5.3. *We have $h_{\mu^\star}(F^{-1}) \geq h_{\mu_\phi}(G)$.*

Proof. Using the well-known Brin–Katok theorem, we can compute the entropy by the formula (see [**26**]):

$$h_{\mu_\phi}(G) = \lim_{\epsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log \frac{1}{\mu_\phi(\mathcal{B}_G^n(X, \epsilon))}$$

for almost every X in Λ with respect to μ_ϕ, where $\mathcal{B}_G^n(X, \epsilon)$ denotes the dynamical ball of G centered at X. with radius ϵ and length n. We note that, given $\epsilon > 0$, by the continuity of π (on a compact space), there exists $0 \leq \delta \leq \epsilon$ such that

$$\pi(\mathcal{B}(Z, \delta)) \subset \mathcal{B}(\pi(Z), \epsilon) \quad \text{for every } Z \in \Omega.$$

Using this fact it is straightforward to check that

$$\mathcal{B}_{F^{-1}}^n(Y, \delta) \subset \pi^{-1}(\mathcal{B}_G^n(X, \epsilon)) \quad \text{for every } Y \in \pi^{-1}(X).$$

By the definition of the measure $\mu^\star$ we have

$$\mu^\star(\mathcal{B}_{F^{-1}}^n(Y, \delta)) \leq \mu^\star(\pi^{-1}(\mathcal{B}_G^n(X, \epsilon))) = \mu_\phi(\mathcal{B}_G^n(X, \epsilon)).$$

If we consider $M \subset \Lambda$ a full measure set with respect to μ_ϕ, then $\pi^{-1}(M) \subset \Omega$ satisfies $\mu^\star(\pi^{-1}(M)) = \mu_\phi(M) = 1$. Thus,

$$h_{\mu_\phi}(G) \leq \lim_{\epsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log \frac{1}{\mu_\phi(\mathcal{B}_G^n(X, \epsilon))} = h_{\mu^\star}(F^{-1}).$$ □

Consider $\varphi := \phi \circ \pi$. Thus, φ is a real function on Ω.

PROPOSITION 5.2. *The probability measure $\mu^\star$ is an equilibrium state associated with the system (F^{-1}, φ).*

Proof. Let $\overline{\mu}$ be an F^{-1}-invariant probability measure on Ω. Consider a probability measure on Λ, defined by $\eta := \overline{\mu} \circ \pi^{-1}$. We have,

$$h_{\overline{\mu}}(F^{-1}) + \int \phi \circ \pi \, d\overline{\mu} \leq h_\eta(G) + \int \phi \, d\eta + \int_\Lambda h_{\text{top}}(F^{-1}, \pi^{-1}(X)) \, d\mu(X).$$

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This follows from a result of Ledrappier and Walters [**14**]. For a proof of the statement see [**3**, Theorem 4.1]. According to Lemma 5.1 we have

$$\int_{\Lambda} h_{\text{top}}(F^{-1}, \pi^{-1}(X))\, d\mu(X) = 0.$$

By the semiconjugacy, η is G-invariant, thus

$$h_{\eta}(G) + \int \phi\, d\eta \leq P_{\text{top}}(G, \phi).$$

Going back to the inequality we get

$$h_{\overline{\mu}}(F^{-1}) + \int \phi \circ \pi\, d\overline{\mu} \leq P_{\text{top}}(G, \phi).$$

Moreover, the last inequality holds for any F^{-1}-invariant measure:

$$P_{\text{top}}(F^{-1}, \varphi) \leq P_{\text{top}}(G, \phi).$$

On the other hand, π is a semiconjugacy, which implies (see [**6**])

$$P_{\text{top}}(G, \phi) \leq P_{\text{top}}(F^{-1}, \varphi).$$

Therefore, the topological pressures of both systems coincide. Note that, by a change of variables, we get

$$\int \phi\, d\mu_{\phi} = \int \phi \circ \pi\, d\mu^{\star} = \int \varphi\, d\mu^{\star}.$$

Using the fact that $h_{\mu_{\star}}(F^{-1}) \geq h_{\mu_{\phi}}(G)$, proved in Proposition 5.3, and the fact that the topological pressures coincide, we have

$$h_{\mu^{\star}}(F^{-1}) + \int \phi \circ \pi\, d\mu^{\star} \geq h_{\mu_{\phi}}(G) + \int \phi\, d\mu_{\phi} = P_{\text{top}}(F^{-1}, \varphi).$$

Thus, $\mu^{\star}$ is an equilibrium state associated to (F^{-1}, φ). □

5.2. *Uniqueness.* In the last section we constructed a probability measure related to the equilibrium state on the space Λ. We proved that this measure is actually an equilibrium state for the horseshoe associated with a potential which projects onto the potential on Λ.

Consider the equilibrium state $\mu^{\star}$ for (F^{-1}, φ) obtained in the §5.1. Assume that there is another equilibrium state $\eta^{\star}$ which is different from $\mu^{\star}$. We define a probability measure on Λ by the push-forward of $\eta^{\star}$:

$$\eta(A) := \eta^{\star}(\pi^{-1}(A)) \quad \text{for all Borelian sets } A.$$

We will prove that the measure η differs from the unique equilibrium state on Λ, given by Theorem B.

LEMMA 5.4. *The probability measure η is different from the equilibrium state μ_{ϕ}.*

Proof. We suppose that $\eta^{\star} \neq \mu^{\star}$ which implies that there exists an $A \in \mathcal{A}$ satisfying $\eta^{\star}(A) \neq \mu^{\star}(A)$. The σ-algebra $\mathcal{A}$, which coincides with the Borelians on Λ, was obtained as $\mathcal{A} = \mathcal{A}_0 \cup F^{-1}\mathcal{A}_0 \cup \cdots \cup F^{-n}\mathcal{A}_0 \cup \cdots$. Thus, there exist $A_0 \in \mathcal{A}_0$ and $n_0 \in \mathbb{N}$ such that $A = F^{-n_0}(A_0)$.

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Remember also that $\mathcal{A}_0 = \pi^{-1}(\mathcal{A}_\Lambda)$ and therefore there exists $B \in \mathcal{A}_\Lambda$ such that $A_0 = \pi^{-1}(B)$. Using that $\mu^\star$ is invariant under F^{-1} we have

$$\eta(B) = \eta^\star(\pi^{-1}(B)) = \eta^\star(A_0) = \eta^\star(F^{-n_0}(A_0)) = \eta^\star(A).$$

On the other hand, we have

$$\mu_\phi(B) = \mu_\phi(\pi(A_0)) = \mu^\star(A_0) = \mu^\star(F^{-n_0}A_0) = \mu^\star(A).$$

Since $\eta^\star(A) \neq \mu^\star(A)$ we have that $\eta(B) \neq \mu_\phi(B)$ and the lemma is proven. □

By assuming the existence of another equilibrium state for the horseshoe, we have defined a probability measure on Λ, denoted by η, that does not coincide with the unique equilibrium state on Λ. We are going to prove that η is an equilibrium state on Λ. This will contradict the uniqueness of equilibrium states for the system (G, ϕ) and therefore we will prove that there is only one equilibrium state for the horseshoe associated with the potential φ.

LEMMA 5.5. *The probability measure η is an equilibrium state associated with the system (G, ϕ).*

Proof. We have to prove that

$$h_\eta(G) + \int \phi \, d\eta \geq P_{\text{top}}(G, \phi).$$

In §5.1 we proved that $P_{\text{top}}(G, \phi) = P_{\text{top}}(F^{-1}, \varphi)$. In addition, we proved that $\eta^\star$ is an equilibrium state to (F^{-1}, φ). Thus, we have

$$P_{\text{top}}(G, \phi) = P_{\text{top}}(F^{-1}, \varphi) = h_{\eta^\star}(F^{-1}) + \int \varphi \, d\eta^\star.$$

Then it is enough to prove

$$h_\eta(G) + \int \phi \, d\eta \geq h_{\eta^\star}(F^{-1}) + \int \varphi \, d\eta^\star.$$

The integrals in the inequality coincide (by a change of variables). By Lemma 5.3 we have $h_\eta(G) \geq h_{\eta^\star}(F^{-1})$ which completes the proof. □

Thus, if the horseshoe admits two different equilibrium states associated with φ we derive that the system (G, ϕ) also admits two different equilibrium states. Since it is not possible, by Theorem B, the equilibrium state for the horseshoe is unique.

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