*Euro. Jnl of Applied Mathematics* (2001), *vol.* 12, *pp.* 625–644. Printed in the United Kingdom

© 2001 Cambridge University Press

# On a singular Sturm-Liouville problem involving an advanced functional differential equation

B. VAN BRUNT[1], G. C. WAKE[2] and H. K. KIM[1]

[1]*Department of Mathematics, Massey University, New Zealand*
[2]*Department of Mathematics and Statistics, The University of Canterbury, New Zealand*

(*Received in revised form 19 March 2001*)

Solutions to a boundary-value problem involving a second-order linear functional differential equation with an advanced argument are investigated in this paper. The boundary conditions imposed on the differential equation are analogous to conditions defining various singular Sturm-Liouville problems, and if an eigenvalue parameter is introduced certain properties of the spectrum can be deduced having analogues with the classical problem. Dirichlet series solutions are constructed for the problem and it is established that the spectrum contains an infinite number of real positive eigenvalues. A Laplace transform analysis of the problem then reveals that the spectrum does not generically consist of isolated points and that there may be an infinite number of eigenfunctions corresponding to a given eigenvalue. In contrast, it is also shown that there is a subset of eigenvalues that correspond to the zeros of an entire function for which the corresponding eigenfunctions are unique.

## 1 Introduction

In this paper we study a functional differential equation of the form

$$y''(x) - ay'(x) - by(x) + \lambda y(\alpha x) = 0, \tag{1.1}$$

subject to the boundary conditions

$$y(0) = 0, \quad \lim_{x\to\infty} y(x) = 0, \quad \lim_{x\to\infty} y'(x) = 0 \tag{1.2}$$

and the 'normalizing' condition

$$\int_0^\infty y(x)\,dx = 1. \tag{1.3}$$

Here, $\alpha$, $a$ and $b$ are constants such that $\alpha > 1$, $a > 0$, and $b > 0$. Equation (1.1) is an advanced functional differential equation. Boundary-value problems such as this one arise in models of cell growth which include the effects of dispersion. In these models, $y$ plays the rôle of a probability density function (cf. Cooper *et al.* [18]). The first order analogue of this problem has been studied by Hall [8] and Hall & Wake [9], for the specific choice of $\lambda = b\alpha$, and Kim [13] for general values of $\lambda$.

We note that the first order analogue of equation (1.1), i.e. the pantograph equation,

$$y'(x) = Ay(x) + By(\alpha x)$$

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

has been studied by numerous researchers [15, 5, 12, 7, 14]. Generalized versions of the pantograph equation have been studied by Carr & Dyson [3], Derfel & Iserles [4] and Iserles [11]. Iserles gives several other references to the pantograph equation and applications.

We can regard the above boundary-value problem as a type of singular Sturm-Liouville problem involving a second-order advanced functional differential equation. In analogy with the classical Sturm-Liouville theory, we shall refer to the special values of $\lambda$, which yield solutions as eigenvalues, and the corresponding solutions as eigenfunctions.

Let $\mathscr{L}$ denote the operator defined by

$$\mathscr{L}y = y''(x) - ay'(x) - by(x).$$

The functional differential equation (1.1) can be written

$$\mathscr{L}y + \lambda y(\alpha x) = 0.$$

Associated with the operator $\mathscr{L}$ is the operator $\mathscr{L}_\alpha$ defined by

$$\mathscr{L}_\alpha y = y''(x/\alpha) - ay'(x/\alpha) - by(x/\alpha),$$

and equation (1.1) can thus be written as

$$\mathscr{L}_\alpha y + \lambda y(x) = 0.$$

In this sense, $\lambda$ plays the rôle of an eigenvalue parameter.

The values of $\lambda$ for which equations (1.1) and (1.2) have a non-trivial solution describe the values at which cell-growth models of cohorts (distributed by size $x$ and undergoing fission simultaneously) admit Steady-Size Distribution (SSD). That is, the number distribution $n(x,t)$ over size $x$ grows asymptotically like

$$n(x,t) \sim y(x)N(t),$$

for $t$ large. The right-hand side of this equation is called the SSD-solution. Thus, the spectrum of our problem and the corresponding eigenfunctions are the cases for which SSD behaviour for $n$ can occur. This matches the pattern observed experimentally (cf. Hall & Wake [9]).

One approach to this type of problem consists of first determining the Green's function associated with the operator $\mathscr{L}$ (or $\mathscr{L}_\alpha$), and regarding the $\lambda y$ term as a 'forcing function'. Given that a Green's function exists to the problem, say $G(x,\xi)$, a functional differential equation such as equation (1.1) can be written as a Fredholm integral equation,

$$\begin{aligned} y(x) &= -\lambda \int_0^\infty G(x,\xi)y(\alpha\xi)\,d\xi \\ &= -\frac{\lambda}{\alpha}\int_0^\infty G(x,\xi/\alpha)y(\xi)\,d\xi. \end{aligned}$$

Unfortunately, the kernel is not symmetric, and the integral equation is singular. As it turns out, it is often easier to work with the original functional differential equation. As with the ordinary differential equation analogue, a series solution can be found for the problem and this eases the analysis in some cases. In the next section, we consider first the simple case when $a = 0$, and study the eigenvalues and eigenfunctions. This special

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

case has the merit that eigenvalues and eigenfunctions can be found explicitly, and thus it illustrates the general theory, which is discussed later. The solutions constructed in § 2 are of a special type, and there is no *a priori* reason to limit potential solutions to those expressible as a Dirichlet series. In § 3, we investigate the associated Laplace transform problem and determine a general solution. This solution is then used in the fourth section to deduce certain results concerning the distribution of eigenvalues and the uniqueness of eigenfunctions.

## 2 A Dirichlet series solution

One can solve functional differential equations of the form (1.1) using Laplace transforms. We outline this procedure in the next section. Here, we simply note that this motivates our choice of solution form as a Dirichlet series. In fact, there is an extensive Dirichlet series methodology developed for the pantograph equation (cf. [7, 11]).

In this section we construct solutions to the boundary-value problem described in the introduction. A direct construction of the solution is simpler than using Laplace transforms, but the method does not preclude non-Dirichlet series solutions and so no results regarding uniqueness can be asserted. We address non-Dirichlet series solutions in § 4. We begin here with the simple case when $a = 0$, and then move to the more general problem.

Consider the boundary-value problem consisting of the differential equation

$$y''(x) - by(x) + \lambda y(\alpha x) = 0, \tag{2.1}$$

and conditions (1.2) and (1.3). We seek a solution to this problem of the form

$$y(x) = \sum_{n=0}^{\infty} c_n e^{-\alpha^n r x}, \tag{2.2}$$

where $r$ and the coefficients $c_n$ are constants to be determined. Substituting the above series into equation (2.1) leads to the recurrence relation

$$\frac{c_n}{c_{n-1}} = \frac{-\lambda}{b(\alpha^{2n} - 1)}, \tag{2.3}$$

for $n = 1, 2, \ldots$, and the indicial equation

$$r^2 - b = 0\,. \tag{2.4}$$

Although there are two solutions to the indicial equation, only the choice $r = \sqrt{b}$ leads to a convergent series.

Equation (2.3), for the choice $r = \sqrt{b}$, indicates that $y(x)$ can be expressed as

$$y(x) = c_0 \left( e^{-\sqrt{b}x} + \sum_{n=1}^{\infty} \frac{(-\frac{\lambda}{b})^n}{\prod_{m=1}^{n}(\alpha^{2m} - 1)} e^{-\alpha^n \sqrt{b} x} \right).$$

The constant $c_0$ can be determined by the condition (1.3), i.e.

$$c_0 = \sqrt{b} \left( 1 + \sum_{n=1}^{\infty} \frac{(-\frac{\lambda}{b\alpha})^n}{\prod_{m=1}^{n}(\alpha^{2m} - 1)} \right)^{-1}.$$

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

Note that though the manipulations leading to above solution are formal, the resulting series is uniformly convergent on $(0,\infty)$, and thus term-by-term differentiation etc. is justified. At $x = 0$,

$$y(0) = c_0 \left( 1 + \sum_{n=1}^{\infty} \frac{(-\frac{\lambda}{b})^n}{\prod_{m=1}^{n}(\alpha^{2m}-1)} \right), \tag{2.5}$$

and it is clear that for general values of $\lambda$ we have $y(0) \neq 0$; thus, condition (1.2) cannot in general be satisfied for this class of solutions. There are, however, certain values of $\lambda$ which do satisfy this initial condition.

Let $q = 1/\alpha^2$, then

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{(-\frac{\lambda}{b})^n}{\prod_{m=1}^{n}(\alpha^{2m}-1)} &= 1 + \sum_{n=1}^{\infty} \frac{(-\frac{\lambda}{b})^n q^{\frac{n(n+1)}{2}}}{\prod_{m=1}^{n}(1-q^m)} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-\frac{\lambda q}{b})^n q^{\frac{n(n-1)}{2}}}{\prod_{m=1}^{n}(1-q^m)} \\ &\equiv F_0(\lambda). \end{aligned}$$

The final expression can be converted into an infinite product using the Euler identity (cf. Andrews [2])

$$1 + \sum_{n=1}^{\infty} \frac{t^n q^{\frac{n(n-1)}{2}}}{\prod_{m=1}^{n}(1-q^m)} = \prod_{n=0}^{\infty}(1+tq^n);$$

hence,

$$F_0(\lambda) = \prod_{n=0}^{\infty} \left( 1 - \frac{\lambda q^{n+1}}{b} \right) = \prod_{n=0}^{\infty} \left( 1 - \frac{\lambda}{b\alpha^{2(n+1)}} \right). \tag{2.6}$$

Note that the denominator of $c_0$ also can be expressed as an infinite product, i.e.

$$c_0 = \sqrt{b} \left( \prod_{n=0}^{\infty} \left( 1 - \frac{\lambda}{b\alpha^{2n+3}} \right) \right)^{-1}. \tag{2.7}$$

Now, $F_0(\lambda)$ is an entire function the zeros of which correspond to the values of $\lambda$ for which a solution of the form (2.2) to the boundary-value problem exists. These values will be referred to as eigenvalues. The eigenvalues generated by these solutions are thus

$$\lambda = b\alpha^{2j},$$

for $j = 1, 2, \ldots$, and to each eigenvalue there is associated a corresponding eigenfunction that satisfies the functional differential equation (2.1) subject to the conditions (1.2) and (1.3). The $j$th eigenfunction $y_j(x)$ is of the form

$$y_j(x) = c_{0j} \left( e^{-\sqrt{b}x} + \sum_{m=1}^{\infty} \frac{(-\alpha^{2j})^m}{\prod_{k=1}^{m}(\alpha^{2k}-1)} e^{-\alpha^m \sqrt{b}x} \right), \tag{2.8}$$

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

where

$$c_{0j} = \sqrt{b}\left(\prod_{m=0}^{\infty}\left(1 - \frac{\alpha^{2j}}{\alpha^{2m+3}}\right)\right)^{-1}. \tag{2.9}$$

We now return to the general boundary-value problem, where $a > 0$. If we seek solutions to equation (1.1) of the form (2.2), we arrive at the recurrence relation

$$\frac{c_n}{c_{n-1}} = \frac{-\lambda}{\alpha^{2n}r^2 + a\alpha^n r - b}, \tag{2.10}$$

for $n = 1, 2, \ldots$, and the indicial equation

$$r^2 + ar - b = 0. \tag{2.11}$$

We have assumed $a > 0$ and $b > 0$, so the indicial equation has a positive root, say $r_1$ and a negative root, say $r_2$. The only solution of the indicial equation that leads to a convergent series is

$$r_1 = \frac{-a + \sqrt{a^2 + 4b}}{2}.$$

A solution to equation (1.1) is thus of the form

$$y(x) = c_0\left(e^{-r_1 x} + \sum_{n=1}^{\infty}\frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{2m}r_1^2 + a\alpha^m r_1 - b)}e^{-\alpha^n r_1 x}\right). \tag{2.12}$$

Condition (1.3) implies that

$$c_0 = r_1\left(1 + \sum_{n=1}^{\infty}\frac{(-\frac{\lambda}{\alpha})^n}{\prod_{m=1}^{n}(\alpha^{2m}r_1^2 + a\alpha^m r_1 - b)}\right)^{-1}, \tag{2.13}$$

and relations (2.12) and (2.13) imply that

$$y(0) = c_0 F(\lambda),$$

where

$$F(\lambda) \equiv 1 + \sum_{n=1}^{\infty}\frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{2m}r_1^2 + a\alpha^m r_1 - b)}. \tag{2.14}$$

The normalizing condition (1.3) precludes the case $c_0 = 0$ (the trivial solution), and thus the condition $y(0) = 0$ implies that $F(\lambda) = 0$. The eigenvalues are the zeros of the entire function $F$. Unlike the simple case, there is no conspicuous partition formula to convert $F$ into an infinite product and thereby find eigenvalues. Nonetheless, we can readily deduce the existence of an infinite sequence of real eigenvalues $\{\lambda_n\}$, such that $\lambda_n \to \infty$ as $n \to \infty$. Equation (2.12) implies that $y'(0) = -r_1 a_0 F(\alpha\lambda)$ and $y''(0) = -r_1^2 a_0 F(\alpha^2\lambda)$. The differential equation (1.1) thus produces the functional relation

$$r_1^2 F(\alpha^2\lambda) + ar_1 F(\alpha\lambda) + (\lambda - b)F(\lambda) = 0. \tag{2.15}$$

At $\lambda = b$ we have

$$r_1^2 F(\alpha^2 b) + ar_1 F(\alpha b) = 0,$$

and hence $F$ must change sign in the interval $(\alpha b, \alpha^2 b)$, or $F(\alpha^2 b) = F(\alpha b) = 0$. In any case

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

there is a zero of $F$, say $\lambda_1$, in the interval $[\alpha b, \alpha^2 b)$. Now, at $\lambda = \lambda_1$,

$$r_1^2 F(\alpha^2 \lambda_1) + a r_1 F(\alpha \lambda_1) = 0,$$

so that a similar argument implies the existence of a second zero $\lambda_2$ in the interval $[\alpha\lambda_1, \alpha^2\lambda_1)$. We can continue in this manner and establish the requisite sequence of eigenvalues and thus a set of eigenfunctions that satisfy the boundary-value problem for the appropriate eigenvalue. Kim [13] has studied some of the basic properties of the eigenfunctions and established results concerning the zeros of the eigenfunctions.

## 3 Laplace transform analysis

The Dirichlet series solution given in § 2 shows that solutions exist to the boundary value problem for certain values of $\lambda$, but there may be other eigenvalues not associated with the zeros of $F$ (non-Dirichlet series solutions), and for a given eigenvalue there may be more than one eigenfunction. To address these questions we study the corresponding Laplace transform problem.

Suppose that $y$ is a solution to equation (1.1) with eigenvalue $\lambda$ satisfying conditions (1.2) and (1.3). The function $y$ is continuous and bounded on the interval $[0, \infty)$, and hence the Laplace transform

$$Y(s) = \int_0^\infty e^{-sx} y(x)\, dx \tag{3.1}$$

is defined for all $s \geqslant 0$. Moreover, $Y$ must be holomorphic in the half plane $P_0 = \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$. Condition (1.3) indicates that

$$Y(0) = 1. \tag{3.2}$$

If we apply the Laplace transform to both sides of equation (1.1), we get the functional relation

$$Y(s) = \frac{\frac{-\lambda}{\alpha} Y(s/\alpha) + \frac{\lambda}{\alpha} - b}{(s + r_1)(s + r_2)}, \tag{3.3}$$

where $r_1$ and $r_2$ are the positive and negative roots, respectively, to the indicial equation (2.11). Here, we have used the relation

$$y'(0) = \frac{\lambda}{\alpha} - b, \tag{3.4}$$

which can be derived by integrating both sides of equation (1.1) from 0 to $\infty$ and applying conditions (1.2) and (1.3).

In this section, we construct a general solution to equation (3.3) satisfying condition (3.2). We shall assume throughout the section that $a > 0$. We focus first on solutions that are holomorphic at $s = 0$ and then construct general solutions. The solutions are valid for a broad class of $\lambda \in \mathbb{C}$, but not all of these solutions correspond to solutions of the original boundary-value problem. The requirement that $Y$ be holomorphic in $P_0$ filters out some solutions that do not correspond to solutions of the boundary-value problem, and to some extent restricts the potential eigenvalues.

Equation (3.3) is an example of a class of equations which is known as 'non-homogeneous $q$-difference equations' with $q = 1/\alpha < 1$. A non-homogeneous $q$-difference

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

equation is of the form

$$f(s)Y(qs) + g(s)Y(s) + h(s) = 0.$$

These functional equations have been studied Adams [1] in addition to other researchers for the cases where the coefficients are holomorphic at $s = 0$. In particular, Adams shows there are meromorphic solutions to the initial-value problems such as (3.2), (3.3) that are holomorphic at $s = 0$. We show this directly for initial value problem (3.2), (3.3) by first considering the associated first order retarded functional differential equation, and then appealing to an existence and uniqueness theorem for such equations. The meromorphic character of the solution holomorphic at $s = 0$ is then detailed for this particular problem.

**Lemma 3.1** *There exists a solution to equation* (3.3) *satisfying condition* (3.2) *unique among functions holomorphic at* $s = 0$ *provided* $\lambda \neq \alpha^m b$ *for any integer* $m \geqslant 2$.

**Proof** Differentiating equation (3.3) gives the functional differential equation

$$(s + r_1)(s + r_2)Y'(s) + (s + a)Y(s) + \frac{\lambda}{\alpha^2}Y'(s/\alpha) = 0. \tag{3.5}$$

Now $r_1 \neq 0$ and $r_2 \neq 0$, and hence equation (3.5) is a retarded functional differential equation of the form

$$Y'(s) = H(s, Y(s), Y'(s/\alpha)),$$

where $H$ is a function entire in $Y(s)$ and $Y'(s/\alpha)$, and holomorphic in $s$ for $|s|$ sufficiently small.

Flamant [6] showed that initial-value problems such as (3.2), (3.5) have a unique solution among functions holomorphic in a neighbourhood of $z = 0$, if the equations define the derivatives of $Y$ at $s = 0$ uniquely. A more general result concerning this type of initial-value problem has been obtained by van Brunt [17], based on the analysis of Rogers [16]. We may thus assert the existence of a local solution to the differential equation (3.5) satisfying the initial condition (3.2) that is unique among functions holomorphic at $s = 0$, provided the initial condition and the differential equation define the derivatives of $Y$ at $s = 0$ uniquely. Equation (3.5) indicates that $Y'(0)$ is determined uniquely unless $\lambda = \alpha^2 b$. For this choice of $\lambda$, we get the condition $aY(0) = 0$, which cannot be satisfied because neither factor is zero. The second derivative at $s = 0$ can be obtained by differentiating equation (3.5). The resulting equation will have a unique solution for $Y''(0)$, provided $Y'(0)$ is uniquely determined and $\lambda \neq \alpha^3 b$. We can continue in this manner to determine the higher derivatives of $Y$ at $s = 0$. The $n$th derivative of $Y$ at $s = 0$ is determined by differentiating the equation (3.3) $n$ times, and using the values of the lower order derivatives to solve for $Y^{(n)}(0)$. As long as the lower order derivatives are determined uniquely $Y^{(n)}(0)$ is determined uniquely unless $\lambda = \alpha^{n+1} b$. At this value of $\lambda$, the terms containing $Y^n(0)$ vanish and a compatibility condition among the parameters $\alpha$, $a$ and $b$ must be imposed to satisfy the equation. □

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

**Lemma 3.2** *The solution $Y$ to equation* (3.3) *that is holomorphic at the origin can be continued to produce a function meromorphic in* $\mathbb{C}$. *If $Y$ has any singularities, then they must be simple poles at $s = -\alpha^m r_1$ or $s = -\alpha^m r_2$ for some $m = 0, 1, 2, \ldots$.*

**Proof** We know that there is a $\delta > 0$ such that $Y$ is holomorphic in the disk $|s| < \delta$. Equation (3.3) implies that for any $n \geqslant 0$

$$Y(s) = \frac{(-1)^{n+1}\left(\frac{\lambda}{\alpha}\right)^{n+1} Y(s/\alpha^{n+1})}{\prod_{m=0}^{n}(\frac{s}{\alpha^m}+r_1)(\frac{s}{\alpha^m}+r_2)} + \left(\frac{\lambda}{\alpha}-b\right)\sum_{j=0}^{n-1}\frac{(-1)^j\left(\frac{\lambda}{\alpha}\right)^j}{\prod_{m=0}^{j}(\frac{s}{\alpha^m}+r_1)(\frac{s}{\alpha^m}+r_2)}.$$

Choose any complex number $s$ and choose $n$ so large that $|s/\alpha^{n+1}| < \delta$. If $s \neq -\alpha^m r_1$ and $s \neq \alpha^m r_2$ for all $m = 0, 1, 2, \ldots, n$, then the above expression indicates that $Y$ must be holomorphic at $s$. If $s = -\alpha^m r_1$ or $s = -\alpha^m r_2$ for some $m = 0, 1, 2, \ldots, n$ then $Y$ has 'at worst' a simple pole at $s$. Now $s$ is an arbitrary number in the complex plane, so that $Y$ must be meromorphic in $\mathbb{C}$, and the only potential singularities that $Y$ may have are simple poles. □

Given that the solution $Y$ of Lemma 3.1 must be meromorphic in $\mathbb{C}$ with perhaps simple poles in the sets $\{-\alpha^m r_1\}$, $\{-\alpha^m r_2\}$, we anticipate a global solution to equation (3.3) of the form

$$Y(s) = \sum_{n=0}^{\infty}\frac{c_n}{s+r_1\alpha^n} + \sum_{n=0}^{\infty}\frac{d_n}{s+r_2\alpha^n} \tag{3.6}$$

(a Mittag-Leffler expansion for $Y$). Substituting the above expansion into equation (3.3) and equating principal parts leads to the relations

$$c_n = \frac{-\lambda c_{n-1}}{r_1(\alpha^n-1)(r_1\alpha^n-r_2)}, \quad d_n = \frac{-\lambda d_{n-1}}{r_2(\alpha^n-1)(r_2\alpha^n-r_1)},$$

for $n = 1, 2, \ldots$. The coefficients $c_0$ and $d_0$ must satisfy the relations

$$d_0 = \frac{\frac{\lambda}{\alpha}-b}{r_1-r_2} - \frac{\lambda}{r_1-r_2}\sum_{n=0}^{\infty}\left\{\frac{c_n}{r_1\alpha^{n+1}-r_2} + \frac{d_n}{r_2(\alpha^{n+1}-1)}\right\} \tag{3.7}$$

and

$$c_0 = \frac{-(\frac{\lambda}{\alpha}-b)}{r_1-r_2} + \frac{\lambda}{r_1-r_2}\sum_{n=0}^{\infty}\left\{\frac{c_n}{r_1(\alpha^{n+1}-1)} + \frac{d_n}{r_2\alpha^{n+1}-r_1}\right\}. \tag{3.8}$$

Let

$$F(\lambda) = 1 + \sum_{n=1}^{\infty}\frac{(-\lambda)^n}{\prod_{m=1}^{n} r_1(\alpha^m-1)(r_1\alpha^m-r_2)}$$

(cf. equation (2.14)) and

$$G(\lambda) = 1 + \sum_{n=1}^{\infty}\frac{(-\lambda)^n}{\prod_{m=1}^{n} r_2(\alpha^m-1)(r_2\alpha^m-r_1)}.$$

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

Then,

$$\sum_{n=0}^{\infty} c_n = c_0 F(\lambda), \quad \sum_{n=0}^{\infty} d_n = d_0 G(\lambda),$$

and adding equations (3.7) and (3.8) gives

$$c_0 + d_0 = \lambda \sum_{n=0}^{\infty} \left( \frac{c_{n+1}}{-\lambda} + \frac{d_{n+1}}{-\lambda} \right),$$

i.e.

$$c_0 F(\lambda) + d_0 G(\lambda) = 0. \tag{3.9}$$

Now, a direct calculation using the recurrence relations and the definitions of $F$ and $G$ yields

$$\begin{aligned} &\frac{-\lambda}{r_1 - r_2} \sum_{n=0}^{\infty} \left( \frac{c_n}{r_1 \alpha^{n+1} - r_2} + \frac{d_n}{r_2(\alpha^{n+1} - 1)} \right) \\ &= d_0 + \frac{c_0 r_1 F(\alpha\lambda) + d_0 r_2 G(\alpha\lambda)}{r_1 - r_2}, \end{aligned}$$

and hence equation (3.7) is equivalent to the condition

$$c_0 r_1 F(\alpha\lambda) + d_0 r_2 G(\alpha\lambda) + \frac{\lambda}{\alpha} - b = 0. \tag{3.10}$$

For a fixed value of $\lambda$, equations (3.9) and (3.10) form a system of two linear equations for the unknown constants $c_0$ and $d_0$. The constants are determined uniquely provided

$$r_1 F(\alpha\lambda) G(\lambda) - r_2 G(\alpha\lambda) F(\lambda) \neq 0. \tag{3.11}$$

It can be verified directly or indirectly through the original differential equation (as was done in § 2) that the functions $F$ and $G$ satisfy the following functional identities for all $\lambda \in \mathbb{C}$:

$$r_1^2 F(\alpha\lambda) + a r_1 F(\lambda) + \left( \frac{\lambda}{\alpha} - b \right) F(\lambda/\alpha) = 0, \tag{3.12}$$

$$r_2^2 G(\alpha\lambda) + a r_2 G(\lambda) + \left( \frac{\lambda}{\alpha} - b \right) G(\lambda/\alpha) = 0. \tag{3.13}$$

These relations imply that

$$\begin{aligned} &r_1 F(\alpha\lambda) G(\lambda) - r_2 G(\alpha\lambda) F(\lambda) \\ &= \frac{\frac{\lambda}{\alpha} - b}{r_1 r_2} \left( r_1 F(\lambda) G(\lambda/\alpha) - r_2 F(\lambda/\alpha) G(\lambda) \right) \\ &= \frac{(\frac{\lambda}{\alpha} - b)(\frac{\lambda}{\alpha^2} - b)}{r_1^2 r_2^2} \left( r_1 F(\lambda/\alpha) G(\lambda/\alpha^2) - r_2 F(\lambda/\alpha^2) G(\lambda/\alpha) \right) \\ &\vdots \\ &= \left( \prod_{m=1}^{n} (1 - \frac{\lambda}{\alpha^m b}) \right) \left( r_1 F(\lambda/\alpha^{n-1}) G(\lambda/\alpha^n) - r_2 F(\lambda/\alpha^n) G(\lambda/\alpha^{n-1}) \right). \end{aligned}$$

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

As $n \to \infty$ we have that $F(\lambda/\alpha^n) \to F(0) = 1$ and $G(\lambda/\alpha^n) \to G(0) = 1$, and hence

$$r_1 F(\alpha\lambda)G(\lambda) - r_2 G(\alpha\lambda)F(\lambda) = (r_1 - r_2)\prod_{m=1}^{\infty}\left(1 - \frac{\lambda}{\alpha^m b}\right). \tag{3.14}$$

Since $r_1 - r_2 > 0$, condition (3.11) is satisfied for all $\lambda \in \mathbb{C}$ unless $\lambda = \alpha^m b$ for some positive integer $m$.

In summary, provided that $\lambda \neq \alpha^m b$ for any positive integer $m$, the Mittag-Leffler expansion (3.6) is the only solution to equation (3.3) satisfying condition (3.2) that is holomorphic at the origin. The solution (3.6) is valid for all $\lambda$ except when $\lambda = \alpha^m b$; however, not all values of $\lambda$ correspond to solution of the original boundary-value problem. The condition that $Y$ be holomorphic in the half plane $P_0$ indicates that $d_0 = 0$, and hence

$$Y(s) = \sum_{n=0}^{\infty} \frac{c_n}{s + r_1\alpha^n}.$$

If we invert the above transform then we get the Dirichlet series solution (2.2). Equation (3.9) implies that $\lambda$ must be a zero of $F$ in agreement with the results of the previous section. The Dirichlet series solutions are thus characterized by Laplace transforms holomorphic in the half plane $P_0$ and at $s = 0$.

If we retain the assumption that $\lambda \neq \alpha^m b$ for any positive integer $m$, but drop the condition that $Y$ be holomorphic at the origin and require only that the solution be meromorphic in $P_0$, the Mittag-Leffler solution is still valid; however, there is no guarantee of uniqueness. Let $Y$ denote a solution to equation (3.3) satisfying condition (3.2) , $Y_h$ denote the Mittag-Leffler solution, and $Z = Y - Y_h$. Then $Z$ must satisfy the equation

$$Z(s) = \frac{\frac{\lambda}{\alpha b}Z(s/\alpha)}{(1 + s/r_1)(1 + s/r_2)}, \tag{3.15}$$

and

$$Z(0) = 0. \tag{3.16}$$

The function $Z$ must be meromorphic in $P_0$ and using an argument similar to that used to show Lemma 3.2 we see that 'at worst' $Z$ has simple poles in $P_0$ at $s = -r_2\alpha^m$ for $m = 0, 1, 2, \ldots$. In fact, equation (3.15) can be solved explicitly for $Z$.

**Lemma 3.3** *If $|\lambda| \leqslant \alpha b$, then the only solution to equation* (3.15) *meromorphic in $P_0$ is the trivial solution $Z = 0$.*

**Proof** Equation (3.15) implies that for any $s \in P_0$ at which $Z$ is holomorphic

$$Z(s) = \frac{\left(\frac{\lambda}{\alpha b}\right)^n Z(s/\alpha^n)}{\prod_{m=0}^{n-1}(1 + \frac{s}{\alpha^m r_1})(1 + \frac{s}{\alpha^m r_2})}.$$

Now $(\lambda/(\alpha b))^n \leqslant 1$ for all $n$ and $Z(s/\alpha^n) \to 0$ as $n \to \infty$; moreover, the infinite product

$$\prod_{m=0}^{\infty}\left(1 + \frac{s}{\alpha^m r_1}\right)\left(1 + \frac{s}{\alpha^m r_2}\right)$$

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

is finite and non-zero. Consequently,

$$Z(s) = \lim_{n\to\infty} \left\{ \frac{\left(\frac{\lambda}{\alpha b}\right)^n Z(s/\alpha^n)}{\prod_{m=0}^{n-1}(1+\frac{s}{\alpha^m r_1})(1+\frac{s}{\alpha^m r_2})} \right\} = 0,$$

and hence $Z$ must be identically zero in $P_0$. □

Suppose now that $|\lambda| > \alpha b$, and let

$$Z(s) = f(s)W(s),$$

where

$$W(s) = \frac{s^{\kappa_0}}{\prod_{n=0}^{\infty}(1+\frac{s}{r_1\alpha^n})(1+\frac{s}{r_2\alpha^n})},$$

and

$$\kappa_0 = \frac{\ln\left|\frac{\lambda}{\alpha b}\right| + i\phi}{\ln(\alpha)}.$$

Here, we use the principle branch of $\log(z)$, $\mathrm{Log}(z)$, $-\pi < \phi = \arg(\lambda/\alpha b) < \pi$, and $s^{\kappa_0} = e^{\kappa_0 \mathrm{Log}(s)}$. (The above case precludes consideration of the case when $\lambda$ is a pure real number that is negative, but a suitable choice of a branch of $\log(z)$ can be made to accommodate this case.) Note that $\mathrm{Re}(\kappa_0) > 0$, and since we have assumed that $\lambda \neq \alpha^m b$ for any $m = 2, 3, \ldots$, $\kappa_0$ cannot be an integer. Substituting the above expression for $Z$ into equation (3.15) gives the relation

$$f(s) = f(s/\alpha).$$

Let $\mathrm{Log}(s) = \eta$ and $f(s) = g(\eta)$. Then the above equation for $f$ implies that

$$g(\eta) = g(\eta - \ln(\alpha)),$$

so that $g$ must be periodic with period $\ln(\alpha)$. The completeness of the Fourier series basis implies that $f$ can be expressed in the form

$$f(s) = \sum_{-\infty}^{\infty} w_n \exp\left(\frac{2n\pi i}{\ln(\alpha)}\mathrm{Log}(s)\right), \tag{3.17}$$

where the $w_n$'s are constants, and we can thus derive the following solution to equation (3.15) satisfying (3.16):

$$Z(s) = \frac{s^{\kappa_0}\sum_{-\infty}^{\infty} w_n \exp(\frac{2n\pi i}{\ln(\alpha)}\mathrm{Log}(s))}{\prod_{n=0}^{\infty}(1+\frac{s}{r_1\alpha^n})(1+\frac{s}{r_2\alpha^n})}. \tag{3.18}$$

The above solution is in fact general since any branch of $\log(z)$ leads to a solution of the form (3.18).

We note here that Flamant [6] considered a similar problem leading to a series involving logarithms. In addition, Kato & McLeod [12] derive a series of the form (3.18) in connexion with the pantograph equation with constant coefficients and a retarded argument.

In summary, we have the following result regarding solutions to the Laplace problem.

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

**Lemma 3.4** *Suppose that* $\lambda \neq \alpha^m b$ *for* $m = 1, 2, \ldots$. *Then the general solution to equation* (3.3) *satisfying condition* (3.2) *is of the form*

$$Y(s) = \sum_{n=0}^{\infty} \frac{c_n}{s + r_1 \alpha^n} + \sum_{n=0}^{\infty} \frac{d_n}{s + r_2 \alpha^n}, \tag{3.19}$$

*if* $|\lambda| \leqslant \alpha b$; *and of the form*

$$Y(s) = \sum_{n=0}^{\infty} \frac{c_n}{s + r_1 \alpha^n} + \sum_{n=0}^{\infty} \frac{d_n}{s + r_2 \alpha^n} + \frac{s^{\kappa_0} \sum_{-\infty}^{\infty} w_n \exp(\frac{2n\pi i}{\ln(\alpha)} Log(s))}{\prod_{n=0}^{\infty}(1 + \frac{s}{r_1 \alpha^n})(1 + \frac{s}{r_2 \alpha^n})}, \tag{3.20}$$

*if* $|\lambda| > \alpha b$. *Here, the constants* $c_n$, $d_n$ *and* $\kappa_0$ *are as defined earlier and the* $w_n$ *are constants.*

Finally, we note that the existence/uniqueness result of Lemma 3.1 includes the case $\lambda = \alpha b$. For this value of $\lambda$ the unique solution holomorphic at the origin is

$$Y(s) = \frac{1}{\prod_{n=0}^{\infty}(1 + \frac{s}{r_1 \alpha^n})(1 + \frac{s}{r_2 \alpha^n})}. \tag{3.21}$$

Lemma 3.3 implies this is the only solution to equation (3.3) satisfying condition (3.2) meromorphic in $P_0$.

## 4 Distribution of eigenvalues and uniqueness

The Laplace transform analysis of the previous section can be used to glean some information regarding the distribution of eigenvalues and the uniqueness of the corresponding eigenfunctions. As in the previous section we focus exclusively on the case $a > 0$.

**Lemma 4.1** *Suppose that* $\lambda$ *is an eigenvalue for the boundary-value problem and that* $|\lambda| \leqslant \alpha b$, $\lambda \neq \alpha b$. *Then* $F(\lambda) = 0$, *where* $F$ *is the entire function defined by equation* (2.14).

**Proof** If $y$ is a solution to the boundary-value problem, then its Laplace transform $Y$ must be holomorphic in the half plane $P_0$. Since $|\lambda| \leqslant \alpha b$ and $\lambda \neq \alpha b$, the Laplace transform must be of the form defined by equation (3.19). Hence, $d_n = 0$ for each $n = 0, 1, 2, \ldots$ (i.e. $d_0 = 0$) in order to meet the holomorphy condition. The normalizing condition implies that $c_0 \neq 0$ and thus equation (3.9) implies that $F(\lambda) = 0$. □

**Lemma 4.2** *The number* $\alpha b$ *cannot be an eigenvalue.*

**Proof** Suppose that $\lambda = \alpha b$ is an eigenvalue. Then any corresponding eigenfunction $y$ must have a Laplace transform $Y$ that is holomorphic in $P_0$. But the only function satisfying equation (3.3) and condition (3.2) is given by equation (3.21), and this function must have poles in $P_0$. □

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

**Remarks**

(1) Lemma 4.2 is essentially the same result obtained by Cooper *et al.* [18] by a different method.

(2) The function $F$ is not zero at $\alpha b$. In fact, the Euler identity shows that

$$F(\alpha b) = \left( \prod_{m=1}^{\infty} (1 - \frac{r_2}{r_1 \alpha^m}) \right)^{-1} > 0,$$

since $r_2/r_1 < 0$.

(3) It is not known if $F$ has any complex zeros in the disc $|\lambda| \leqslant \alpha b$.

If we restrict ourselves to real eigenvalues, a stronger result can be asserted.

**Theorem 4.3** *There are no eigenvalues in the interval* $[-\alpha b, \alpha b]$.

**Proof** Suppose that there is an eigenvalue $\lambda \in [-\alpha b, 0]$. Then Lemma 4.1 implies that $F(\lambda) = 0$. But $F(\lambda) \geqslant 1$ for all $\lambda \leqslant 0$ since the coefficients of $(-\lambda)^n$ are all positive. Hence, there are no eigenvalues in the interval $[-\alpha b, 0]$.

Suppose now that there is an eigenvalue in the interval $(0, \alpha b)$. Let

$$\sigma(x) = \int_x^{\infty} y(\xi)\, d\xi. \tag{4.1}$$

Then, equation (1.1) implies that

$$\sigma''(x) - a\sigma'(x) - b\sigma(x) + \frac{\lambda}{\alpha}\sigma(\alpha x) = 0, \tag{4.2}$$

and conditions (1.2) and (1.3) along with the definition of $\sigma$ imply that

$$\sigma(0) = 1, \quad \sigma'(0) = 0,$$

$$\lim_{x\to\infty} \sigma(x) = 0, \quad \lim_{x\to\infty} \sigma'(x) = 0.$$

Equation (4.2) and the above boundary conditions at $x = 0$ imply that

$$\sigma''(0) = b - \frac{\lambda}{\alpha} > 0.$$

Since $\lim_{x\to\infty} \sigma(x) = 0$, the above inequality implies that $\sigma$ must have a global maximum at some point $x_M \in (0, \infty)$ at which $\sigma'(x_M) = 0$, $\sigma''(x_M) \leqslant 0$, and $\sigma(x_M) > 1$. Equation (4.2) implies that at $x_M$

$$\sigma''(x_M) - b\sigma(x_M) + \frac{\lambda}{\alpha}\sigma(\alpha x_M) = 0,$$

so that

$$b\sigma(x_M) - \frac{\lambda}{\alpha}\sigma(\alpha x_M) \leqslant 0.$$

But $\lambda/\alpha < b$, and the above inequality indicates that $\sigma(\alpha x_M) > \sigma(x_M)$ contradicting the definition of $x_M$. Therefore, there are no eigenvalues in the interval $(0, \alpha b)$. Lemma 4.2 indicates that $\alpha b$ cannot be an eigenvalue, and thus the theorem is established. □

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

If $|\lambda| > \alpha b$, the situation concerning potential eigenvalues is less clear. We know from the arguments at the end of § 2 that $F$ has an infinite number of zeros on the positive real axis, but $F$ may have complex zeros, and there may be other eigenvalues not associated with the zeros of $F$. *Prima facie* it seems as if the general solution for the Laplace transform $Y$ given by equation (3.20) must have poles in $P_0$ unless the $d_n$ and $w_n$ are zero. In fact, however, it is possible to construct solutions such that these constants are not all zero, yet the singularity at each point $-r_2\alpha^n \in P_0$, $n = 0, 1, 2, \ldots$ is removable. Formally, we have the following result:

**Lemma 4.4** *Let $\lambda \in \mathbb{C}$ be any number such that:*

(1) $|\lambda| > \alpha b$;

(2) $\lambda \neq \alpha^m b$ *for any integer* $m \geqslant 2$;

(3) $F(\lambda) \neq 0$.

*Then there exists a non zero constant $d_0$ such that the Laplace transform $Y$ defined by equation* (3.20) *is holomorphic in $P_0$.*

**Proof** In the open half plane $P_0$, the function $Y$ can have 'at worst' simple poles at $-r_2\alpha^n$. It is sufficient to show that there is a $d_0 \neq 0$ such that

$$\operatorname{Res}(Y; -r_2\alpha^n) = \lim_{s \to -r_2\alpha^n} \left(1 + \frac{s}{r_2\alpha^n}\right) Y(s) = 0$$

for each $n = 0, 1, 2, \ldots$. Let $f(s)$ be defined as in equation (3.17). Then $f(-r_2\alpha^n) = f(-r_2)$ for each $n = 0, 1, 2, \ldots$. For $n = 0$, the singularity at $-r_2$ is removable provided

$$d_0 = \frac{(-r_2)^{\kappa_0+1} f(-r_2)}{\prod_{m=0}^{\infty}(1 - \frac{r_2}{r_1\alpha^m}) \prod_{m=1}^{\infty}(1 - \frac{1}{\alpha^m})}. \tag{4.3}$$

For a given $\lambda$ satisfying the conditions of the theorem, however, $d_0$ is uniquely determined by equations (3.9) and (3.10). These equations and equation (3.14) imply that

$$d_0 = \frac{bF(\lambda)}{(r_2 - r_1) \prod_{m=1}^{\infty}(1 - \frac{\lambda}{\alpha^{m+1}b})}. \tag{4.4}$$

The constants $w_n$ are at this stage arbitrary and we may thus choose them so that the two expressions for $d_0$ agree. With such a choice of $w_n$ we thus have that

$$\operatorname{Res}(Y; -r_2) = 0.$$

Now suppose that $n \geqslant 1$. The contribution to the residue at $-r_2\alpha^n$ from the Mittag-Leffler solution $Y_h$ is

$$\begin{aligned} \operatorname{Res}(Y_h; -r_2\alpha^n) &= \frac{d_n}{r_2\alpha^n} \\ &= \frac{\left(\frac{\lambda}{\alpha b}\right)^n d_0}{r_2 \prod_{m=1}^{n}(1 - \alpha^m)(1 - \frac{r_2}{r_1}\alpha^m)}. \end{aligned}$$

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

The contribution to the residue from the $Z$ terms is

$$\begin{aligned}\mathrm{Res}(Z\,;-r_2\alpha^n) &= \frac{(-r_2\alpha^n)^{\kappa_0} f(-r_2\alpha^n)}{\prod_{m=0}^{\infty}(1-\frac{r_2\alpha^n}{r_1\alpha^m})\prod_{\substack{m=0\\ m\neq n}}^{\infty}(1-\frac{\alpha^n}{\alpha^m})} \\ &= \frac{(-r_2\alpha^n)^{\kappa_0} f(-r_2)}{\prod_{m=1}^{n}(1-\alpha^m)(1-\frac{r_2}{r_1}\alpha^m)\prod_{m=0}^{\infty}(1-\frac{r_2}{r_1\alpha^m})\prod_{m=1}^{\infty}(1-\frac{1}{\alpha^m})}.\end{aligned}$$

Using the expression (4.3) for $d_0$ we see that

$$\mathrm{Res}(Z\,;-r_2\alpha^n) = -\frac{(\alpha^n)^{\kappa_0} d_0}{r_2\prod_{m=1}^{n}(1-\alpha^m)(1-\frac{r_2}{r_1}\alpha^m)},$$

and using the definition of $\kappa_0$ we see that

$$\mathrm{Res}(Y_h + Z\,;-r_2\alpha^n) = \mathrm{Res}(Y\,;-r_2\alpha^n) = 0.$$

□

Under the conditions of Lemma 4.4 we may construct a solution to equation (3.3) satisfying condition (3.2) that is holomorphic in the right half plane $P_0$ having a branch point at $s = 0$. The simplest such solution would be to choose $w_n = 0$ for each $n = \pm 1, \pm 2, \ldots$, and let

$$w_0 = \frac{bF(\lambda)\prod_{m=0}^{\infty}(1-\frac{r_2}{r_1\alpha^m})\prod_{m=1}^{\infty}(1-\frac{1}{\alpha^m})}{(r_2-r_1)(-r_2)^{\kappa_0+1}\prod_{m=1}^{\infty}(1-\frac{\lambda}{\alpha^{m+1}b})}, \tag{4.5}$$

but there are clearly an infinite number of solutions.

Although solutions exist to the equations defining the Laplace transform that are holomorphic in $P_0$, these solutions need not correspond to solutions to the boundary-value problem. It remains to show that the transforms can be inverted. We focus on the simplest transform with a branch point given by

$$\begin{aligned}Y_0(s) = &\sum_{n=0}^{\infty}\frac{c_n}{s+r_1\alpha^n} + \sum_{n=0}^{\infty}\frac{d_n}{s+r_2\alpha^n} \\ &+\frac{s^{\kappa_0}w_0}{\prod_{k=0}^{\infty}(1+\frac{s}{r_1\alpha^k})(1+\frac{s}{r_2\alpha^k})},\end{aligned} \tag{4.6}$$

where $w_0$ is as defined by equation (4.5), and the $c_n$ and $d_n$ are the constants for the Mittag-Leffler solution. Note that under the conditions of Lemma 4.4 the constant $c_0$ is determined uniquely by equations (3.9) and (3.10) since $d_0$ is known. Hence all the $c_n$'s are uniquely determined and such that $Y$ meets the condition $Y(0) = 1$.

Let $H^2(P_0)$ denote the Hardy space of functions $g$ holomorphic in the half plane $P_0$ such that for any fixed $\mu > 0$ the (Lebesgue) integral

$$M_\mu^2(g) = \int_{-\infty}^{\infty} |g(\mu + i\tau)|^2\, d\tau$$

is finite. Given that $Y_0$ is holomorphic in $P_0$, it is straightforward to show that $Y_0 \in H^2(P_0)$. The significance of this observation is that the Paley-Wiener theorem (cf. Hoffman [10])

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

can be invoked to prove that there is a unique $y \in L^2[0,\infty)$ such that

$$Y_0(s) = \int_0^\infty y(x)e^{-sx}\,dx.$$

The Laplace transform is thus invertible, but we know only that $y \in L^2[0,\infty)$. The condition $\lim_{s\to 0^+} Y_0(s) = 1$ can be used to show that $y$ is integrable, but the solution guaranteed by the Paley-Wiener theorem need not be a classical solution, e.g. the solution may not be in $C^2(0,\infty)$ and it may not be differentiable at points such as $x = 0$. Similar arguments can be made for more general choices of the constants $w_n$ to construct other solutions. In this sense, the spectrum includes intervals of the form $(\alpha^m b, \alpha^{m+1}b)$, and is thus not discrete.

There are, however, certain values of $\lambda$, in a countable set corresponding to the zeros of $F$, that do produce unique solutions.

**Theorem 4.5** *Suppose that $F(\lambda) = 0$ and that $\lambda \neq \alpha^m b$ for any integer $m \geqslant 2$. Then there exists a unique solution to the boundary value problem.*

**Proof** If $F(\lambda) = 0$, then $\lambda \neq \alpha b$, and equation (3.9) implies that $d_0 G(\lambda) = 0$. Equation (3.14) indicates that $G(\lambda) \neq 0$ since $\lambda \neq \alpha^m b$ for any integer $m \geqslant 1$; hence, $d_0 = 0$. Therefore $d_k = 0$ for all $k = 0, 1, 2, \ldots$, and consequently $w_n = 0$ for all integers $n$. The only Laplace transform holomorphic in $P_0$ is thus

$$Y(s) = \sum_{n=0}^{\infty} \frac{c_n}{s + r_1\alpha^n}.$$

The constant $c_0$ is determined uniquely by equation (3.10) and hence the $c_n$'s are all determined uniquely. The above transform can be inverted to get the Dirichlet series solution discussed in § 2. □

We know that $F(\alpha b) > 0$, and the functional relation (2.15) indicates that $F(\alpha^2 b) < 0$. Depending on the parameters $a$, $b$, and $\alpha$, however, it is possible that $F(\alpha^m b) = 0$ for some $m \geqslant 3$. For example, the functional relation (2.15) implies that, for $\lambda = b$,

$$F(\alpha^2 b) = -\frac{a}{r_1} F(\alpha b),$$

and, for $\lambda = \alpha b$,

$$\begin{aligned} r_1^2 F(\alpha^3 b) &= -a r_1 F(\alpha^2 b) - b(\alpha - 1) F(\alpha b) \\ &= \left(a^2 - b(\alpha - 1)\right) F(\alpha b), \end{aligned}$$

so that $F(\alpha^3 b) = 0$ if $a^2 = b(\alpha - 1)$. If $F(\alpha^m b) = 0$ for some $m \geqslant 3$ then $F(\alpha^{m+1} b) \neq 0$, for otherwise the functional relation would imply that $F(\alpha^k b) = 0$ for all $k = 2, 3, \ldots$ leading to the contradiction that $F(\alpha^2 b) = 0$.

Equation (3.14) implies that if $F(\alpha^m b) = 0$ for some $m \geqslant 3$ then $G(\alpha^m b) = 0$, since $F(\alpha^{m+1} b) \neq 0$. The Mittag-Leffler solution can thus be salvaged, but $c_0$ and $d_0$ need satisfy only equation (3.10) so that an infinite number of solutions for the Laplace transform are available. In essence, we do not have uniqueness of solutions for these cases, though the Dirichlet series solution of § 2 is among the solutions.

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

Although generically there may be an infinite number of solutions for a given value of $\lambda$, for real positive eigenvalues certain uniqueness results can be gleaned by imposing further integrability conditions on the solutions. This is perhaps not unexpected as uniqueness to solutions of the pantograph equation with an advanced argument is linked to growth/decay conditions (cf. [12]).

**Theorem 4.6** *Suppose that $\lambda$ is a real eigenvalue such that $\alpha^n b < \lambda < \alpha^{n+1} b$ for some $n = 1, 2, \ldots$. Then there exists at most one solution to the boundary-value problem that is $n+1$ times integrable on the interval $[0, \infty)$.*

**Proof** We begin first with the case $n = 1$ and $\alpha b < \lambda < \alpha^2 b$. Suppose that $y_1$ and $y_2$ are two solutions that are twice integrable on the interval $[0, \infty)$. Let

$$\delta_0(x) = y_1(x) - y_2(x),$$
$$\delta_1(x) = \int_x^\infty \delta_0(\xi)\, d\xi,$$
$$\delta_2(x) = \int_x^\infty \delta_1(\xi)\, d\xi.$$

Equation (1.1) along with conditions (1.2) and (1.3) imply that

$$\delta_1''(x) - a\delta_1'(x) - b\delta_1(x) + \frac{\lambda}{\alpha}\delta_1(\alpha x) = 0, \tag{4.7}$$

and

$$\delta_1(0) = 0, \quad \lim_{x\to\infty} \delta_1(x) = 0, \quad \lim_{x\to\infty} \delta_1'(x) = 0. \tag{4.8}$$

Now, $b < \lambda/\alpha < \alpha b$, and the proof of Theorem 4.3 can be modified (or $\delta_0$ suitably scaled) to show that there are no integrable solutions to this problem unless

$$\delta_2(0) = \int_0^\infty \delta_1(\xi)\, d\xi = 0. \tag{4.9}$$

Equation (4.7) and conditions (4.8) imply that

$$\delta_2''(x) - a\delta_2'(x) - b\delta_2(x) + \frac{\lambda}{\alpha^2}\delta_2(\alpha x) = 0, \tag{4.10}$$

and the definition of $\delta_2$ indicates that

$$\lim_{x\to\infty} \delta_2(x) = 0. \tag{4.11}$$

We now show that $\delta_2 = 0$ for all $x \in [0, \infty)$. Suppose that $\delta_2$ is not identically zero in the interval $[0, \infty)$. Then equations (4.9) and (4.11) imply that $\delta_2$ must attain either a global positive maximum or a global negative minimum in the interval $(0, \infty)$. Without loss of generality it can be supposed that $\delta_2$ attains a global maximum at some point $x_M \in (0, \infty)$ (i.e. we could always redefine $\delta_0$ as $y_2 - y_1$). Now, $\delta_2$ is twice continuously differentiable in $(0, \infty)$, and consequently $\delta_2''(x_M) \leqslant 0$, $\delta_2'(x_M) = 0$, and $\delta_2(x_M) > 0$. Hence,

$$b\delta_2(x_M) - \frac{\lambda}{\alpha^2}\delta_2(\alpha x_M) \leqslant 0.$$

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

But $\lambda/\alpha^2 < b$, and therefore $\delta_2(\alpha x_M) > \delta_2(x_M)$, contradicting the definition of $x_M$. We thus conclude that $\delta_2(x) = 0$ for all $x \in [0, \infty)$. This means that $\delta_1(x) = \delta_0(x) = 0$ for all $x \in [0, \infty)$, and consequently $y_1 = y_2$.

Let us now consider the case $n = 2$ and $\alpha^2 b < \lambda < \alpha^3 b$. Let $\delta_k(x)$, $k = 0, 1, 2$ be as defined earlier, and let

$$\delta_3 = \int_x^\infty \delta_2(\xi)\, d\xi.$$

The function $\delta_1$ must satisfy equation (4.7) and conditions (4.8) for $\alpha b < \lambda/\alpha < \alpha^2 b$, but the boundary-value problem for $\delta_1$ corresponds to that for $\delta_0$ in the case $n = 1$ considered earlier. We thus conclude that $\delta_2$ satisfies equation (4.9) for this case. The function $\delta_2$ must also satisfy equation (4.10) with $b < \lambda/\alpha^2 < \alpha b$, along with condition (4.11). Again we can use the proof of Theorem 4.3 to assert that there are no integrable solutions to this boundary-value problem, unless

$$\delta_3(0) = \int_0^\infty \delta_2(\xi)\, d\xi = 0. \tag{4.12}$$

The function $\delta_3$ must therefore satisfy the equation

$$\delta_3''(x) - a\delta_3'(x) - b\delta_3(x) + \frac{\lambda}{\alpha^3}\delta_3(\alpha x) = 0, \tag{4.13}$$

along with condition (4.12), and the condition

$$\lim_{x\to\infty} \delta_3(x) = 0. \tag{4.14}$$

Now, $\lambda/\alpha^3 < b$, and the same argument used to show that $\delta_2 = 0$ in the case $n = 1$ can be used here to show that $\delta_3(x) = 0$ for all $x \in [0, \infty)$. We thus conclude that $y_1 = y_2$.

The above arguments outline an inductive proof for the theorem. Suppose that the uniqueness result is true for all $n \leqslant N$. Then, for $\alpha^{N+1} b < \lambda < \alpha^{N+2} b$ we can construct a family of boundary-value problems of the form

$$\delta_k''(x) - a\delta_k'(x) - b\delta_k(x) + \frac{\lambda}{\alpha^k}\delta_k(\alpha x) = 0,$$

$$\delta_k(0) = I_k, \quad \lim_{x\to\infty}(x) = 0$$

for $k = 1, 2, \ldots, N + 2$. Here $\delta_0(x)$ is as defined earlier, and for $k = 1, 2, \ldots, N + 2$

$$\delta_k(x) = \int_x^\infty \delta_{k-1}(\xi)\, d\xi.$$

The functions $\delta_k$ are all smooth well-defined under the hypothesis that $\delta_0$ is at least $N+2$ times integrable. The numbers $I_k, k \geqslant 2$ are not determined directly by the conditions (1.2) and (1.3) and at this stage are unknown ($I_1 = 0$ is known): we show that they must be zero. By hypothesis the result is true for $n = N$; the boundary-value problem for $\delta_1$, however also corresponds to this case and hence we conclude that $I_2 = 0$. Similarly, the boundary-value problem for $\delta_2$ corresponds to the case $n = N - 1$ and we conclude that $I_3 = 0$. We can continue in this manner until eventually we conclude that $I_{N+2} = 0$. The boundary-value problem for $\delta_{N+2}$ now has the conditions $\delta_{N+2}(0) = 0$ and $\lim_{x\to\infty} \delta_2(x) = 0$, and here $\lambda/\alpha^{N+2} < b$. We can thus use the same argument used in the cases $n = 1$ and $n = 2$

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

above to show that $\delta_{N+2}(x) = 0$ for all $x \in [0, \infty)$, and hence $y_1 = y_2$. The result is thus true for all $n \leqslant N + 1$ and by induction for all positive integers $n$. □

**Corollary 4.7** *Suppose that $\lambda = \alpha^m b$ for some integer $m \geqslant 2$. Then there exists at most one solution to the boundary-value problem that is $m$ times integrable on the interval $[0, \infty)$.*

**Proof** The proof of this result follows along the same lines as that for Theorem 4.6. We can use Lemma 4.2 and avoid an appeal to the proof technique of Theorem 4.3. The only notable difference occurs when we consider the boundary-value problem

$$\delta_m''(x) - a\delta_m'(x) - b\delta_m(x) + \frac{\lambda}{\alpha^m}\delta_m(\alpha x) = 0$$

$$\delta_m(0) = 0, \quad \lim_{x\to\infty} \delta_m(x) = 0.$$

We can argue as before that $\delta_m$ must have a global maximum at some point $x_M \in (0, \infty)$. This leads to the inequality

$$\delta_m(x_M) - \delta_m(\alpha x_M) \leqslant 0$$

(since $\lambda/\alpha^m = b$), and it may be that $\delta_m''(x_M) = 0$ and $\delta_m(x_M) = \delta_m(\alpha x_M)$. This does not contradict the definition of $x_M$, but now the same arguments can be applied to the 'other' maximum point $\alpha x_M$. Again, we must have that $\delta_m''(\alpha x_M) = 0$ and $\delta_m(\alpha x_M) = \delta_m(\alpha^2 x_M)$. We can repeat these arguments *ad infinitum* and thus construct a sequence of point $x_M, \alpha x_M, \alpha^2 x_M, \ldots$ such that $\delta_m(x_M) = \delta_m(\alpha x_M) = \cdots \delta_m(\alpha^k x_M) = \cdots$. Now, $\lim_{k\to\infty} \delta_m(\alpha^k x_M) = \delta_m(x_M) > 0$ and hence we have the contradiction that $\lim_{x\to\infty} \delta_m(x) \neq 0$. □

Note that the above Theorem and Corollary do not establish the existence of $n + 1$ times integrable solutions only that there can be at most one of them for a given $\lambda$. For the case when $F(\lambda) = 0$, the Dirichlet series is clearly the $n + 1$ times integrable solution to the problem.

Finally, we note that the case when $\lambda = \alpha^m b$ for some $m \geqslant 2$ and $F(\alpha^m b) \neq 0$ remains to be explored. For this case we cannot salvage the Mittag-Leffler solution for the Laplace transform and it is not clear whether equation (3.3) even has solutions satisfying condition (3.2). As remarked in the proof of Lemma 3.1, not all the derivatives of $Y$ at $s = 0$ can be determined in this case unless some compatibility conditions are satisfied to avoid contradictions. These conditions amount to the function $F$ vanishing at $\lambda = \alpha^m b$. If the compatibility conditions are satisfied, then the derivatives are defined but not uniquely. If the conditions are not satisfied then we conclude that there are no solutions $Y$ holomorphic at the origin. Certainly if solutions exists for these cases, then they are not unique as the $Z$ term defined by equation (3.18) can always be introduced. Note that $\kappa_0$ is a positive integer in this case.

## References

[1] Adams, C. R. (1931) Linear $q$-difference equations. *Bull. Amer. Math. Soc.* June, 361–400.
[2] Andrews, G. (1971) *Number Theory*. Saunders.

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press

[3] Carr J. & Dyson J. (1975) The matrix functional equation $y'(x) = Ay(\lambda x) + By(x)$. *Proc. Roy. Soc. Edinburgh*, **75**(A), 5–22.
[4] Derfel, G. & Iserles, A. (1997) The pantograph equation in the complex plane. *J. Math. Anal. & Applic.* **213**, 117–132.
[5] Fox, L., Mayers, D. F., Ockendon, J. R. & Tayler, A. B. (1971) On a functional differential equation. *J. Inst. Math. Appl.* **8**, 271–307.
[6] Flamant, P. (1924) Sur une équation différentielle fonctionnelle linéare. *Palermo Rend.*, **48**, 135–208.
[7] Fredrickson, P. O. (1971) Dirichlet series solution for certain functional differential equations. In: M. Uradbe (ed.), *Japan-United States Seminar on Ordinary Differential and Functional Equations: Lecture Notes in Mathematics 243*, pp. 247–254. Springer-Verlag.
[8] Hall, A. J. (1991) Steady Size Distributions in Cell Populations. PhD Thesis, Massey University.
[9] Hall, A. J. & Wake, G. C. (1989) A functional differential equation arising in the modelling of cell-growth. *J. Aust. Math. Soc. Ser. B*, **30**, 424–435.
[10] Hoffman, K. (1962) *Banach Spaces of Analytic Functions.* Prentice-Hall.
[11] Iserles, A. (1993) On the generalized pantograph functional-differential equation. *Euro. J. Appl. Math.* **4**, 1–38.
[12] Kato, T. & McLeod, J. B. (1971) The functional differential equation $y'(x) = ay(\lambda x) + by(x)$. *Bull. Amer. Math. Soc.* **77**, 891–937.
[13] Kim, H. K. (1998) Advanced Second Order Functional Differential Equations. PhD thesis, Massey University.
[14] Morris, G. R., Feldstein, A. & Bowen, E. W. (1972) The Phragmén-Lindelöf principle and a class of functional differential equations. *Ordinary Differential Equations*, pp. 513–540. Academic Press.
[15] Ockendon, J. R. & Tayler, A. B. (1971) The dynamics of a current collection system for an electric locomotive. *Proc. Roy. Soc. Lond. A*, **322**, 447–468.
[16] Rogers, J. (1988) Existence, uniqueness, and construction of the solution of a system of ordinary functional differential equations, with application to the design of perfectly focusing symmetric lenses. *IMA J. Appl. Math.*, **41**, 105–134.
[17] van Brunt, B. (1993) An existence, uniqueness, and analyticity theorem for a class of functional differential equations. *NZ J. Math.*, **22**, 101–107.
[18] Wake, G. C., Cooper, S., Kim, H. K. & van Brunt, B. (2000) Functional differential equations for cell-growth models with dispersion. *Comm. Appl. Analysis*, **4**, 561–573.

https://doi.org/10.1017/S0956792501004624 Published online by Cambridge University Press